ON THE CONTINUATION OF INTERFEROGRAMS BEYOND THE DOMAIN OF DEFINITION

V.A. Tartakovskii

Institute of Atmospheric Optics,
Siberian Branch of the Russian Academy of Sciences, Tomsk,
Received September 18, 1993

The method of continuation of individual scans of the interferogram is proposed, according to which the continuation is constructed from the fragments of scans themselves. In doing so the smoothness functionals are minimized on the set of scan readings. The results of numerical experiments on the interferogram inversion are given.

Application of the Fourier and Hilbert transforms to invert interferograms[1–3] was a natural step forward as well as in the case of solution of many other problems in which the analysis was transferred from the coordinate to the frequency space. In this case the determination of the arrangement of extremal lines in the interferogram was replaced by the more exhaustive analysis of the object field after its filtration in the trigonometric basis using the Hilbert filter.

Some problems arise when the algorithm based on the Hilbert transform (H) is applied. It is their successful solution that opens up the possibility to achieve a high accuracy reconstruction of the object phase higher than $\lambda/100$.

One of these problems discussed in this paper is the inconsistency between the domain of definition of the trigonometric basis functions, i.e., that of the H operator, which has the only boundary at infinity and the bounded domain often being, in addition, multiple–connected, where the interferogram is specified. This inconsistence manifests itself as boundary bursts in wave function and especially in its phase in the process of the interferogram inversion. Therefore the problem arises of the optimal continuation of the interferogram beyond the domain of definition which can be solved using the iteration method (R. Gershberg, 1986, Ref.4). To do this, the algorithm of the fast Fourier transform (FFT) is most often applied. However, the FFT iterations of the two–dimensional data array take a lot of time and, besides, the additional two–dimensional array is required to place the imaginary part of spectrum of the function under transformation. Even more essential drawback is the absence of assurance in the iteration process convergence for discrete objects in general case, especially in the presence of noise.[5] It is of great importance to choose the correct zero approximation as well.

Two–dimensional matrix of readings of the interferogram is redundant with respect to the object phase, because the number of readings is chosen proceeding from the frequency of the space carrier, namely, the number of interference bands, and on the frequency properties of the noise. Therefore the idea occurs to analyze the interferogram only in its individual sections from which the phase sections are retrieved. Their composition represents a two–dimensional function. The sections of the interferogram, the scans, should have the causal Fourier transform for the object field with respect to the parameter of the scanning line.

It was found in numerical experiments[3] that the phase monotony or the possibility to bring it to this form due to parity and periodicity of the cosine function is sufficient for the causality of the object field either immediately or after the stretching – compressing transformation. This provides considerable opportunities for choosing the rectilinear and curvilinear scans in the interferogram.

If the chosen scanning curve is closed and pertains to the domain of definition of the interferogram the problem of continuation is absent because the scan is the periodic function of the scanning parameter. However, usually the scans are discontinued at the external boundaries of the domain of definition of the interferogram. Besides, when the domain is multiple–connected, the discontinuities appear at internal boundaries. As the result a scan consists of some bounded fragments.

Further we obtain the criteria of the optimal continuation. Let $u(x)$ be the bounded function defined for all x, and the scan of the interferogram contains the known fragments of this function. Let $\tilde{u}(x)$ be coincident with $u(x)$ within the limits of the scan. Let us introduce the integrable square function $\Omega(x)$ posessing the low frequency spectrum which does not overlap with the spectra of the functions $\tilde{u}(x)$ and $u(x)$. Let us consider the equality following from the properties of the Hilbert transform

$$\int\limits_{-\infty}^{\infty} \Omega^2(x)\,(u(x) - \tilde{u}(x))^2\,dx =$$

$$= \int\limits_{-\infty}^{\infty} \Omega^2(x)\,\big(H\,[u(x) - \tilde{u}(x)]\big)^2\,dx\,. \qquad (1)$$

