# POTENTIALITIES OF THE METHODS OF POSTDETECTOR PROCESSING OF IMAGES OF INCOHERENTLY ILLUMINATED OBJECTS OBSERVED THROUGH A TURBULENT ATMOSPHERE

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*Optical transfer function (OTF) of the turbulent atmosphere and telescopic receiving optical system are considered theoretically using different methods of postdetector processing of images of incoherently illuminated objects observed through the turbulent atmosphere. When approximating the transmission function of a receiving lens by a Gaussian shape simple analytical expressions are obtained for OTFs of the turbulent atmosphere and a telescope. The following image processing methods are examined: the averaged image recording ("very long" averaging times) and short-exposure images ("very short" averaging times), namely, Labeyrie and Knox–Thompson methods and the method of triple correlation of the image intensity. Potentialities and applicability limits of the methods under different conditions of optical radiation propagation through the turbulent atmosphere are discussed. The influence of the finiteness of the inner scale of turbulence on the OTFs under consideration is also estimated.*

Problem on reconstructing images distorted due to the effects of atmospheric turbulence has been widely studied since 1960s. In recent decade, the technical revolution caused by vigorous development of computer engineering occurs in the field of image processing. Improvement reached in quality of optical and electronic components and serious reduction of its cost together with a wide use of micro– and mini–computers and corresponding peripheral devices allowed analysis and processing of an image to be done in real time with a computer and optical digital systems.

Processing of the distorted images is needed practically in all ranges of the electromagnetic waves: radio (radar and radiometry), optical (classical optics and astronomy, infrared thermovision), $X$–rays ($X$–ray astronomy, roentgenography). On the whole, the reconstruction of distored images is a research branch on development of methods and facilities to compensate for distortions introduced into images by various systems during the process of image construction. In particular, a possibility exists to introduce a compensating action both before (adaptive optics) and after the image recording (postdetector image processing). The optical image distortions appear not only because of the imperfections of the recorders (for example, aberrations of receiving optics) but also because of optical inhomogeneity of a propagation medium (for example, turbulence of the atmosphere). In this case, the aberrations of optical systems lead to defocusing and geometrical distortions; the atmospheric turbulence deteriorates resolution of the obtained images by more than an order in optical astronomy.

As known,[1,2] random variations in the dielectric constant of air cause fluctuations of the parameters of optical waves propagating through the turbulent atmosphere. Just these fluctuations cause the fluctuations of illumination in the image space of a receiving optical system. In majority of cases the combined effect of both the atmosphere and receiving optical system may be considered as a random linear filtration. In this case, it is sufficient to use optical transfer function (OTF) when describing the "atmosphere – receiving optical system" complex.[3] The optical transfer function is defined as the Fourier transform of the intensity distribution of light from a point source in the object space in the image space of a receiving optical system. In this paper we present some results of a theoretical study of the OTFs of a turbulent atmosphere and telescopic receiving optical system for different methods of postdetector processing of images of incoherently illuminated objects observed through the turbulent atmosphere, namely, the averaged image recording method and short–exposure image processing (the Labeyrie and Knox–Thompson methods and the method of triple correlation of image intensity).

The optical wave field behind a receiving lens in the sharp–image plane can be written according to the Huygens–Kirchhoff method[4]

$$U_g(\rho,\, t) = \frac{k \exp\left( i\, k\, F + \dfrac{i\, k}{2\, F}\, \rho^2 \right)}{2\pi\, i\, F} \times$$

$$\times \int\limits_{-\infty}^{\infty}\!\!\int \mathrm{d}\rho'\; U(\rho',\, t)\; K(\rho') \exp\left\{ -\frac{i\, k}{F}\, \rho\, \rho' \right\}. \qquad (1)$$

Here $U(\rho',\, t)$ is the field of an optical wave incident on the input aperture of the receiving system; $K(\rho)$ is the transmission function of the optical receiving system; $F$ is the focal length of the receiving lens; $k = 2\pi/\lambda$, $\lambda$ is the optical radiation free–space wavelength; $\rho'$ and $\rho$ are the transverse coordinates within the input aperture and in the sharp–image plane of the receiving lens, respectively; and, $t$ is time. Using Eq. (1), we write the optical wave intensity in the telescope focal plane as follows:

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$$I_g(\rho,\ t) = U_g(\rho,\ t)\ U_g^*(\rho,\ t) = \frac{k^2}{4\pi^2\ F^2} \int\int\limits_{-\infty}^{\infty}\int\int \mathrm{d}\rho'\ \mathrm{d}\rho'' \times$$

$$\times\, U(\rho',\ t)\, U^*(\rho'',\ t) K(\rho')\ K^*(\rho'') \exp \left\{ -\frac{i\,k}{F}\ \rho\ (\rho' - \rho'') \right\}. \quad (2)$$

Because the "instantaneous" value of OTF of the optical system is the Fourier transform of random intensity in the sharp–image plane of this system, we obtain from Eq. (2)

$$M(\mathbf{p},\ t) = \int\limits_{-\infty}^{\infty}\int \mathrm{d}\rho\ I_g(\rho,\ t) \exp \left\{ -\frac{i\,k}{F}\ \rho\ \rho' \right\}, \quad (3)$$

where $M(\mathbf{p},\ t)$ is the OTF of the optical system, and $\mathbf{p}$ is the spatial scale. By substituting Eq. (2) into Eq. (3) and integrating the latter over $\rho$ and $\rho''$, we obtain the following expression for OTF of the optical system[5]:

$$M(\mathbf{p},\ t) = \int\limits_{-\infty}^{\infty}\int \mathrm{d}\rho'\ U(\rho',\ t)\ U^*(\rho' + \mathbf{p},\ t)\ K(\rho')\ K^*(\rho' + \mathbf{p}). \quad (4)$$

