# SOLUTION OF THE EQUATION FOR THE COHERENCE FUNCTION IN THE RAY APPROXIMATION

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*The ray approximation is constructed for the solution of the equation for the second–order coherence function. Based on the solution obtained in this approximation the self–actions are compared of coherent and partially coherent beams with equal Fresnel's numbers. The relation of this approximation with the ray approximations constructed for the solution of the small–angle radiative transfer equation is discussed.*

The equation for the second–order coherence function can be used to describe the propagation of partially coherent radiation in a refraction medium

$$\frac{\partial\Gamma}{\partial z}+\frac{1}{ki}\nabla_{\mathbf{R}}\nabla_{\rho}\Gamma(z,\mathbf{R},\rho)+$$
$$+\frac{k}{2i}\left[\varepsilon\left(z,\mathbf{R}+\frac{\rho}{2}\right)-\varepsilon\left(z,\mathbf{R}-\frac{\rho}{2}\right)\right]\Gamma=0\,,\qquad(1)$$

where $k$ is the wave number, $z$ is the coordinate along the propagation axis, $\mathbf{R}$ and $\rho$ are the summed and difference coordinates in a plane perpendicular to the propagation axis, and $\varepsilon$ is the perturbation of the dielectric constant.

For a linear refraction medium Eq. (1) is the rigourous consequence of the parabolic equation. For a nonlinear medium Eq. (1) may be derived from the parabolic equation under certain conditions (see, e.g., Refs. 1–3) enabling us to separate the product of the fluctuating dielectric constant and field by averaging.

Without dwelling on this question, we note that in a physical sense it means that the induced (nonlinear) fluctuations of the dielectric constant unsignificantly affect the field fluctuations.

Possible ways of obtaining the analytical solutions of Eq. (1) are rather limited. Numerical solution of the given equation is a nontrivial problem due to its multidimensionality, because the coherence function depends on five spatial variables. In the papers that have been published the numerical solutions of this equation are given for the axisymmetric problems whose dimensionality is reduced to four.[4]

In this paper the ray approximation is applied for the solution of the given equation.

Let us represent the coherence function in the form

$$\Gamma(z,\mathbf{R},\rho)=\gamma(z,\mathbf{R},\rho)\exp\left(i\Phi(z,\mathbf{R},\rho)\right)\,,$$

where $\gamma$ and $\Phi$ are the real functions. Substituting this relation into Eq. (1) and assuming real and imaginary parts of this equation to be equal to zero we derive the following system of equations:

$$\frac{\partial\gamma}{\partial z}+\frac{1}{\kappa}\left[\nabla_{\mathbf{R}}\gamma\nabla_{\rho}\Phi+\gamma\nabla_{\mathbf{R}}\nabla_{\rho}\Phi+\nabla_{\rho}\gamma\nabla_{\mathbf{R}}\Phi\right]=0\,,\qquad(2)$$

$$\gamma\frac{\partial\Phi}{\partial z}+\frac{1}{k}\gamma\nabla_{\mathbf{R}}\Phi\nabla_{\rho}\Phi=\frac{1}{\kappa}\nabla_{\mathbf{R}}\nabla_{\rho}\gamma+$$
$$+\frac{\kappa}{2}\left[\varepsilon\left(z,\mathbf{R}+\frac{\rho}{2}\right)-\varepsilon\left(z,\mathbf{R}-\frac{\rho}{2}\right)\right]\gamma\,.\qquad(3)$$

Acting on Eq. (3) by the operator $\nabla_{\rho}$ and approaching $\rho$ to zero we derive

$$\frac{\partial\theta}{\partial z}+\theta\nabla_{\mathbf{R}}\theta=\frac{1}{2}\nabla_{\mathbf{R}}\varepsilon(z,\mathbf{R})+\left(\frac{1}{\kappa^2\gamma}\nabla_{\rho}\nabla_{\mathbf{R}}\nabla_{\rho}\gamma\right)\Big|_{\rho=0}\,,\qquad(3a)$$

where
$\theta=\kappa^{-1}\nabla_{\rho}\Phi|_{\rho=0}$ .

It is well known that the average energy flux density $P$ is related with the coherence function by the following formula[10]

$$i\kappa\mathbf{P}(z,\mathbf{R})=\nabla_{\mathbf{r}}\Gamma(z,\mathbf{R},\mathbf{r})|_{\mathbf{r}=0}\,,$$

where $\mathbf{r}=\{\zeta,\rho\}$, $z$ and $\mathbf{R}$ are the summed coordinates, and $\zeta$ and $\rho$ are the difference coordinates. Taking into consideration the fact that $\gamma$ is the even function while $\Phi$ is the odd function of the difference argument $\mathbf{r}$ we derive

$$\kappa\mathbf{P}(z,\mathbf{R})=\gamma(z,\mathbf{R},\mathbf{r}=0)\nabla_{\mathbf{r}}\Phi|_{\mathbf{r}=0}=W(z,\mathbf{R})\nabla_{\mathbf{r}}\Phi|_{\mathbf{r}=0}\,,$$

where $W(z,\mathbf{R})=\gamma(z,\mathbf{R},\mathbf{r}=0)$ is the average intensity of the radiation.

