NEUTRINO OSCILLATIONS IN HETEROGENEOUS MEDIA

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Algebraic formalism is proposed to describe the neutrino oscillations in heterogeneous media when traditional approaches based on perturbation theory are inefficient. Exact solutions of corresponding equations are obtained for models of variations in the density of a medium occurring at the Sun and the Earth.

1. INTRODUCTION

In the mid–1960s academician U.Kh. Kopvillem proposed the new view on physics of coherent processes of radiation interaction with matter (such as induction, avalanche, and echo). The progress that has been made towards the classification of elementary particles (hadrons) according to the $SU(3)$ symmetry by that time led him to an idea to classify various effects of echo using the intimate properties of the symmetry underlying their nature and ciphered in the Hamiltonian of the process, namely, in the dynamic Lie algebra that is engendered by it. This assumption implies conceptually a theoretical description and an experimental search for echo effects in different physical groups of the Lie algebra.[1,2] This idea acquiered its own mathematical form and physical content in Refs. 3–6.

Algebraic basis plays a major role in modern physics. It is applied to the study of various objects in nature. By the algebraic basis is meant a principle of studying rather than a method for obtaining the specific results. The methods are considerably different depending on the goals to be sought; however, they are always based on application of some or other properties of algebraic structure. In this paper we apply the Lie algebra method[7,8] to the study of nonstationary neutrino oscillations in heterogeneous media.

The available experimental and theoretical data on the existence of neutrino mass (quark–lepton symmetry, unified superstring theory, etc.) verify the hypothesis of neutrino mixing.[9] In the framework of this hypothesis the neutrino states with specific flavors are superpositions of the states with specific masses. This is in many ways similar to the effects of mixing of K° and K mesons and of quarks and is caused by a certain new interaction leading to the oscillations of the neutrino flavor (i.e., to the transitions of the type $\nu_e - \nu_y$, $\nu_e - \nu_\tau$, etc.) and hence to the explicit nonconservation of a lepton charge. Such fundamental consequences of the hypothesis of neutrino mixing as nonzero mass and nonconservation of the lepton charge gain interest in search for $\nu - \nu$ oscillations in the beams of reactor and solar neutrinos.

Here we study $\nu - \nu$ oscillations in the formalism of dynamic symmetry which considers in a natural manner the effect of the medium whose density varies along the path of neutrino beam propagation, effects of absorption in it, and the existence of the arbitrary number of different types of neutrinos as well as possibile transitions in the magnetic field with simultaneous change of the flavor and helicity. If we restrict ourselves to the case of oscillations in the system of two types of neutrinos, i.e., $\nu_e - \nu_\mu$, $\nu - \overline{\nu}$, $\nu_{eL} - \nu_{\mu R}$, and $\nu_L - \overline{\nu}_R$, then $SU(2)$ will be the group of dynamic symmetry of such a process. When N types of neutrinos are available, $SU(N)$ is the total group of symmetry; however, in the majority of important cases the symmetry can be reduced to $SU(2)$, $SO(3)$, $SO(3, 1)$, $SO(4)$, and other groups with the comparatively small number of parameters.

The algebraic method developed in the present paper for studying the nonstationary quantum processes[10] enables the exact solutions of dynamical problems to be found when other methods are inefficient. In particular, the above–mentioned is possible in such cases in which it is difficult to use perturbation theory as well as in the case of fast and irregular variations in the field parameters (the amplitude, frequency, and phase) in physics of magnetic and optical resonances and variations in the density of a medium engendering neutrino oscillations. In this paper the given approach is used not only as the new formalism for describing the neutrino oscillations but also as a method for obtaining the new solutions of this problem for realistic models of variations in the density of heterogeneous media.

It is easy to show in an explicit form how from the hypothesis of mixing it follows that the neutrino oscillations can appear in vacuum. Let us expand an arbitrary neutrino state in terms of the eigenstates of weak interactions with specific momenta p_ν

$$|\nu\rangle = \sum_\xi \Psi_\xi \, |\nu_\xi\rangle \,, \tag{1}$$

here the subscript ξ denotes flavor, ξ = e, μ, τ, ...