Since the optical wave field $U(\rho,\ t)$ is random (because of the dielectric fluctuations of the turbulent atmosphere), the instantaneous value of OTF $M(\mathbf{p},\ t)$ (Eq. (4)) is a random value rapidly varying with time. If the object image is recorded during the time when the atmospheric turbulence is "frozen", i.e., during $\sim 10^{-3}$ s, then practically each recorded image can be considered as a random intensity distribution.

In the case of recording averaged image ("very long" exposure) which corresponds to the averaging times longer than the time of "frozen" turbulence (the exposure time of $\sim$ 10...100 s), the OTF of the turbulent atmosphere and receiving optical system can be written as follows:

$$\langle M(\mathbf{p})\rangle = \lim_{\tau\to\infty} \int\limits_0^{\tau} \mathrm{d}\ t\ M(\mathbf{p},\ t) =$$

$$= \int\limits_{-\infty}^{\infty}\int \mathrm{d}\rho'\ \langle U(\rho',\ t)\ U^*(\rho' + \mathbf{p},\ t)\rangle\ K(\rho')\ K^*(\rho' + \mathbf{p}), \quad (5)$$

where $\tau$ is the averaging time (the exposure time), and $\Gamma_2(\rho',\ \rho' + \mathbf{p};\ t,\ t) = \langle U(\rho',\ t) U^*(\rho' + \mathbf{p},\ t)\rangle$ is the second–order mutual–coherence function of the incident optical wave.

The incident optical wave is assumed to be plane (for example, radiation from a star). Then

$$U(\rho,\ t) = U_0\ \exp\{\psi(\rho,\ t)\}\ .$$

Here $U_0$ is the amplitude of the incident optical wave; $\psi(\rho,\ t) = \chi(\rho,\ t) + iS(\rho,\ t)$ describes fluctuations of the optical wave complex phase; $\chi(\rho,\ t)$ and $S(\rho,\ t)$ are the fluctuations of the amplitude logarithm and optical wave phase, respectively. As known,[4] the probability distribution for $\chi(\rho,\ t)$ and $S(\rho,\ t)$ obey the normal law in the region of weak intensity fluctuations of the optical radiation propagating through the turbulent atmosphere, then

$$\Gamma_2(\rho_1,\ \rho_2;\ t_1,\ t_2) = U_0^2 \exp \left\{ -\frac{1}{2}\ D(\rho_1 - \rho_2,\ t_1 - t_2) \right\}, \quad (6)$$

where $D(\rho,\ t)$ is the spatiotemporal structure function of the complex phase fluctuations of the plane optical wave.[4] By substituting Eq. (6) into Eq. (5), we obtain the OTF of the turbulent atmosphere and receiving optical system for "very long" exposures

$$\langle M(\mathbf{p})\rangle = M_{\mathrm{atm}}(\mathbf{p})\ M_{\mathrm{lens}}(\mathbf{p}),$$

where $M_{\mathrm{atm}}(\mathbf{p}) = \Gamma_2(\mathbf{p})$ is the OTF of the turbulent atmosphere, $M_{\mathrm{lens}}(\mathbf{p}) = \int\limits_{-\infty}^{\infty}\int \mathrm{d}\rho\ K(\rho)\ K^*(\rho + \mathbf{p})$ is the OTF the receiving optical system; i.e., as known,[1] in this case the contributions of the atmospheric turbulence and receiving optical system are factorable. The spatial structure function of the complex–phase fluctuations of a plane optical wave propagating through the turbulent atmosphere has the following form[4]:

$$D(\rho)\ \simeq \begin{cases} 0.45\ k^2\ C_\varepsilon^2\ L\ \kappa_m^{1/3}\ \rho^2 & \text{for } \rho \ll l_0, \\ 0.73\ k^2\ C_\varepsilon^2\ L\ \rho^{5/3} & \text{for } l_0 \ll \rho \ll L_0, \end{cases}$$

where $C_\varepsilon^2$ is the ground value of the structure parameter of fluctuations of the dielectric constant in the turbulent atmosphere; $L = C_\varepsilon^{-2} \int\limits_0^{\infty} \mathrm{d}\,x\ C_\varepsilon^2(x,\ \theta)$ is the effective thickness of an optically active layer of the atmospheric turbulence, $C_\varepsilon^2(x,\ \theta)$ is the altitude profile of the structure parameter of the dielectric constant fluctuations in turbulent atmosphere depending on the zenith angle $\theta$ ($\theta \in [-\pi/2,\ \pi/2]$), $\kappa_m = 5.92/l_0$, $l_0$ is the inner scale of the atmospheric turbulence, and $L_0$ is the outer scale of the atmospheric turbulence.

