LENS TRANSFORM IN CALCULATIONS OF COHERENT PARAXIAL BEAM PROPAGATION

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The theory of so–called "lens" transform known also as Talanov coordinate transform is discussed. It gives the possibility for the numerical modeling of focused beams propagation along very short paths and collimated beams propagation along very long path, reducing the problem on propagation of the initial beam to the problem on propagation of the equivalent beam with different initial curvature of the wave front. The formula for conversion of the refractive index field along the initial beam path to the refractive index field along the equivalent beam path is derived. This formula is used to recalculate the parameters of a medium in the problem on nonstationary thermal blooming.

1. INTRODUCTION

Methods of numerical simulation appear to be convenient for studying of nonlinear effects arising in transferring of high powerful radiation through the atmosphere or other media. But the efficiency of modern computers restricts the range of the paths which can be calculated for acceptable time intervals. These restrictions are caused by both the wide range of beam diameter variations and by high–frequency spatial oscillations in the complex amplitude of the optical wave. The former factor manifests itself as the beam propagates along the path and is typical mostly of collimated beams. The latter one is typical of focused beams with the large Fresnel number of emitting aperture. It influences the process of numerical calculation at the very beginning if it is not masked by defocusing effect of the inhomogeneous medium.

The "lens" transform known also as the Talanov coordinate system transform enables us to extend to some degree the area of calculated problems by reducing the problem on propagation of an initial beam to the problem on propagation of an equivalent beam with different initial curvature of the wave front. It was shown in Ref. 1 that the inhomogeneous parabolic equation, which describes paraxial beam propagation through the medium with the quadratic Kerr effect, is invariant to the change of variables binding the complex amplitudes of the beams focused by lenses of different focal lengths. This transform is also reported to be applicable for some other types of nonlinearity including a few nonstationary problems. But it should be stressed that the invariance of an initial equation or a system of equations describing the beam propagation is not necessary for the numerical simulation. Therefore the Talanov transform can be applied to a wider range of problems by recalculating parameters of a medium depending on the longitudinal coordinate with preservation of the unambiguous relation between the original and equivalent beams.

J. Wallace[3] applied this method for solving the problem on stationary thermal blooming taking into account the kinetic cooling effect. Bradley and Herrman[2] used the coordinate system in which the Gaussian beam propagating in vacuum kept the size and phase unchanged in the problem on thermal blooming of both continuous and pulsed radiation. Later Sziclas and Siegman[4] showed that the lens transform is one of the limiting cases of the more general complex coordinate transform. Fleck with co–authors[5] applied the Talanov transform to the complicated nonstationary problem on thermal blooming taking into account the effects of transonic beam scanning, forced and free convection as well as atmospheric turbulence. In the same article the more general form of transform was developed distinguished by the independent transformation of the wave front curvature along the longitudinal coordinates x and y. However in Ref. 5 the Talanov transform was applied not to the initial system of equations but to the diffraction steps of calculating scheme so that after every diffraction step of the splitting method scheme the inverse transform was accomplished and the heat transfer equation was solved for the initial beam rather than for the equivalent one. In the recently published article by Ustinov[6] the Talanov transform was generalized to the case of stationary thermal blooming including the transonic beam scanning.

This article is devoted to generalization of the Talanov transform to the case of nonstationary thermal blooming. The problem is completely solved for the equivalent beam both at the diffraction steps and between them. Of course, as was done in Ref. 5, the lens transform could be applied only to the diffraction steps of calculating scheme, but for thermal blooming along vertical and slant atmospheric paths we were interested in using the varying (increasing) integration steps along the longitudinal coordinate. They were chosen so that every step was accompanied by approximately equal phase distortions. To meet this condition it was necessary to know the profile of the thermal blooming parameter for the equivalent beam rather than for the initial one.

Besides, we were interested in generalization of the Talanov transform to a wider range of problems. That is why we derived first the transform formula for the arbitrary distribution of the refractive index in the linear case (that was not explicitly reported in the above–mentioned articles) and then the formulas for the transformation of the atmospheric parameters profiles. Thus, having separated the optical part of the problem from the substantial equation, we got the possibility for dealing solely with the substantial equation that simplified the problem on generalization of

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the transform to other regimes of the thermal blooming and other nonlinearity mechanisms.