From the vectors $|\nu_\xi\rangle$ let us form such linear combinations, denoted by the Latin subscripts, which satisfy the following equation for the eigenvalues:

$$H|\nu_j\rangle \equiv (H_0 + H_{\text{mix}})\,|\nu_j\rangle = E_j|\nu_j\rangle \,, \quad j = 1, 2, 3, \dots \,. \tag{2}$$

We add the term H_{mix}, being responsible for mixing of neutrinos of different types, to the standard Hamiltonian H_0 which is diagonal on the flavor basis

$$\langle\nu_\xi|H_0|\nu_\xi\rangle = P_\nu \,, \quad \langle\nu_\xi|H_0|\nu_{\xi}\rangle = 0 \,, \tag{3}$$

The vectors with the Latin subscripts characterize the states with specific masses. The transition from the flavor basis to the energy basis is prescribed by a certain unitary transform

$$|\nu_\xi\rangle = \sum_j S_{\xi j}|\nu_j\rangle \,, \quad S_{\xi j} \equiv \langle\nu_j|\nu_\xi\rangle \,. \tag{4}$$

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The probability of oscillation transition of the ξ–type neutrino at $t = 0$ to the η–type neutrino by the instant t can be calculated by the solution of the temporal Schrödinger wave equation with stationary Hamiltonian $H_0 + H_{\text{mix}}$

$$P(\nu_\eta;\, t|\nu_\xi;\, 0) = |\langle\nu_\eta|\, \exp[-i\,t\,(H_0 + H_{\text{mix}})]\,|\nu_\xi\rangle|^2 =$$

$$= |\sum_j S_{\xi j}\,[\exp(-i\,t\,E_j)]S^{+}_{j\eta}| = \sum_j |\langle\nu_j|\nu_\xi\rangle|^2 |\langle\nu_\eta|\nu_j\rangle|^2 +$$

$$+ 2\mathrm{Re} \sum_{j\neq\kappa} \langle\nu_\eta|\nu_j\rangle\langle\nu_\eta|\nu_\kappa\rangle^* \langle\nu_j|\nu_\xi\rangle\langle\nu_\kappa|\nu_\xi\rangle^* \times$$

$$\times \exp[i\,(E_\kappa - E_j)\,t]\,. \tag{5}$$

The second term in the last relation is responsible for the neutrino oscillations sought in vacuum. They are possible if at least for one pair of levels j, κ in the exponent $E_\kappa \neq E_j$ and if at least for one of the pairs j, κ and ξ, η the amplitude coefficient of the exponent is nonzero. Thus in order for the neutrino oscillations can appear in vacuum it is necessary to fulfil the following two conditions:

(1) if at least one type of the neutrino possesses nonzero mass, i.e.,

$$\langle\nu_\xi|H_{\text{mix}}|\nu_\xi\rangle > 0\,, \tag{6}$$

2) if at least one pair of neutrinos of different types is mixed, i.e.,

$$\langle\nu_\xi|H_{\text{mix}}|\nu_\eta\rangle \neq 0\,, \qquad \xi \neq \eta\,. \tag{7}$$

2. COMMON PROPERTIES OF OSCILLATIONS OF TWO TYPES OF NEUTRINOS IN THE MEDIUM

In matter the neutrinos undergo the elastic forward scattering by electrons and nuclei due to weak interaction; in addition, the scattering amplitude is different for different types of neutrinos. Thus the electron neutrino possesses the excess amplitude of elastic scattering by electrons of the medium due to charged currents (compared to the amplitude of scattering by all the targets caused by neutral currents which is the same for all types of neutrinos). The account of neutrino interaction with particulate matter results in the appearance of the new term H_{int} in the Hamiltonian. This term is no longer stationary when neutrinos move through the heterogeneous medium. Since we manipulate the units in which $\hbar = c = 1$ in this case $r \simeq t$ and consequently, the total Hamiltonian becomes the explicit function of the time

$$\mathcal{H}(t) \equiv H_0 + H_{\text{mix}} + H_{\text{int}}\,, \tag{8}$$

in addition, its dependence on time is unambiguously determined by variation in the density of the medium along the propagation path of the neutrino beam. The temporal Schrödinger wave equation with Hamiltonian (8) cannot be solved exactly (in terms of the known special functions) for arbitrary variation in the density $\rho(r)$ of the medium even in the simplest case of two types of neutrinos. In this paper we obtain the classes of exact solutions of this problem for different laws modeling the variation in $\rho(r)$ in the natural media by the method of dynamic symmetry.