So, the normalized transfer function of the turbulent atmosphere and a telescope in the case of observation of an averaged image can be presented as follows:

$$\tau_1(\mathbf{p}) = \langle M(\mathbf{p})\rangle / \langle M(0)\rangle = \frac{M_{\mathrm{lens}}(\mathbf{p})}{M_{\mathrm{lens}}(0)} \exp \left\{ -\frac{1}{2}\ D(p) \right\} =$$

$$= \frac{M_{\mathrm{lens}}(\mathbf{p})}{M_{\mathrm{lens}}(0)} \exp \left\{ -\left(\frac{p}{\rho_c}\right)^{\gamma} \right\}, \quad (7)$$

where

$$\gamma = \begin{cases} 2 & \text{for } p < l_0, \\ 5/3 & \text{for } p > l_0; \end{cases} \qquad \rho_c = \begin{cases} \rho_m & \text{for } p \ll l_0, \\ \rho_0 & \text{for } p \gg l_0; \end{cases}$$

$\rho_c$ is the coherence radius of the plane optical wave in the turbulent atmosphere, whereas $\rho_m = (0.225 k^2 C_\varepsilon^2 L \kappa_m^{1/3})^{-1/2}$ and $\rho_0 = (0.365 k^2 C_\varepsilon^2 L)^{-3/5}$ are the values of the coherence radius of the plane optical wave for $D(l_0) \gg 1$ and $D(l_0) \ll 1$ that corresponds to the cases when $l_0 \gg \rho_c$ and $l_0 \ll \rho_c$, respectively. Since the characteristic scale of the OTF variations, $M_{\mathrm{lens}}(\mathbf{p})$, is determined by the radius $R$ of a receiving optical system, two cases may be considered:

(1) when the receiving optical system radius is smaller than the coherence radius of the optical wave ($R \ll \rho_c$)

$$\tau_1(\mathbf{p}) \simeq \frac{M_{\text{lens}}(\mathbf{p})}{M_{\text{lens}}(0)} \left\{1 - \left(\frac{p}{\rho_c}\right)^{\gamma}\right\}, \quad (8)$$

(2) when the receiving optical system radius is larger than the coherence radius of the optical wave ($R \gg \rho_c$)

$$\tau_1(\mathbf{p}) \simeq \exp\left\{-\left(\frac{p}{\rho_c}\right)^{\gamma}\right\}. \quad (9)$$

The image observed during time shorter than time of medium "freezing" ($\sim 10^{-3}$ s) is not averaged ("very short" exposures). To eliminate the distorting influence of random inhomogeneities of the turbulent atmosphere, different methods of processing the short–exposure images are used: the Labeyrie[6] and Knox–Thompson[7] methods or the method of triple correlation of the image intensity.[8–10] For "very long" exposures one measures the distribution of average field intensity in the focal plane of a telescopic receiving optical system, while methods of processing the short–exposure images make it possible to measure statistical characteristics of the optical image intensity fluctuations. They are the variance of intensity fluctuations in the Labeyrie method, the spatial correlation of intensity fluctuations in the Knox–Thompson method, and the third moment of the image intensity for the special choice of observation points in the method of triple correlation of image intensity. All these methods will be considered in sequence starting from the Labeyrie method. The OTF in the Labeyrie method is found from the second moment of the "instantaneous" OTF of the turbulent atmosphere and a telescopic system (Eq. (4))

$$\tau_2(\mathbf{p}) = \langle M(\mathbf{p}, t)\, M^*(\mathbf{p}, t)\rangle / \langle M(0)\rangle^2, \quad (10)$$

where

$$\langle M(\mathbf{p}, t)\, M^*(\mathbf{p}, t)\rangle =$$

$$= \iiiint\limits_{-\infty}^{\infty} d\rho'\, d\rho'' \langle U(\rho', t)\, U^*(\rho' + \mathbf{p}, t) \times$$

$$\times\, U^*(\rho'', t)\, U(\rho'' + \mathbf{p}, t)\rangle\, K(\rho')\, K^*(\rho' + \mathbf{p})\, K^*(\rho'')\, K(\rho'' + \mathbf{p})$$

is the second moment of the "instantaneous" OTF of the turbulent atmosphere and a telescope. By making the same suppositions as before when deriving the second–order mutual–coherence function and assuming that $\langle \chi^2(\rho, t)\rangle \ll 1$ (this condition is well fulfilled for a plane wave propagating along the paths through the whole atmosphere with at zenith angles $\theta \le 80°$), we obtain

$$\langle U(\rho', t)\, U^*(\rho' + \mathbf{p}, t)\, U^*(\rho'', t)\, U(\rho'' + \mathbf{p}, t)\rangle \simeq U_0^4 \exp\{-D(\mathbf{p}) +$$
$$+\frac{1}{2} D(\rho' - \rho'' - \mathbf{p}) + \frac{1}{2} D(\rho' - \rho'' + \mathbf{p}) - D(\rho' - \rho'')\}. \quad (11)$$