Let the function $\Omega(x)$ turn into unity within the limits of the scan and into zero beyond of it. The left–hand side of Eq. (1) goes to zero in this case, and the spectrum of the function $\Omega(x)$ becomes wider. When the spectrum becomes so wide that begins to overlap the spectra of $\tilde{u}(x)$ and $u(x)$, the equality (1) becomes invalid. In order to the invalidity of the equality can manifest itself at the smallest possible value of its left side it is necessary to ensure the most possible narrow–

0235-6880/93/12 898-05 $02.00

bandness of the functions $u(x)$ and $\tilde{u}(x)$. *A posteriori*, it is feasible only to reduce the width of spectrum of the function $\tilde{u}(x)$ due to the more smooth joining of individual scan fragments of the interferogram within the domain of definition, the more smooth continuation beyond the domain, and the choice of the functional dependence providing this smoothness. Thus, in order to the Hilbert transform of the continued function can correspond most closely to the true one it is necessary for the continuation operation to ensure the minimal width of the spectral band of the function $\tilde{u}(x)$ when the fragments of $\tilde{u}(x)$ are specified. Therein lies the criteria of the optimum.

Next consider the structural limitations contributed by the numerical methods. The most effective and technically supplied algorithm for the Fourier transform of the set of readings is the FFT. The Hilbert transform and the other types of linear filters are readily realized on its basis. However, the FFT algorithm implies that the readings are given at a circle, hence they may correspond only to the periodic function. Therefore the scan of the interferogram should be continued onto the whole infinite axis also periodically. In addition it should be kept in mind that the FFT algorithm is processing the arrays of the quite definite length. Usually this value is equal to the integer power of two, namely, 2^m, but for the rarely used Singleton algorithm (R.C. Singleton, 1968, Ref. 6) the allowed lengths represent sufficiently dense set.

A priori information on the properties of an interferogram consists in that the phase difference of the object field and the reference fields contains the considerable linear, quadratic, or close to them component and it is the component which yields the fringed structure of the interferogram. The linear continuation of such component conserving the continuity both of it itself and of its derivative satisfies the criteria of optimum introduced before, because the width of spectrum $s(k)$ of the signal $u(x)$ with respect to the central frequency k_c is determined by the derivative of the signal phase $\phi(x)$. For the signal with a constant amplitude $\exp i\phi(x)$ the following relation is valid

$$\frac{1}{2\pi T}\sum_{k=-\infty}^{\infty}(k-k_c)^2\,|s(k)|^2=\frac{1}{T}\int_0^T\phi'^2(x)\,dx-\left(\frac{1}{T}\int_0^T\phi'(x)\,dx\right)^2, \quad (2)$$

where T is the period. Consequently, the condition should be fulfilled for appearance of additional interference fringes or their parts, the width of which should be close to the width of the nearest given interference fringes after the continuation of the interferogram. The known theorem on the convergence of the Fourier series connects the speed of convergence of the series with a number m of the continuous derivatives of the function under transformation

$$s(k)=o\,(1/k^{m+1})\,. \quad (3)$$

It is clear from this expression that in the process of continuation the jumps between the known fragments $u(x)$ and obtained as a result of continuation $\tilde{u}(x)$ are undesirable. However, this situation is typical for the iteration method of continuation, namely, either a large number of iterations takes place and we have the smooth joining of fragments with the availability of convergence or only a few number of iterations is carried out and we have a discontinuity between the functions $u(x)$ and $\tilde{u}(x)$ in the point of their joining.

In the paper by A. Spik, 1987, Ref. 7, the technique is described of supplementing the interference fringes in individual scans of the interferogram by the sine curve, the initial phase, frequency and amplitude of which are determined using the known fragments. According to this method it is necessary to determine the positions of the extremal points in the scan. However, in the presence of the noise this operation is incorrect and the smoothness of the continued function is unavailable.

In this paper the way of continuation of individual scans of the interferogram is proposed, according to which the continuation is constructed from the fragments of the scans themselves. In doing so some smoothness functionals are minimized at the set of readings of the scan.