It was first noted in Ref. 11 that the behavior of neutrino oscillations in the homogeneous medium is different compared to vacuum due to the interaction of the neutrino with charged currents. In so doing the oscillation length and angle of mixing in the medium are functions of its density. Then it was shown in Refs. 12 and 13 that in the medium of variable density the effect of neutrino oscillations can be intensified in the process of propagation of the neutrino beam through the resonance layers. Thus the heterogeneous medium initiates oscillations and in so doing plays the role of the external resonance electromagnetic field in physics of magnetic and optical resonances. According to the available data, the mass m_{ν_e} is far beyond the limits of sensitivity of laboratory experiments and therefore the observed resonance oscillations of neutrinos in the heterogeneous media provide the unique possibility for measuring m_{ν_e}.

The temporal Schrödinger wave equation for two types of neutrinos, in which the effect of matter was taken into account, was considered in Refs. 11–13 for different modes of oscillations. On the basis of flavors it has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\Psi_e\\ \Psi_\mu\end{pmatrix} = -\,i\begin{pmatrix}h_e(t) & \bar{h}\\ \bar{h} & h_\mu(t)\end{pmatrix}\begin{pmatrix}\Psi_e\\ \Psi_\mu\end{pmatrix}. \tag{9}$$

If we omit the terms caused by neutral currents which make the identical contributions to the wave functions, then the matrix elements of the Hamiltonian will be equal to[12]

$$h \equiv h_e(t) - h_\mu(t) = \frac{2\pi}{l_\nu}\left(\cos 2\Theta - \frac{l_\nu}{l_0}\right),$$

$$\bar{h} = \frac{\pi}{l_\nu}\sin 2\Theta\,, \tag{10}$$

where Θ is the angle of mixing in vacuum, $|l_\nu| \equiv 4\pi\rho/\Delta m^2$ is the length of oscillations in vacuum, and $\Delta m^2 \equiv m_1^2 - m_2^2$. The characteristic length for the medium has the form

$$l_0^{-1} \equiv \frac{\mathrm{r}}{2\pi\, m_N\, p}\sum_\kappa (f_{e\kappa} - f_{\mu\kappa})\, n_\kappa\,, \tag{11}$$

where $f_{\xi\kappa}$ is the amplitude of forward scattering of the ν_ξ–type neutrinos by the kth constituent of matter at the zero angle; ξ = e, μ; κ = e, p, n; and, n_κ is the particle number of the kth constituent per a nucleon of mass m_N. Traditionally, the equation is solved in the oscillation transition probability from ν_e at $t = 0$ to ν_e at the point $r \simeq t$. The solution of this equation is derived from Eq. (9) and has the form

$$\dddot{p} - \frac{\dot{h}}{h}\ddot{p} + \left(h^2 + 4\bar{h}^2\right)\dot{p} - 2\frac{\dot{h}}{h}\bar{h}^2\,(2\,p - 1) = 0\,, \tag{12}$$

where $p(0) = 1$, $\dot{p}(0) = 0$, $\ddot{p}(0) = -\,2\,\bar{h}^2$, i.e., ν_e is created in the source.

For the medium with constant density ($\dot{h} = 0$) the solution of Eq. (12) has the simple form[11]

$$p = 1 - A\sin^2(\pi\, r/l_m)\,, \tag{13}$$

where A determines the oscillation depth

$$A \equiv \sin^2 2\Theta_m = (\overline{h}\, l_m/\pi)^2 , \tag{14}$$

and l_m is the oscillation length in the medium

$$l_m^2 \equiv 4\pi^2/(h^2 + 4\overline{h}^2) . \tag{15}$$

The angle of mixing Θ_m in the medium is given by formula (14) and using Eqs. (10) and (15) it can be represented in the form

$$\sin^2 2\Theta_m = \sin^2 2\Theta\, [(\cos\Theta - l_\nu/l_0)^2 + \sin^2 2\Theta]^{-1} . \tag{16}$$