Analysis of Eqs. (10) and (11) shows that further analysis needs for the concrete form of the transmission function of the receiving optical system $K(\rho)$. The fluctuating wave is assumed to be incident on a round lens with the area $S = \pi R^2$, where $R$ is the radius of receiving aperture. As is shown in Ref. 4, square–law exponent is a good approximation of the transmission function of the receiving aperture in this case. This fact and frequent use of the apodization filters to improve an image quality,[11] allow us to choose the transmission function of the receiving aperture in the form

$$K(\rho) = K_0 \exp\left\{-\frac{\rho^2}{2R^2}\right\}, \quad (12)$$

where $K_0$ is the telescope amplitude transmission on the optical axis. In this case, the normalizing factor in Eq. (10) is $\langle M(0)\rangle = \pi U_0^2 K_0^2 R^2$, and $\tau_2(0) = 1$. Simple expression for the OTF in the Labeyrie method may be obtained from Eq. (10) after elementary transformations using Eqs. (11) and (12)

$$\tau_2(\mathbf{p}) = \frac{1}{2\pi R^2} \exp\left\{-\frac{p^2}{2R^2}\right\} \iint\limits_{-\infty}^{\infty} d\rho' \times$$

$$\times \exp\left\{-\frac{(\rho')^2}{2R^2} - D(\mathbf{p}) - D(\rho') + \frac{1}{2} D(\rho' - \mathbf{p}) + \frac{1}{2} D(\rho' + \mathbf{p})\right\}. \quad (13)$$

Asymptotical analysis of Eq. (13) carried out using the expansion of the integrand exponent into a series and computation of the integral over ' yields the following results. Two situations can occur for the receiving apertures with the radius less than the coherence radius of the incident optical wave ($R \ll \rho_c$): $R \gg l_0$ and $R \ll l_0$. Since the maximum values of the inner scale of the atmospheric turbulence are 1...2 cm, the case of receiving apertures satisfying the condition $R \ll l_0$ belongs to an exotic type. Under these conditions the OTF of the Labeyrie method equals

$$\tau_2(\mathbf{p}) \simeq \begin{cases} \tau_0^2(\mathbf{p}) \left\{1 - \frac{1}{6} (k_m R)^2 (p/\rho_m)^2\right\} & \text{for } p \ll l_0, \\ 0 & \text{for } p \gg l_0, \end{cases}$$

where $\tau_0(\mathbf{p}) = \exp[-p^2/(4R^2)]$ is the normalized OTF of the telescope for the transmission function of its entrance pupil (12). Comparison of the obtained expression with Eq. (8) shows that for small apertures it is possible to remove practically completely the image distortions introduced by the atmospheric turbulence. The case of the receiving apertures, which are large as compared to the inner scale of the atmospheric turbulence ($R \gg l_0$) is of greater practical interest. In this case for $p \ll l_0$

$$\tau_2(\mathbf{p}) \simeq \tau_0^2(\mathbf{p})\, \{1 - 2(p/\rho_m)^2\},$$

and for $l_0 \ll p \ll R$

$$\tau_2(\mathbf{p}) \simeq \tau_0^2(\mathbf{p})\, \{1 - 2(p/\rho_0)^{5/3}\, [1 - \alpha(p / R)^{1/3}]\},$$

where $\alpha = 0.70$.

Bahavior of the OTF at $p \gg R$ is not considered here and hereafter since it has no practical importance. Two cases may be also considered for the receiving apertures with the radii larger than the coherence radius of the received optical wave ($R \gg \rho_c$), namely, $\rho_c \ll R \ll l_0$ and $R \gg l_0$. In the first case ($p \ll l_0$)

$$\tau_2(\mathbf{p}) \simeq \tau_0^2(\mathbf{p}) \left\{\exp\left[-2\left(\frac{p}{\rho_m}\right)^2\right]\left[1 + 2\left(\frac{p}{\rho_m}\right)^2\right] + \frac{1}{2}\left(\frac{\rho_m}{R}\right)^2\right\},$$

i.e., as for $R \ll l_0$, $R \ll \rho_c$, an increase in the telescope resolution in the atmosphere is observed over all region of spatial scales recorded with given apertures as compared to the observation of the distribution an image average intensity (Eq. (9)). In turn, two situations are separated for the receiving apertures with $R \gg l_0$ in dependence on the ratio of the inner scale of the atmospheric turbulence and coherence radius of a plane optical wave in the turbulent atmosphere:

(1) if $l_0 \gg \rho_c$ then for $p \ll l_0$

$$\tau_2(\mathbf{p}) \simeq \tau_0^2(\mathbf{p}) \exp\left[-2\left(\frac{p}{\rho_m}\right)^2\right],$$

and for $l_0 \ll p \ll R$

$$\tau_2(\mathbf{p}) \simeq \tau_0^2(\mathbf{p}) \left\{\exp\left[-2\left(\frac{p}{\rho_0}\right)^{5/3}\right]\times\right.$$

$$\left.\times\left[1 + 2\alpha\left(\frac{p}{\rho_0}\right)^{5/3}\left(\frac{p}{R}\right)^{1/3}\right] + \frac{1}{4}(\rho_m/R)^2\right\};$$

(2) if $l_0 \ll \rho_c$, then for $p \ll l_0$

$$\tau_2(\mathbf{p}) \simeq \tau_0^2(\mathbf{p}) \left\{1 - 2\left(\frac{p}{\rho_m}\right)^2\right\},$$

and for $l_0 \ll p \ll \rho_c$

$$\tau_2(\mathbf{p}) \simeq \tau_0^2(\mathbf{p}) \left\{1 - 2\left(\frac{p}{\rho_0}\right)^{5/3}[1 - \alpha(p/R)^{1/3}]\right\},$$

while for $\rho_c \ll p \ll R$

$$\tau_2(\mathbf{p}) \simeq \tau_0^2(\mathbf{p}) \left\{\exp\left[-2\left(\frac{p}{\rho_0}\right)^{5/3}\right]\right.$$

$$\left.\times\left[1 + 2\alpha\left(\frac{p}{\rho_0}\right)^{5/3}\left(\frac{p}{R}\right)^{1/3}\right] + 2^{-11/5}\left(\frac{\rho_0}{R}\right)^2\right\}.$$