Let us consider one of possible algorithms of realization of this method. We construct the continuation making shifts of the scan fragments beyond the domain of definition. Let the continued scan be defined in the interval $[1, n]$. Let us at first continue the right edge of the scan on some number of readings r, which ensures the minimum of the functional

$$\underset{r\in(k,n-k)}{L(r)}=\sum_{i=n-k+1}^{n}|u(i)-u(i-r)|=\min\,. \quad (4)$$

Then let us continue the left edge of the scan on l readings under condition

$$\underset{l\in(k,n-k)}{L(l)}=\sum_{i=1}^{k}|u(i)-u(i+l)|=\min\,. \quad (5)$$

The search of the minimum can be made simply taking readings one by one. Simultaneously the separations are minimized both between the two functions and between their differences $\Delta u(i)$. For example, at $k = 2$ we obtain

$$\min=|u(1)-u(l+1)|+|u(2)-u(l+2)|>|[u(2)-u(1)]-[u(l+2)-u(l+1)]|=|\Delta u(1)-\Delta u(l+1)|\,. \quad (6)$$

The newly formed edges are joining between each other in the similar way to obtain the periodic function, but the smoothness functional depends now on two parameters

$$\underset{\substack{p\in(0,n-k)\\ q\in(0,l-k)}}{L(p,q)}=\sum_{i=1}^{k}|u(r+i+p-1)-u(i+q)|\,.$$

In the same way, one by one, the conditional minimum $L(p, q)$ is found, so that the length of the continued scan $nc = n + l + p + q$ belongs to the Singleton set of numbers. Therefore the sufficient density of these numbers in the interval $[n, 3n]$, nc is of principle importance. It is this quantity on which the efficiency of the method depends. For the most widely used range from $m = 7$ to $m = 8$ these numbers are listed in Table I.

TABLE I.

128	130	132	133	135	136	138	140	143	144	147
150	152	153	154	156	160	161	162	165	168	169
170	171	175	176	180	182	184	187	189	190	192
195	196	198	200	204	207	208	209	210	216	220
224	225	228	234	240	242	243	245	250	252	256

Figure 1 demonstrates the exponentional increase of the Singleton numbers and, consequently, the feasibility of continuation on definite interval decreases beginning from some point. The feasibility of continuation can be determined with the use of the ratio of the quantity of the Singleton numbers in a given interval to the length of this interval, see Fig. 2.

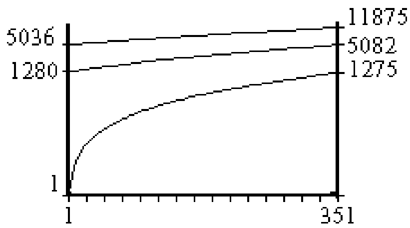

FIG. 1. The Singleton numbers. The numbers in the abscissa axis correspond to the lower curve. The rest curves have the intervals of definition [352, 702] *and* [703, 1053], *respectively. The logarithm scale corresponds to ordinate axis.*

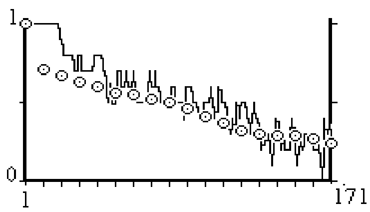

FIG. 2. The feasibility of the scan continuation on 10 readings (circles) and on 100 readings (curve). The reading number from which the continuation begins is shown in the abscissa axis.

The conclusion following from the above description is that the method meets the introduced criteria of optimum under location of k and n in the class of functions of the continued scan and within the framework of the Singleton algorithm.

A number of numerical experiments was made to check out the method and the algorithm described above. The first experiment consisted in calculation of the model interferogram. It had the 6.3 interference fringes, the root–mean square (rms) deviation of the object phase was equal to 0.5. The phase was prescribed by the part of the series containing 21 Zernike polinomials, the power spectrum was unimodular and changed from zero at the edges to unity at the point corresponding to the spectral aberration. The order of the reading matrix was equal to 77, the number of quantization levels was equal to 128. The interferogram was inverted according to the method of Ref. 3 on the basis of the Hilbert transform using the proposed algorithm of continuation and without it. The last is possible because the number 77 is just the Singleton number. The rms deviation of the retrieved phase from the initial one was calculated without taking into account the band of various width at the boundary of the matrix. Sixteen experiments were made. It can be seen from Table II that the continuation decreases essentially, by the order of magnitude, the boundary bursts which occupied at the boundary the fringe of 10% wide as compared with the dimension of the interferogram.