It follows from this that the dependence of the parameter of mixing in the medium in the form of Eq. (16) on the density of matter exhibits resonance at small $\sin^2 2\Theta$: $l_\nu/l_0 \sim \rho p$. Equation (16) reaches the maximum of $\Theta_m = 45°$ at

$$l_\nu/l_0 = \cos 2\Theta , \tag{17}$$

i.e., the process of mixing in matter exhibits maximum. From Eqs. (17), (11), and (16) it is easy to calculate the resonance density of matter[12]

$$\rho_R = m_N\, (\Delta\, m)^2 (\cos 2\Theta)\, (2\sqrt{2}\; G_F\, p)^{-1} \tag{18}$$

and the half–width of the resonance

$$\Delta\, \rho_R = \rho_R \tan 2\Theta , \tag{19}$$

here G_F is the Fermi constant. Thus the medium can intensify mixing and increase the probability of transition from one type of neutrinos to another one.

An alternative opportunity for describing the neutrino oscillations in the heterogeneous media is associated with the use of the Schrödinger wave equation on the basis of eigenstates of neutrinos in the medium $|\nu_{im}>$, $i = 1, 2$, which is related to the flavor basis by the transform

$$S = \begin{pmatrix} \cos\Theta_m & \sin\Theta_m \\ -\sin\Theta_m & \cos\Theta_m \end{pmatrix} , \tag{20}$$

which makes Hamiltonian (9) diagonal

$$S^{-1}\mathcal{H}S = \mathcal{H}_{\text{diag}} . \tag{21}$$

The Schrödinger wave equation acquires the form[13]

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \Psi_{1m} \\ \Psi_{2m} \end{pmatrix} = -i \begin{pmatrix} h_1 & -i\,\dot{\Theta}_m \\ i\,\dot{\Theta}_m & h_2 \end{pmatrix} \begin{pmatrix} \Psi_{1m} \\ \Psi_{2m} \end{pmatrix} , \tag{22}$$

in which the diagonal elements can be found using Eqs. (10) and (11)

$$h_{1,2} = \frac{1}{2}\, [h_e + h_\mu \pm (h^2 + 4\overline{h}^2)^{1/2}] . \tag{23}$$

In the heterogeneous medium the angle Θ_m is unsteady, which leads to the change of flavor of the eigenstates $|\nu_{im}>$. The relation for Θ_m as a function of ρ has the form[13]

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\Theta_m = \frac{1}{2\rho_R}\,\frac{\tan 2\Theta}{(1 - \rho/\rho_R)^2 + \tan^2 2\Theta}\,\frac{\mathrm{d}\rho}{\mathrm{d}t} . \tag{24}$$

Thus the likely solutions of Eqs. (9) or (22) depend strongly on the type of the function of variation in the density of the medium versus distance. The adiabatic density variation was considered in Ref. 13 in detail. It can be determined by the condition

$$|\dot{\Theta}_m| \ll |h_{1,2}| = 2\pi/l_m , \tag{25}$$

under which the nondiagonal elements of matrix (22) can be neglected. The purpose of this paper is the derivation of the exact solutions of equations (9) and (22) (including those for more than two types of neutrinos) for models of variation in the density of the medium occurring at the Sun and Earth. The exact results provide the conditions of complete transformation of neutrinos from one type to another to be determined. The field of applicability of such solutions is not limited by the criteria for adiabatic behavior or, *vice versa*, discontinuities in the density.

3. ALGEBRAIC FORMALISM

In this section we consider the algebraic formalism of dynamic symmetry[10] as applied to the theory of neutrino oscillations in the medium. The Schrödinger wave equation for N types of neutrinos can be written in the general form

$$i\,\frac{\mathrm{d}}{\mathrm{d}t}\,\Psi(t) = \mathcal{H}(t)\, \Psi(t) , \tag{26}$$

where $\Psi(t)$ is the N–dimensional vector of probability amplitudes of transitions, which is represented on the flavor or eigenstate basis in the medium. The algebraic method of solution of Eq. (26) is based on the Hamiltonian expansion

$$\mathcal{H}(t) = \sum_{j=1}^{n} h_j(t)\, \hat{H}_j , \tag{27}$$

on the basis $\{H_j,\ j = 1, \ldots, n\}$ of a certain representation of the dynamic n–dimensional Lie algebra L_n with the coefficients $h_j(t)$ and on the corresponding parametrization of the evolution operator

$$U(t, 0) = \Psi(t)\, [\Psi(0)]^{-1} , \tag{28}$$

which obeys the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\, U = \mathcal{H}(t)\, U , \quad U(0, 0) = \mathrm{I} . \tag{29}$$