Thus, processing of a set of "very short"–exposure images by Labeyrie method makes it possible to eliminate a part of the image distortions introduced by the atmospheric turbulence as compared with the recording of averaged image (Eqs. (8) and (9)). Moreover, the higher is the level of turbulent fluctuations, the greater is the gain. The improvement is especially essential for small receiving apertures ($R < l_0$) and for large spatial scales in the case of large receiving apertures ($p > \rho_c$ for $R > l_0$, $R > \rho_c$). Resolution of small scales ($p < l_0$) for large apertures ($R > l_0$) when processing by the Labeyrie method is not higher than the resolution when recording averaged images.

In the Labeyrie method, the intensity spectrum modulus of the recorded image is measured that does not allow, even in principle, the initial image to be reconstructed exactly. At the same time, the exact knowledge of the phase of the image intensity spectrum only makes it possible to reconstruct the initial image completely. The image intensity spectrum modulus can be reconstructed from the known values of the intensity spectrum phase. The methods of Knox–Thompson and triple correlation of the image intensity allow both the modulus and phase of the recorded image to be measured.

When an image is processed using the Knox–Thompson method, the OTF of an optical system is equal to

$$\tau_3(\mathbf{p}_1, \mathbf{p}_2) = \langle M(\mathbf{p}_1, t)\, M^*(\mathbf{p}_2, t)\rangle / \langle M(0)\rangle^2, \qquad (14)$$

where

$$\langle M(\mathbf{p}_1, t)\, M^*(\mathbf{p}_2, t)\rangle = \int\int_{-\infty}^{\infty}\int\int d\rho'\, d\rho'' \times$$

$$\times \langle U(\rho', t)\, U^*(\rho' + \mathbf{p}_1, t) U^*(\rho'', t)\, U(\rho'' + \mathbf{p}_2, t)\rangle \times$$

$$\times K(\rho')\, K^*(\rho' + \mathbf{p}_1)\, K^*(\rho'')\, K(\rho'' + \mathbf{p}_2)$$

is the correlation function of the "instantaneous" OTF of the turbulent atmosphere and a telescope (Eq. (4)). Having written the fourth moment of a plane wave field in approximation which was used for the analysis of the methods considered above, substituting the expression (12) into Eq. (14) for the function $K(\rho)$, and integrating over one of the variables, we derive the OTF of Knox–Thompson method

$$\tau_3(\mathbf{p}_1,\mathbf{p}_2) = \frac{1}{2\pi R^2}\exp\left\{-\frac{3p_1^2+3p_2^2-2\mathbf{p}_1\mathbf{p}_2}{8R^2}\right\}\int\int_{-\infty}^{\infty} d\rho \exp\left\{-\frac{\rho^2}{2R^2}-\right.$$

$$-\frac{(\mathbf{p}_1-\mathbf{p}_2)\mathbf{r}}{2R^2} - \frac{1}{2}D(\mathbf{p}_1) - \frac{1}{2}D(\mathbf{p}_2) - \frac{1}{2}D(\rho) - \frac{1}{2}D(\rho + \mathbf{p}_1 - \mathbf{p}_2) +$$

$$\left. + \frac{1}{2}D(\mathbf{r} - \mathbf{p}_2) + \frac{1}{2}D(\rho + \mathbf{p}_1)\right\}. \qquad (15)$$

Analyzing Eq. (14) asymptotically using a standard method of calculating of multiplex integrals,[4] we can show that for small receiving apertures with $R \ll l_0$, $R \ll \rho_c$ at $\{p_1, p_2\} \ll l_0$

$$\tau_3(\mathbf{p}_1, \mathbf{p}_2) \simeq \tau_0(\mathbf{p}_1)\, \tau_0(\mathbf{p}_2) \left\{1 - \left(\frac{|\mathbf{p}_1 - \mathbf{p}_2|}{\rho_m}\right)^2\right\},$$

and for $\rho_c \ll R \ll l_0$, when $\{p_1, p_2\} \ll l_0$

$$\tau_3(\mathbf{p}_1, \mathbf{p}_2) \simeq \tau_0(\mathbf{p}_1)\, \tau_0(\mathbf{p}_2) \left\{\exp\left[-\left(\frac{p_1}{\rho_m}\right)^2 - \left(\frac{p_2}{\rho_m}\right)^2\right]\times\right.$$

$$\left.\times\left[1 + 2\frac{\mathbf{p}_1\mathbf{p}_2}{\rho_m^2}\right] + \frac{1}{4}\left(\frac{\rho_m}{R}\right)^2 \exp\left[-\left(\frac{|\mathbf{p}_1 - \mathbf{p}_2|}{\rho_m}\right)^2\right]\right\}.$$