2. FORMULA FOR ARBITRARY LINEAR MEDIUM

Let a paraxial beam with linear polarization of the electromagnetic field vectors be considered. The complex amplitude $U(\rho, z)$ of the wave propagating along the OZ axis in vacuum is described by the parabolic wave equation

$$2\,i\,k\,\frac{\partial U}{\partial z} = \Delta_{\perp} U\ , \tag{1}$$

where $k = 2\pi/\lambda$ is the wave number, λ is the wavelength, $\Delta_{\perp} = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the transverse Laplacian. By integrating this equation by means of the Fourier transform or the Green's function we obtain the known convolution integral

$$U(\rho, z) = \frac{i}{\lambda z} \int\int d^2\rho'\ U_0(\rho') \exp\left\{-\frac{i}{2}\frac{k}{z}(\rho - \rho')^2\right\}\ . \tag{2}$$

Consider two optical systems consisting of thin focusing lenses in the plane $z = 0$ with focal lengths f_1 and f_2. Let two identical waves with the complex amplitude $U_0(\rho)$ be incident on the input pupils of these systems. Then at the distance z_1 from the input pupil plane of the first optical system the relation for the amplitude is

$$U_1(\rho, z_1) = \frac{i}{1 z_1} \exp\left\{-\frac{i\,k}{2\,z_1}\,r^2\right\} \times$$

$$\times \int\int d^2\rho'\ U_0(\rho') \exp\left\{i\frac{k}{z_1}\rho\rho'\right\} \exp\left\{\frac{i\,k}{2}\,\rho'^2\left(\frac{1}{f_1} - \frac{1}{z_1}\right)\right\}\ . \tag{3}$$

We take into account here, that wave propagation through the thin focusing lens of the focal length f_1 corresponds to the multiplication of the complex amplitude by the phase factor $\exp\left\{\frac{i}{2}\,\rho'^2\,\frac{k}{f_1}\right\}$. Analogously to the second optical system

$$U_2(\rho, z_2) = \frac{i}{\lambda z_2} \exp\left\{-\frac{i\,k}{2\,z_2}\,\rho^2\right\} \times$$

$$\times \int\int d^2\rho'\ U_0(\rho') \exp\left\{i\frac{k}{z_2}\,\rho\,\rho'\right\} \exp\left\{\frac{i\,k}{2}\,\rho'^2\left(\frac{1}{f_2} - \frac{1}{z_2}\right)\right\}. \tag{4}$$

In particular case $z_1 = f_1$ and $z_2 = f_2$, the correlation between U_1 and U_2 can be easily derived from Eqs. (3) and (4)

$$U_2(\rho, f_2) = U_1\left(\rho\frac{f_1}{f_2}, f_1\right)\frac{f_1}{f_2}\exp\left\{-\frac{i}{2}\,\rho^2 k\left(\frac{1}{f_2} - \frac{1}{f_1}\right)\right\}\ . \tag{5}$$

The formula for intensities can be obtained from Eq. (5)

$$I_2(\rho, f_2) = (f_1/f_2)^2\, I_1\,(\rho\, f_1/f_2, f_1)\ . \tag{6}$$

Thus, the amplitude distributions differ from each other by the factor and the scale, and the phase fronts differ by the quadratic phase difference proportional to the difference in optical lens powers, namely $1/f_2 - 1/f_1$. In general case $z_1 \neq f_1$ and $z_2 \neq f_2$, we can express one field in terms of another if the relation is true

$$1/f_1 - 1/z_1 = 1/f_2 - 1/z_2\ . \tag{7}$$

By denoting $\delta = 1/f_2 - 1/f_1$, it is easy to obtain

$$z_1 = z_2/(1 - \delta\, z_2)\ ; \tag{8}$$

$$z_2 = z_1/(1 + \delta\, z_1)\ ; \tag{9}$$

If z_1 and z_2 meet relations (8) and (9), then from Eqs. (3) and (4) we have

$$U_2(\rho, z_2) = \frac{1}{1 - \delta\, z_2}\, U_1\left(\frac{\rho}{1 - \delta\, z_2}, z_1\right)\exp\left\{\frac{i\,k}{2}\,\rho^2\,\frac{\delta}{1 - \delta\, z_2}\right\}. \tag{10}$$