Various methods of parametrization are possible

$$U = F\,\{\exp\,[g_j(t)\, \hat{H}_j]\} , \tag{30}$$

where $F\{\ldots\}$ denotes the additive multiplicative or combined form of the generator exponents L_n (see Ref. 10). The evolution operator can be found in the explicit form by solving the system of n nonlinear differential equations of the first order for the group parameters g

$$h_j(t) = M_{ij}(g_i, g_j)\, \frac{\mathrm{d}}{\mathrm{d}t}\, g_j , \quad i, j = 1, \ldots, n . \tag{31}$$

The form of system (31) depends on the structure of expansion coefficients $h_j(t)$ of dynamic algebra L_n, and parametrization method (30). The algorithms for the solution of linear evolution equation (29) for the basic types of dynamic symmetry were developed in Refs. 10 and 14–16.

The group $SU(2)$ carries the entire information about the dynamics of oscillations of two types of neutrinos in the heterogeneous media and of N types of neutrinos under the certain conditions. In this case the Hamiltonian is expanded on the basis, generally speaking, of N–dimensional representation $SU(2)$

$$\mathcal{H}(t) = h_0(t)\,R_0 + h_-(t)\,R_- + h_+(t)\,R_+ \tag{32}$$

with generators satisfying the commutation relations

$$[R_+, R_-] = 2\,R_0\ ,\ \ [R_0, R_\pm] = 2\,R_\pm\ . \tag{33}$$

In so doing for the multiplicative parametrization of $SU(2)$

$$U = \exp g_0\,R_0\,\exp g_-\,R_-\,\exp g_+\,R_+ \tag{34}$$

the system of equations (31) is reduced to the unique equation in $g = \exp(g_0/2)$ (see Ref. 14)

$$\ddot{g} - \frac{\dot{h}_+}{h_+}\dot{g} + \frac{1}{2}\left[-\dot{h}_0 + h_0\frac{\dot{h}_+}{h_+} - \frac{1}{2}(4h_-\,h_+ + h_0^2)\right]g = 0\ . \tag{35}$$

The group parameters of $SU(2)$ are related to each other by the relations[16]

$$g_- = \frac{\dot{g}_0 - h_0}{2\,h_+}\exp g_0\ ,\qquad g_+ = \int\limits_0^t h_+\exp(-g_0)\,\mathrm{d}t'\ , \tag{36}$$

and fit the initial conditions

$$g_0(0) = g_\pm(0) = 0\ ,\ \ \dot{g}_0(0) = h_0(0)\ ,\ \ \dot{g}_\pm(0) = h_\pm(0)\ .$$

Basic equation (35) can be simplified by changing to the rotating coordinate system according to the relation

$$U(t,\,0) = \exp\left[-iR_0\int\limits_0^t h_0(t')\,\mathrm{d}t'\right]\tilde{U}(t,\,0)\ . \tag{37}$$

In what follows the operator $\tilde{U}$ fits Eq. (29) with the Hamiltonian

$$\tilde{\mathcal{H}}(t,0) = h_-\,R_-\,\exp\left[-i\int\limits_0^t h_0(t')\,\mathrm{d}t'\right] +$$

$$+ h_+\,R_+\exp\left[i\int\limits_0^t h_0(t')\,\mathrm{d}t'\right], \tag{38}$$

and for the quantity $x = \exp(\tilde{g}_0/2)$ parametrized analogously to Eq. (34) we obtain the equation

$$\ddot{x} - \left(\frac{\dot{h}}{h_+} + i\,h_0\right)\dot{x} - h_-\,h_+\,x = 0 \tag{39}$$

with the corresponding initial conditions.