For telescopes with the aperture radius larger than the inner scale of the atmospheric turbulence but less than the coherence radius of a plane wave in the turbulent atmosphere, at $|\mathbf{p}_1 + \mathbf{p}_2| \ll l_0$

$$\tau_3(\mathbf{p}_1, \mathbf{p}_2) \simeq \tau_0(\mathbf{p}_1)\, \tau_0(\mathbf{p}_2) \times$$

$$\times \left\{1 - \left(\frac{p_1}{\rho_m}\right)^2 - \left(\frac{p_2}{\rho_m}\right)^2 + 2^{7/6}(k_m R)^{-1/3}\frac{\mathbf{p}_1\mathbf{p}_2}{\rho_m^2}\right\}, \qquad (16)$$

and at $l_0 \ll |\mathbf{p}_1 + \mathbf{p}_2| \ll R$

$$\tau_3(\mathbf{p}_1, \mathbf{p}_2) \simeq \tau_0(\mathbf{p}_1)\tau_0(\mathbf{p}_2) \left\{1 - \left(\frac{p_1}{\rho_0}\right)^{5/3}\left[1 - \alpha\left(\frac{p_1}{R}\right)^{1/3}\right] - \right.$$

$$\left. - \left(\frac{p_2}{\rho_0}\right)^{5/3}\left[1 - a\left(\frac{p_2}{R}\right)^{1/3}\right] - \alpha\left(\frac{|\mathbf{p}_1 - \mathbf{p}_2|}{\rho_0}\right)^{5/3}\left(\frac{|\mathbf{p}_1 - \mathbf{p}_2|}{R}\right)^{1/3}\right\} \qquad 17)$$

If the receiving telescope aperture radius exceeds the coherence radius of the plane wave then for $|\mathbf{p}_1 + \mathbf{p}_2| \ll l_0$ the OTF of Knox–Thompson method is described by Eq. (16), and for $l_0 \ll |\mathbf{p}_1 + \mathbf{p}_2| \ll \rho_c$ it is described by Eq. (17). For $\rho_c \ll \mathbf{p}_1 + \mathbf{p}_2 | \ll R$ the following asymptotic dependence takes place

$$\tau_3(\mathbf{p}_1, \mathbf{p}_2) \simeq \tau_0(\mathbf{p}_1)\, \tau_0(\mathbf{p}_2) \left\{ \exp\left[ -\left(\frac{p_1}{\rho_0}\right)^{5/3} - \left(\frac{p_2}{\rho_0}\right)^{5/3} \right] \times \right.$$

$$\left. \times \left[ 1 + 2\alpha \frac{\mathbf{p}_1\, \mathbf{p}_2}{\rho_0^{5/3} R^{1/3}} \right] + 2^{-11/5} \left(\frac{\rho_0}{R}\right)^2 \exp\left[ -\frac{(\mathbf{p}_1 - \mathbf{p}_2)^2}{\rho_0^2} \right] \right\}. \tag{18}$$

It is natural that for $\mathbf{p}_1 = \mathbf{p}_2$ the OTF of Knox–Thompson method and that of the Labeyrie method are identical, i.e., these methods are equivalent from the point of view of measuring the intensity spectrum modulus of an image recorded. As follows from Eqs. (16)–(18), if the characteristic linear scale $\tau_3(\mathbf{p}_1, \mathbf{p}_2)$ (with respect to the difference of arguments $|\mathbf{p}_1 - \mathbf{p}_2|$) equals to the coherence radius of a plane wave then unique information on the intensity spectrum phase of an image can be obtained only within the limits of a speckle. A problem of "joining" the image spectrum phase from the neighboring speckles within the framework of the present method remains unsolved. Also it is clear from Eqs. (17) and (18) that since processing of the "very short"–exposure images compensates for the wave front slopes in the areas which are larger than or comparable to the receiving aperture dimensions then it leads to noticeable improvement of the image quality on the speckle scales.

Let us finally consider the method of triple correlation of the image intensity. Its OFT is determined from the third moment of the "istantaneous" OTF of turbulent atmosphere and the telescope (Eq. (4)) at the points $\mathbf{p}_1$, $\mathbf{p}_2$ and $-\mathbf{p}_1$, $-\mathbf{p}_2$

$$\tau_4(\mathbf{p}_1, \mathbf{p}_2) = \langle M(\mathbf{p}_1, t) M(\mathbf{p}_2, t)\, M(-\mathbf{p}_1 - \mathbf{p}_2, t)\rangle / \langle M(0)\rangle^3, \tag{19}$$

where

$$\langle M(\mathbf{p}_1, t)\, M(\mathbf{p}_2, t)\, M(-\mathbf{p}_1 - \mathbf{p}_2, t)\rangle =$$

$$= \int\int\int_{-\infty}^{\infty}\int\int\int d\rho'\, d\rho''\, d\rho'''\, \langle U(\rho', t)\, U^*(\rho' + \mathbf{p}_1, t) \times$$

$$\times\, U(\rho'', t)\, U^*(\rho'' + \mathbf{p}_2, t)\, U(\rho''', t)\, U^*(\rho''' - \mathbf{p}_1 - \mathbf{p}_2, t)\rangle \times$$

$$\times\, K(\rho')\, K^*(\rho' + \mathbf{p}_1)\, K(\rho'')\, K^*(\rho'' + \mathbf{p}_2)\, K(\rho''')\, K^*(\rho''' - \mathbf{p}_1 - \mathbf{p}_2).$$