By substituting (9) into (10) and omitting the subscript on z we have

$$U_2(\rho, z) = \frac{1}{1 - \delta\, z}\, U_1\left(\frac{\rho}{1 - \delta\, z}, \frac{z}{1 - \delta\, z}\right)\exp\left\{\frac{i\,k}{2}\,\rho^2\,\frac{\delta}{1 - \delta\, z}\right\}. \tag{11}$$

Similarly

$$U_1(\rho, z) = \frac{1}{1 + \delta\, z}\, U_2\left(\frac{\rho}{1 + \delta\, z}, \frac{z}{1 + \delta\, z}\right)\exp\left\{\frac{i\,k}{2}\,\rho^2\,\frac{-\delta}{1 + \delta\, z}\right\}. \tag{12}$$

Thus, it is shown, that in the plane $z = 0$ the fields differing from each other by the quadratic phase factor $\exp\left\{\frac{i}{2}\, k\delta\rho^2\right\}$ are related by Eqs. (11) and (12) under diffraction in vacuum or optically homogeneous medium. That gives the possibility to express one field in terms of another in the cross sections satisfying to Eq. (6).

In optically inhomogeneous medium where the refractive index is the coordinate–dependent function the complex amplitude of propagating beam is described by the nonuniform equation

$$2\,i\,k\,\frac{\partial U}{\partial z} = \Delta_{\perp} U + 2\,k^2\,\tilde{n}(\rho, z)\,U\ , \tag{13}$$

where $\tilde{n}(\rho, z) = n(\rho, z) - 1 \ll 1$. Let

$$U_1(\rho, z) = U_0(\rho, z) \tag{14}$$

$$U_2(\rho, z) = U_0(\rho, z)\exp\left\{\frac{i}{2}\, k\,\delta\,\rho^2\right\}\ . \tag{15}$$

Then, for the case of propagation in vacuum the fields U_1 and U_2 are related by Eqs. (11) and (12).

Let the field U_1 propagate in medium with the refractive index $n_1(\rho, z)$. Such a distribution of the refractive index $n_2(\rho, z)$ should be found that makes relations (11) and (12) valid for the field U_2 propagating through the optically homogeneous medium with the same distribution.

Thus, let the functions U_1 and U_2 determined over half–space $z > 0$ be related by Eq. (11) and the function $U_1(\rho, z)$ at the same time satisfy the differential equation

$$2\,i\,k\,\frac{\partial U_1}{\partial z} = \Delta_{\perp} U_1 + 2\,k^2\,\tilde{n}_1(\rho, z)\,U_1\ . \tag{16}$$

Such a function $n_2(\rho, z)$ is required to be found that makes the following equation valid for U_2:

$$2\,i\,k\,\frac{\partial U_2}{\partial z} = \Delta_\perp U_2 + \tilde{n}_2(\rho,\,z)\,U_2\,. \qquad (17)$$

By rewriting (16)–(17) to

$$2\,k^2\,\tilde{n}_2(\mathbf{r})\,U_2(\mathbf{r}) = 2\,i\,k\,U'_{2z}(\mathbf{r}) - U''_{2xx}(\mathbf{r}) - U''_{2yy}(\mathbf{r})\,, \qquad (18)$$

$$2\,k^2\,\tilde{n}_1(\mathbf{r})\,U_1(\mathbf{r}) = 2\,i\,k\,U'_{1z}(\mathbf{r}) - U''_{1xx}(\mathbf{r}) - U''_{1yy}(\mathbf{r})\,, \qquad (19)$$

where $r = (\rho,\,z) = (x,\,y,\,z)$, we can reduce the last formula to

$$2\,k^2\,\tilde{n}_1\left(\frac{\mathbf{r}}{1-\delta\,z}\right) = 2\,i\,k\,U'_{1z}\left(\frac{\mathbf{r}}{1-\delta\,z}\right) - - U''_{1xx}\left(\frac{\mathbf{r}}{1-\delta\,z}\right) - U'_{1yy}\left(\frac{\mathbf{r}}{1-\delta\,z}\right) \qquad (20)$$