In order to find the wider possible classes of exact solutions to the evolution problem, we introduce the new variable $z(t)$ and bring Eqs. (35) and (39) into the form being convenient for comparison with the ordinary linear differential equations of the second order.[14,16] For example, we transform Eq. (35) to the form

$$g'' + \frac{g'}{\dot{z}}\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(-\frac{\dot{z}}{2\,h_+}\right) -$$

$$-\frac{g}{2(\dot{z}^2)}\left[h_+\frac{\mathrm{d}}{\mathrm{d}\,t}\frac{h_0}{h_+} + \frac{1}{2}(4\,h_-\,h_+ + h_0^2)\right] = 0\ , \tag{40}$$

where differentiation with respect to z is denoted by the prime. By comparing Eq. (40) or (39) with the equations of mathematical physics (with the Bessel, Legendre, Weber, Whittaker, hypergeometric, and other equations) we determine the classes of exact solutions for the evolution parameters $g_{0,\pm}(t)$ and classes of functions $h_0(t)$ and/or $h_\pm(t)$ for which these solutions are valid.

The above–proposed formalism for description of neutrino oscillations is attractive for a variety of reasons. In its framework the quantum system evolution is described in so much general manner that can be allowed by the laws of quantum mechanics. The solution of the evolution problem is independent of dimensionality of representation of the dynamic Lie group and of the form of its generators. There are no limitations associated with the severe periodicity or adiabatic character of variation in the coefficients $h(t)$. The criteria of the validity of perturbation theory are also not used.

4. *SU*(2) NEUTRINO EVOLUTION IN THE HETEROGENEOUS MEDIA

Even if we restrict ourselves to two types of neutrinos, say, of electron and muon types, then the second–order equation for the probability amplitude follows from the system of Schrödinger equations (9)

$$i\,\ddot{\Psi}_e - (h_e + h_\mu)\,\dot{\Psi}_e - \left[\dot{h}_e - i\,h_e^2 - i\,\overline{h}^2 + i\,(h_e + h_1)\,h_e\right]\Psi_e = 0,$$

whose form is much more complicated than that of Eq. (40) for the evolution parameter in the algebraic formalism.

For $N = 2$ we deal with the two–dimensional representation of $SU(2)$ with generators on the spherical basis

$$R_0 = \frac{1}{2}\begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix},\quad R_+ = \begin{pmatrix}0\,1\\ 0\,0\end{pmatrix},\quad R_+ = \begin{pmatrix}0\,1\\ 0\,0\end{pmatrix} \tag{41}$$

and with Hamiltonian matrix (9)

$$\mathcal{H} = -\,i\,(h_e - h_\mu)\,R_0 - i\,\overline{h}\,R_- - i\,\overline{h}\,R_+ - i\,(h_e + h_\mu)\,\mathrm{I}_2. \tag{42}$$

The contribution of the unit matrix I_2, commutating with all the basis matrices, to the evolution operator is the exponential factor $\exp[i\,(h_e + h_y)\,t]$. As a result, we obtain the Hamiltonian in the form of Eq. (32) with the coefficients

$$h_0(t) \equiv -\,i\,h(t)\ ,\qquad h_+ = h_- \equiv -\,i\,\overline{h} = \mathrm{const}\ . \tag{43}$$

By using transform (37) we derive the simple equation for the evolution parameter

$$\ddot{x} - h(t)\,\dot{x} + \overline{h}^2\,x = 0\ , \tag{44}$$

whose solution describes the dynamics of oscillations of two types of neutrinos according to Eqs. (34), (36), and (37). It follows from Eqs. (10) and (11) that the coefficient h is proportional to the density $\rho(r)$ of the medium. The form of Eq. (44) is convenient for the search for such laws of variation in the density of the medium along the propagation path of a neutrino beam at which the problem admits the exact solutions in terms of elementary or special functions.

(1) The linear law of variation in the density of the medium ($t = r$ and $c = 1$)

$$h(t) = a\,t + b\ , \tag{45}$$

where a and b are the arbitrary parameters. In this case Eq. (44) with variable coefficient (45), by the simple substitution $a\,t + b = t$, is reduced to the equation

$$\frac{d^2x}{d\tau^2} - \frac{\tau}{a}\frac{dx}{d\tau} + \left(\frac{\bar{h}}{a}\right)^2 x = 0 \tag{46}$$

with the solution in terms of the Whittaker function [Ref. 17, Eqs. (2.273.10)].