Here

$$\langle U(\rho', t)\, U^*(\rho' + \mathbf{p}_1, t)\, U(\rho'', t)\, U^*(\rho'' + \mathbf{p}_2, t)\, U(\rho''', t) \times$$

$$\times\, U^*(\rho''' - \mathbf{p}_1 - \mathbf{p}_2, t)\rangle \simeq U_0^6 \exp\Big\{ -\frac{1}{2} D(\mathbf{p}_1) - \frac{1}{2} D(\mathbf{p}_2) - \frac{1}{2} D(\mathbf{p}_1 \mathbf{p}_2) +$$

$$+ \frac{1}{2} D(\rho' - \rho'') + \frac{1}{2} D(\rho' - \rho''') + \frac{1}{2} D(\rho'' - \rho''') - \frac{1}{2} D(\rho' - \rho'' + \mathbf{p}_1) -$$

$$- \frac{1}{2} D(\rho' - \rho'' - \mathbf{p}_2) + \frac{1}{2} D(\rho' - \rho'' + \mathbf{p}_1 - \mathbf{p}_2) - \frac{1}{2} D(\rho' - \rho''' + \mathbf{p}_1) -$$

$$- \frac{1}{2} D(\rho' - \rho''' + \mathbf{p}_1 + \mathbf{p}_2) + \frac{1}{2} D(\rho' - \rho''' + 2\mathbf{p}_1 + \mathbf{p}_2) -$$

$$- \frac{1}{2} D(\rho'' - \rho''' + \mathbf{p}_2) - \frac{1}{2} D(\rho'' - \rho''' + \mathbf{p}_1 + \mathbf{p}_2) +$$

$$+ \frac{1}{2} D(\rho'' - \rho''' + \mathbf{p}_1 + 2\mathbf{p}_2) \Big\}$$

is the sixth moment of a plane wave field obtained in the approximation which is used in this paper for description of the moments of the field of an optical wave propagating through the turbulent atmosphere. As follows from Eq. (19), we have for the receiving apertures which are small as compared with the inner scale of the atmospheric turbulence for $\{p_1, p_2\} \ll l_0$ for $R \ll \rho_c$

$$\tau_4(\mathbf{p}_1, \mathbf{p}_2) \simeq \tau_0(\mathbf{p}_1)\, \tau_0(\mathbf{p}_2)\, \tau_0(\mathbf{p}_1 + \mathbf{p}_2) \times$$

$$\times \left\{ 1 - \frac{1}{12} (\kappa_m R)^2 \left[ \left(\frac{p_1}{\rho_m}\right)^2 + \left(\frac{p_2}{\rho_m}\right)^2 + \left(\frac{|\mathbf{p}_1 + \mathbf{p}_2|}{\rho_m}\right)^2 \right] \right\},$$

and for $R \gg \rho_c$

$$\tau_4(\mathbf{p}_1, \mathbf{p}_2) \simeq \tau_0(\mathbf{p}_1)\, \tau_0(\mathbf{p}_2)\, \tau_0(\mathbf{p}_1 + \mathbf{p}_2) \times$$

$$\times \left\{ \exp\left[ -\left(\frac{p_1}{\rho_m}\right)^2 - \left(\frac{p_2}{\rho_m}\right)^2 - \left(\frac{|\mathbf{p}_1 + \mathbf{p}_2|}{\rho_m}\right)^2 \right] \times \right.$$

$$\times \left[ 1 + 2 \frac{p_1^2 + \mathbf{p}_1\, \mathbf{p}_2 + p_2^2}{\rho_m^2} \right] + \frac{1}{4} \left(\frac{\rho_m}{R}\right)^2 \times$$

$$\left. \times \left[ \exp\left( -2 \frac{p_1^2}{\rho_m^2} \right) + \exp\left( -2 \frac{p_2^2}{\rho_m^2} \right) + \exp\left( -2 \frac{|\mathbf{p}_1 + \mathbf{p}_2|^2}{\rho_m^2} \right) \right] \right\}.$$

If the radius of the receiving aperture is less than the coherence radius of the plane wave in the turbulent atmosphere but larger than the inner scale of the atmospheric turbulence ($l_0 \ll R \ll \rho_c$), then for $\{p_1, p_2\} \ll l_0$

$$\tau_4(\mathbf{p}_1, \mathbf{p}_2) \simeq \tau_0(\mathbf{p}_1)\, \tau_0(\mathbf{p}_2)\, \tau_0(\mathbf{p}_1 + \mathbf{p}_2) \times$$

$$\times \left\{ 1 - \left(\frac{p_1}{\rho_m}\right)^2 - \left(\frac{p_2}{\rho_m}\right)^2 - \left(\frac{|\mathbf{p}_1 + \mathbf{p}_2|}{\rho_m}\right)^2 \right\}, \tag{20}$$

and for $l_0 \ll \{p_1, p_2\} \ll R$

$$\tau_4(\mathbf{p}_1, \mathbf{p}_2) \simeq \tau_0(\mathbf{p}_1)\, \tau_0(\mathbf{p}_2)\, \tau_0(\mathbf{p}_1 + \mathbf{p}_2) \left\{ 1 - \left(\frac{p_1}{\rho_0}\right)^{5/3} \times \right.$$

$$\times \left[ 1 - \alpha \left(\frac{p_1}{R}\right)^{1/3} \right] - \left(\frac{p_2}{\rho_0}\right)^{5/3} \left[ 1 - \alpha \left(\frac{p_2}{R}\right)^{1/3} \right] -$$

$$\left. - \left(\frac{|\mathbf{p}_1 + \mathbf{p}_2|}{\rho_0}\right)^{5/3} \left[ 1 - \alpha \left(\frac{|\mathbf{p}_1 + \mathbf{p}_2|}{R}\right)^{1/3} \right] \right\}. \tag{21}$$