By dividing (18) by (20) and omitting the symbol ~ at $n_{1,2}$, we obtain

$$\frac{n_2(\mathbf{r})}{n_1(\mathbf{r}/(1-\delta\,z))} = \frac{U_1(\mathbf{r}/(1-\delta\,z))}{U_2(\mathbf{r})} \times \times \frac{2\,i\,k\,U'_{2z}(\mathbf{r}) - U''_{2xx}(\mathbf{r}) - U''_{2yy}(\mathbf{r})}{2\,i\,k\,U'_{1z}\left(\frac{r}{1-\delta\,z}\right) - U''_{1xx}\left(\frac{\mathbf{r}}{1-\delta\,z}\right) - U''_{1yy}\left(\frac{\mathbf{r}}{1-\delta\,z}\right)}. \qquad (21)$$

The first fraction in the right side of the equation can be easy calculated from Eq. (11)

$$\frac{U_1(\mathbf{r}/(1-\delta\,z))}{U_2(\mathbf{r})} = (1-\delta\,z)\,\exp\left\{-\frac{i\,k}{2}\,\rho^2\,\frac{\delta}{1-\delta\,z}\right\}. \qquad (22)$$

To calculate the second fraction, Eq (11) should be differentiated with respect to ∂z, $\partial^2 x$, and $\partial^2 y$. By combining the results of differentiation, we have

$$2\,i\,k\,U'_{2z}(\mathbf{r}) - U''_{2xx}(\mathbf{r}) - U''_{2yy}(\mathbf{r}) = \exp\left\{-\frac{i\,k}{2}\,\rho^2\frac{\delta}{1-\delta\,z}\right\}\frac{1}{(1-\delta\,z)^3}\times \times\left[2\,i\,k\,U'_{1z}\left(\frac{\mathbf{r}}{1-\delta\,z}\right) - U''_{1xx}\left(\frac{\mathbf{r}}{1-\delta\,z}\right) - U''_{1yy}\left(\frac{\mathbf{r}}{1-\delta\,z}\right)\right] \qquad (23)$$

so that the second fraction in (21) becomes equal to:

$$\exp\left\{-\frac{i\,k}{2}\,\rho^2\,\frac{\delta}{1-\delta\,z}\right\}\frac{1}{(1-\delta\,z)^3}\,. \qquad (24)$$

By substituting (22) and (24) into (21) we can obtain

$$n_2(\mathbf{r}) = \frac{1}{(1-\delta\,z)^2}\,n_1\left(\frac{1}{1-\delta\,z}\right). \qquad (25)$$

3. GENERALIZATION OF THE TRANSFORM TO THE CASE OF THERMAL BLOOMING ALONG THE INHOMOGENEOUS PATH

Consider the nonstationary heat transfer equation for the absorbing medium moving at a rate $V(z)$ and characterized by the absorption coefficient $\alpha(z)$, density $\rho(z)$, heat capacity $C_p(z)$, and heat conductivity $\chi(z)$. Multiplying it by the refractive index derivative with respect to the temperature $n'_T(z)$ we derive an equation for the refractive index $n(\mathbf{r},\,T) = n'_T(z)\,T(\boldsymbol{r},\,t)$

$$n'_t(\mathbf{r},\,t) + \mathbf{V}(z)\,\nabla n(\mathbf{r},\,t) + \chi(z)\,\Delta_\perp\,n(\mathbf{r},\,t) = q(z)\,I(\mathbf{r},\,t)\,, \qquad (26)$$

where $\Delta_\perp$ is the transverse Laplacian; $q = \alpha\,n'_T/\rho C_p$; $I(\mathbf{r},\,t)$ is the radiation intensity; t is the time.