(2) The exponential law of variation in the density of the medium

$$h(t) = a \exp b\,t + c\ , \tag{47}$$

where a, b, and c are the arbitrary parameters. The substitution of

$$x = y \exp\left[\frac{1}{2}\int_{t_0}^{t} (a\,e^{bt'} + c)\,dt'\right]$$

into Eq. (44) with function (47) brings into the equation

$$\ddot{y} = \left[\frac{a^2}{4}e^{2bt} - \frac{a}{2}(b - c)\,e^{bt} + \frac{c^2}{4} - \bar{h}^2\right]y\ , \tag{48}$$

which transforms into the modified Whittaker equation [Ref. 17, Eqs. (2.273.14)]

$$\frac{d^2y}{dt^2} = \left[\left(\frac{a}{2b}\right)^2 e^{2\tau} + \frac{a(c-b)}{2b^2}e^{\tau} + \left(\frac{c}{2b}\right)^2 - \left(\frac{\bar{h}}{b}\right)^2\right]y \tag{49}$$

owing to the calibration transform $b\,t = \tau$.

To expand the class of functions $h(t)$ admitting the exact solutions of Eq. (44), we introduce the general transform of the independent variable

$$z = \int f(t)\,dt \tag{50}$$

with the arbitrary differentiable and integrable function of the time $f(t)$. In so doing Eq. (44) is transformed into the form

$$f^2x'' + \left(\dot{f} - h\,f\right)x' + \bar{h}^2\,x = 0\ , \tag{51}$$

where differentiation with respect to z is denoted by primes. By selecting the specific form of substitutions (50), we can now find such laws of variation in the density of the medium $h(t)$ for which Eq. (51) is transformed into the ordinary second–order equations with the known exact solutions. Let us restrict ourselves to the functions $h(t) \sim \rho(t)$ modeling the variations in the density at the Sun and Earth.

(3) The variation in the density of the medium according to the law

$$h(t) = a\tanh(t - c)\ , \tag{52}$$

where a and c are the arbitrary parameters. As initial equation, we use

$$(z^2 + 1)\,x'' + (1 - a)\,z\,x' + \bar{h}^2\,x = 0\ , \tag{53}$$

which is equivalent to Eq. (51) under the following conditions:

$$f = \sqrt{z^2 + 1}\ , \tag{54}$$

$$\dot{f} - h\,f = (1 - a)\,z\ . \tag{55}$$

The relation between the variables z and t can be found from Eqs. (54) and (55)

$$z = \sinh(t - c)\ , \tag{56}$$

here c plays the role of the integration constant of Eq. (50). It is easy to ensure that law (52) can actually be derived from condition (55) using Eq. (56). Equation (53), in its turn, using the substitution $z^2 + 1 = y$, is reduced to the hypergeometric Gauss equation

$$y\,(y - 1)\frac{d^2x}{dy^2} + [(\alpha + \beta + 1)\,y - \gamma]\frac{dx}{dy} + \alpha\,\beta\,x = 0 \tag{57}$$

under the following identifications:

$$2\gamma = 1 - a\ , \qquad 2(\alpha + \beta) = -\,a\ , \qquad 4\,\alpha\,\beta = \bar{h}^2\ . \tag{58}$$

Thus the solution of initial equation (44) for variation in the density of the medium versus distance (time) according to law (52) can be expressed in terms of the hypergeometric function.

Note that the spectrum of non–trivial potentials, for which the exact solutions for the probability amplitudes of transitions in the particular case of the neutrinos of two flavors under the condition $h_e = -\,h_\mu$ (see Ref. 13) are well known, is exhausted by three above–considered functions $h(t)$. First, the above–developed formalism for description of neutrino oscillations is not limited by the case $N = 2$ (the obtained solutions can be easily generalized for the case of the $SU(2)$ dynamics with the arbitrary number of types of neutrinos) and condition $h_e = -\,h_\mu$. Second, such a formalism enables the exact solutions for other virtually important laws of variation in the density to be found. Let us show tha

(4) Variation in the density of a medium according to the law

$$h(t) = a\coth(t - c)\ . \tag{59}$$

The following initial equation

$$(z^2 - 1)\,x'' + (1 - a)\,z\,x' + \bar{h}^2\,x = 0\ , \tag{60}$$

$$f = \sqrt{z^2 - 1}\ , \qquad z = \cosh(t - c)\ , \qquad 1 - z^2 = y \tag{61}$$

leads to hypergeometric equation (57) with the parameters given by Eq. (58).