Equation (20) is applicable for telescopes with large input apertures ($R \gg \rho_c$) for $\{p_1, p_2\} \ll l_0$, whereas Eq. (21) is applicable at $l_0 \ll \{p_1, p_2\} \ll \rho_c$; and, for $\rho_c < \{p_1, p_2\} \ll R$ the OTF of the triple correlation method of the image intensity has a form

$$\tau_4(\mathbf{p}_1, \mathbf{p}_2) \simeq \tau_0(\mathbf{p}_1)\, \tau_0(\mathbf{p}_2)\, \tau_0(\mathbf{p}_1 + \mathbf{p}_2) \times$$

$$\times \left\{ \exp\left[ -\left(\frac{p_1}{\rho_0}\right)^{5/3} - \left(\frac{p_2}{\rho_0}\right)^{5/3} + \left(\frac{|\mathbf{p}_1 + \mathbf{p}_2|}{\rho_0}\right)^{5/3} \right] \times \right.$$

$$\times \left[ 1 + 2a \frac{p_1^2 + \mathbf{p}_1\, \mathbf{p}_2 + p_2^2}{r_0^{5/3} R^{1/3}} \right] + 2^{-11/5} \left(\frac{\rho_0}{R}\right)^2 \times$$

$$\times\left[\exp\left(-2\frac{p_1^2}{\rho_0^2}\right)+\exp\left(-2\frac{p_2^2}{\rho_0^2}\right)+\exp\left(-2\frac{|\mathbf{p}_1+\mathbf{p}_2|^2}{\rho_0^2}\right)\right]\Bigg\}. \qquad (22)$$

The modulus of the intensity spectrum of an image is obtained from the third moment of intensity in the following cross sections: (1) $\mathbf{p}_1 = \mathbf{p}$, $\mathbf{p}_2 = 0$, (2) $\mathbf{p}_1 = 0$, $\mathbf{p}_2 = \mathbf{p}$, and (3) $\mathbf{p}_1 = \mathbf{p}$, $\mathbf{p}_2 = -\mathbf{p}$. Using the integral expression (19), we can show that

$$\tau_4(\mathbf{p}, 0)=\tau_4(0, \mathbf{p})= \tau_4(\mathbf{p}, -\mathbf{p}) = \langle M(\mathbf{p})M(-\mathbf{p})M(0)\rangle / \langle M(0)\rangle^3 =$$

$$=\langle M(\mathbf{p})M(-\mathbf{p})\rangle / \langle M(0)\rangle^2=\langle M(\mathbf{p})M^*(\mathbf{p})\rangle / \langle M(0)\rangle^2=\tau_2(\mathbf{p}).$$

Thus, one can see that the modulus of the intensity spectrum of an image can be obtained from the third moment of the image intensity in the same range of values and with the same accuracy as with the use of the Labeyrie and Knox–Thompson methods. Information on the phase of the intensity spectrum of an image is in any of the octants of the four–dimensional space $\{\mathbf{p}_1, \mathbf{p}_2\}$ confined between one of the axes $\mathbf{p}_1 = 0$ or $\mathbf{p}_2 = 0$ and a cross section $\mathbf{p}_1 = \mathbf{p}_2$ or $\mathbf{p}_1 = -\mathbf{p}_2$. Having assumed the optical wave field $U(\rho, t)$ to be distributed according to the normal law, we obtain from the integral expression (19) the following formula relating the OFT of the method of triple correlation of the image intensity to that of Knox–Thompson ($\rho_c \ll R$) method:

$$\tau_4(\mathbf{p}_1, \mathbf{p}_2) \mid \tau_1(\mathbf{p}_1)\, \tau_3(\mathbf{p}_2, \mathbf{p}_1 + \mathbf{p}_2) + \tau_1(\mathbf{p}_2)\, \tau_3(\mathbf{p}_1, \mathbf{p}_1 + \mathbf{p}_2) +$$

$$+\tau_1(\mathbf{p}_1+\mathbf{p}_2)\, \tau_3(\mathbf{p}_1,- \mathbf{p}_2)-2\, \tau_1(\mathbf{p}_1)\tau_1(\mathbf{p}_2)\tau_1(\mathbf{p}_1+\mathbf{p}_2) + O\left[\left(\frac{\rho_c}{R}\right)^4\right].$$

This relation shows in explicit form the fact that the method of triple correlation of the image intensity potentially does not make it possible to obtain more information on the image than the Knox–Thompson method. This conclusion concerns both the measurements of the modulus of the image intensity spectrum and phase. The invariance of the method of triple correlation of the image intensity under shifts of short–exposure images is undoubtly its advantage over the Knox–Thompson method.[12]

The OTFs of the "turbulent atmosphere – telescope" system presented in this paper for different methods of processing the recorded short–exposure images allow the following conclusion to be done. The Knox–Thompson method and the method of triple correlation of the image intensity have the highest potentialities compared to other methods. Moreover, it should be particularly emphasized that these two methods have practically equal potential accuracies of the image reconstruction.

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