We are to obtain the relation between the profiles of the $\mathbf{V}_1$, χ_1, and q_1 parameters of the initial path and the correspondent profiles of $\mathbf{V}_2$, χ_2, and q_2 of the equivalent beam path. For this purpose, the equation for n_1 at the $\mathbf{r}/(1-\delta\,z)$ point is divided by the equation for n_2 at the point of $\mathbf{r}$ taking into account that $I_1(\mathbf{r}/(1-\delta\,z),\,t)/I_2(\mathbf{r},\,t) = (1-\delta\,z)^2$. Partial derivatives of the n_1 function at $\mathbf{r}/(1-\delta\,z)$ can be expressed by the partial derivatives of the n_2 function at the $\mathbf{r}$ point by differentiating formula (25) in form of $n_1(\mathbf{r}/(1-\delta\,z),\,t) = (1-\delta\,z)^2 n_2(\mathbf{r},\,t)$. Let us demand also the dependences of n_1 and n_2 on the time be the same. Omitting the intermediate calculations and reducing $(1-\delta\,z)^2$ in both sides of equation, we obtain

$$\frac{n'_{2t} + (1-\delta\,z)\,\mathbf{V}_1\left(\frac{z}{1-\delta\,z}\right)\nabla n_2 + (1-\delta\,z)^2\,\chi_1\left(\frac{z}{1-\delta\,z}\right)\Delta_\perp n_2}{n'_{2t} + \mathbf{V}_2(z)\nabla n_2 + \chi_2(z)\,\Delta_\perp\,n_2} = = \frac{q_1\left(\frac{z}{1-\delta\,z}\right)}{q_2(z)}\,, \qquad (27)$$

where the arguments $\mathbf{r}$ and t of $n_2(\mathbf{r},\,t)$ have been omitted for brevity. It is easy to see that this equation is valid by the following relations between the initial and equivalent beam path parameters:

$$q_2(z) = q_1\left(\frac{z}{1-\delta\,z}\right), \qquad (28)$$

$$\mathbf{V}_2(z) = (1-\delta\,z)\,\mathbf{V}_1\left(\frac{z}{1-\delta\,z}\right), \text{ and} \qquad (29)$$

$$\chi_2(z) = (1-\delta\,z)^2\chi_1\left(\frac{z}{1-\delta\,z}\right). \qquad (30)$$

If there are no heat transfer and wind on the path, and if parameter q_1 is independent of the longitudinal coordinate (homogeneous path), then these equations will be invariant to the examined transform that corresponds to the results by Talanov.[1]

4. CONCLUSION

The lens transform formula for an arbitrary linear medium derived in this paper enables us to generalize this transform to the cases with complicated nonlinearity mechanism, that was demonstrated for the case of nonstationary thermal blooming along the inhomogeneous path taking the heat conductivity of a medium into account. Another possible application is the transform generalization to the case of randomly inhomogeneous medium, for example, the turbulent atmosphere. As a result of this problem solution, there should be developed the transform formulas for the structural constant of the refractive index fluctuations as well as for outer and inner scales of turbulence while in more general case, the formula for the spectral density transform.

However, derivation of these formulas requires somewhat different approach than that used for nonlinear problems, because the wave propagation through randomly inhomogeneous media is described by a stochastic equation that can be solved only by means of approximate methods. Moreover, even in case of homogeneous and isotropic turbulence along the initial path, the turbulence along the equivalent beam path is to be described in terms of random field with slowly varying average characteristics, that is also an approximate approach. Making formal calculations on the structural function expression for inertial interval of turbulence, we derived the following relation for the structural constant transform:

$$C^2_{n(2)}(z) = \frac{(1-\delta z)^{-2/3}}{(1-\delta z)^4}\, C^2_{n(1)}\left(\frac{z}{1-\delta z}\right). \qquad (31)$$

But we are not sure that this formula gives the valid solution. Apparently, the form of transform of the turbulence parameters along the path should depend on the approximation applied for the propagation equation solution.

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