(5) Variation in the density of a medium according to the law

$$h(t) = a\,\mathrm{sech}(t - c)\ . \tag{62}$$

The initial equation

$$(z^2 + 1)\, x'' + (z - a)\, x' + \overline{h}^2 x = 0 ,$$
$$f = \sqrt{z^2 + 1} , \quad z = \sinh(t - c) \qquad (63)$$

and the substitution $2y = 1 - i\, z$ yields the hypergeometric equation [see Ref. 17, Eq. (2.249)]

$$y\,(y - 1)\, x'' + \left[y - \frac{1}{2}(1 - i\, a) \right] x' + \overline{h}^2 x = 0 , \qquad (64)$$

which acquires standard form (57) under conditions

$$i\,\alpha = \overline{h} , \quad \alpha = -\beta , \quad 2\gamma = 1 - i\, a . \qquad (65)$$

The above–considered laws of variation in the density of the medium for which the $SU(2)$–evolution parameters can be exactly obtained can serve as the good models of variation in the electron density at the Sun: functions (47) and (59) for the external layers and (52) for the internal ones. The variation in the density of the Earth is modeled by combination of laws (45) and (52): the first of them models the linear sections of the function $\rho(r)$ at the Earth and the second – its discontinuities.

By combining different versions of initial equation and substitution (50), we can obtain the variety of laws of variation in the density of the medium versus distance for which sought–for equation (44) could be exactly solved for the $SU(2)$–evolution parameter. Thus by calculating the parameter $g_0 = 2 \ln x$ and then g_+ and g_- according to formulas (36) we can explicitly represent the evolution operator in the factorization form given by Eq. (34). All the quantities required for the description of neutrino oscillations in the heterogeneous media are obtained further by the standard scheme of quantum mechanics. For example, the probability of muon–neutrino transition at $t = 0$ to the electron neutrino at the arbitrary instant can be calculated by the formula

$$p(\nu_e;\, t\,|\,\nu_\mu\,;\, 0) = |\langle\nu_e|\, \exp[g_0(t, 0)\, R_0] \times$$
$$\times \exp[g_-(t)\, R_-]\, \exp[g_+(t)\, R_+]\,|\,\nu_\mu\rangle|^2 . \qquad (66)$$

5. CONCLUSION

The results obtained in Sec. 4 can be generalized for the case of $N\times N$ representation of the group $SU(2)$, i.e., for such cases of mixing and of oscillations of N types of neutrinos in the heterogeneous media which possess this dynamic symmetry. The amplitudes of neutrino transitions being the matrix elements of $N = (2s + 1)$–dimensional irreducible representation $SU(2)$ are expressed in terms of the Jacobi polynomials $P^s_{mn}(\cos\delta)$ for parametrization by the Euler angles φ, δ, and σ which are related to the parameters g as follows:

$$\exp(g_0/2) = \cos(\delta/2)\, \exp[i\,(\varphi + \delta)/2] ,$$
$$g_- = (i/2)\, \sin\delta \cdot \exp i\,\sigma ,$$
$$g_+ = i\, \tan(\delta/2) \cdot \exp(-i\,\sigma) ,$$
$$\cos\delta = 2\, g_-\, g_+ + 1 . \qquad (67)$$

The other possibile generalization of the results of Sec. 4 are connected with the expansion of the dynamic symmetry group. For example, for $N = 3$ with the complete group of symmetry $SU(3)$ I recommend using the regular procedure for finding the corrections for the evolution operator $SU(2)$ in powers of the small coefficient of interaction.[18] At $N = 4$ under certain limitations imposed on the coefficients h of the Hamiltonian matrix, $SO(4)$ and $SO(3, 1)$ are the groups of symmetry. As was shown in Ref. 16, the exact calculation of the evolution operators of these groups is reduced to the solution of two independent $SU(2)$ evolution equations (29).

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