In the process of continuation the number of the interference bands increases. With this purpose the continuation can be made several times for the same reference signal. It could reduce the error of retrieving in the cases when the dispersion and the frequency band of the object phase do not correspond to the carrier frequency of the object field, and this field contains the origin of coordinates in the Fourier plane. To check this out the numerical experiment was made. The interferogram differed from the previous one only by the number of the interference fringes, which was equal to 5.5. Every continuation supplemented 1.5–3 interference fringes.

The decrease and the following increase of the error of the phase retrieving is seen from Table III. The decrease confirms the assumption described above, and the increase in error is connected with the only conditional optimality of the continuation.

TABLE II.

Relative width of the boundary band, %	Normalized rms error of the phase retrieving	
	with continuation	without continuation
0	0.0065(3)	0.068(5)
1	0.0057(2)	0.026(3)
5	0.0057(2)	0.012(2)
10	0.0054(2)	0.008(2)
15	0.0025(1)	0.004(2)
20	0.0025(1)	0.002(1)

TABLE III.

Number of continuations of the interferogram scan	Normalized rms error of the phase retrieving
0	0.456(4)
1	0.136(2)
2	0.124(2)
3	0.108(1)
4	0.212(3)
5	0.283(3)

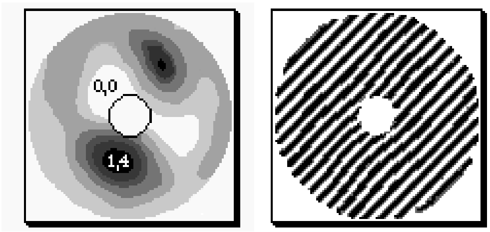

FIG. 3. The reference phase, rad. The interferogram under inversion.

The interferogram defined in the circle with the round central aperture was inverted, see Fig. 3. The number of interference fringes was equal to 18, the rms phase deviation was equal to 0.5, and the number of readings on the diameter was equal to 101. The normalized rms deviation of the multiplicative noise was equal to 0.48. The rest parameters were the same as in the first experiment. This interferogram is double–connected, its scans have different lengths and the same numbers of interference fringes. The scans passing through the central aperture consisted of two fragments which were continued independently. The normalized rms error of retrieving the phase was obtained to be 0.051 (2) that corresponds to the rms error $\lambda/250$ at the peak value $\lambda/20$. The time of inversion of one interferogram using IBM PC 386 was equal to ~ 150 s.

In that way the proposed method of continuation of scans of interferogram beyond the domain of definition solves the formulated problem.

REFERENCES

1. V.A. Tartakovskii, in: *Abstracts of the VI All–Union Symposium on Propagation of Laser Radiation in the Atmosphere,* Tomsk, 1981, Part 3, p. 32–34.
2. M. Takeda, H. Ina, and S. Kobayachi, J. Opt. Soc. Amer. **72**, No. 1, 156–160 (1982).
3. E.A. Vitrichenko, V.P. Lukin, L.A. Pushnoi, and V.A. Tartakovskii, *Problems of Optical Control* (Nauka, Novosibirsk, 1990), 351 pp.
4. C. Roddier and F. Roddier, Apl. Optics, **26**, No. 9, 1668–1673 (1987).
5. R.U. Shafer, Z.M. Mersero, and M.A. Richards, TIIER **69**, No. 4, 34–54 (1981).
6. R.C. Singleton, IEEE Transaction on Audio and Electroacoustics **AU–17**, No. 2, 93–103 (1969).
7. A. Spik, Opt. Appl. **XVII**, No. 4, 349–354 (1987).