Then $\mathbf{n}=\kappa^{-1}\nabla_{\mathbf{r}}\Phi|_{\sigma=0}$ is the unit vector collinear to the direction of the average energy flux density. We assume that the average wave front is perpendicular to the average energy flux density at any point. Hence, we obtain that $\theta$ is the tangential component of the unit vector $\mathbf{n}$ perpendicular to the average wave front. Then for the diffraction ray perpendicular to the average wave front at each point the relation $d\mathbf{R}/dz=\theta$ is valid. Taking this relation into account from Eq. (3a) we derive the equation for the diffraction ray

$$\frac{d^2\mathbf{R}}{dz^2}=\frac{1}{2}\nabla_{\mathbf{R}}\varepsilon+\frac{1}{\kappa^2W}\left(\nabla_{\rho}\nabla_{\mathbf{R}}\nabla_{\rho}\gamma\right)|_{\rho=0}\,.\qquad(4)$$

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Approaching ρ to zero in Eq. (2) and taking into account that γ and Φ are the even and odd functions of the difference argument ρ, respectively, we derive the equation

$$\frac{\partial W}{\partial z} + \nabla_{\mathbf{R}} (\theta W) = 0 , \quad (5)$$

from which it follows that the energy transferred along a ray tube bounded by diffraction rays conserves. Then for variation of the radiation intensity along the diffraction ray we obtain

$$W(z, \mathbf{R}(z)) = W(z = 0, \mathbf{R}_0) / \left| \frac{d\mathbf{R}(z)}{d\mathbf{R}_0} \right| , \quad (6)$$

where the determinant $|d\mathbf{R}(z)/d\mathbf{R}_0|$ is the ratio of the current cross section of the ray tube to the initial one when they both approach to zero, and $\mathbf{R}$ and $\mathbf{R}_0$ are the current and initial transverse coordinates of the diffraction ray $(\mathbf{R}_0 = \mathbf{R}(z = 0))$.

In this way, Eq. (1) is transformed into a system of two equations (4) and (6). There are three unknown variables $\mathbf{R}$, $W$, and γ in these two equations, and additional conditions must be employed. In particular, assuming the radiation to be coherent and representing the radiation field in the form

$$E(z, \mathbf{R}) = A(z, \mathbf{R})\, e^{iS(z, \mathbf{R})} ,$$

we can write

$$\gamma(z, \mathbf{R}, \rho) = A\left(z, \mathbf{R} + \frac{\rho}{2}\right) A\left(z, \mathbf{R} - \frac{\rho}{2}\right) .$$

Substituting this relation into Eq. (4) we derive the well known equation[5,6]

$$\frac{d^2\mathbf{R}}{dz^2} = \frac{1}{2} \nabla_{\mathbf{R}}\, \varepsilon(z, \mathbf{R}) + \frac{1}{2\kappa^2} \nabla_{\mathbf{R}} (A^{-1} \nabla^2_{\mathbf{R}} A(z, \mathbf{R})) , \quad (7)$$

where $A = W^{1/2}$ is the wave amplitude. Following Ref. 5 let us introduce the effective dielectric constant $\varepsilon_{\rm eff} = \varepsilon(z, \mathbf{R}) + \kappa^{-2} A^{-1} \nabla^2_{\mathbf{R}} A$, then

$$\frac{d^2\mathbf{R}}{dz^2} = \frac{1}{2} \nabla_{\mathbf{R}}\, \varepsilon_{\rm ef}(z, \mathbf{R}) . \quad (8)$$

The system of Eqs. (6)–(8) is analogous to the system of equations of geometric optics in quasioptics approximation but has a basic difference. The diffraction term entering into $\varepsilon_{\rm eff}$ prevents the formation of caustics (intersections and collapses of rays). The caustics arise only when we consider the Kerr nonlinearity in the case in which the variation of the dielectric constant is described by the relation $\Delta\varepsilon = \varepsilon_2 |E|^2$, which ignores a real saturation of increase of the dielectric constant with increase of $|E|^2$.

The system of Eqs. (6) and (8) is solvable as far as it contains only two unknowns $\mathbf{R}$ and $W$. It was numerically solved in the ray approximation.[7] However we note that the given system describes the coherent radiation propagation and is identical to the parabolic equation rather than to Eq. (1).

The system of Eqs. (4) and (6) would be solvable if Eq. (4) were independent of γ. We can eliminate this dependence assuming that the modulus of the coherence function γ keeps the Gaussian shape in difference coordinates, i.e.,

$$\gamma(z, \mathbf{R}, \rho) = |\Gamma(z, \mathbf{R}, \rho)| = W(z, \mathbf{R}) \exp\left( -\frac{x^2}{4a_x^2} - \frac{y^2}{4a_y^2} + bxy \right),$$

where $\rho = \{x, y\}$. But this condition is still insufficient. We must also determine a regularity of variation in the parameters $a_x$, $a_y$, and $b$ attendant to the propagation of the radiation through the refraction medium.

We shall assume that the condition

$$\iint_{-\infty}^{\infty} d\rho\, |\Gamma(z, \mathbf{R}, \rho)| = W(z, \mathbf{R}) \cdot s_c(z, \mathbf{R}) = \text{const} , \quad (9)$$

is satisfied along the diffraction ray, where $s_c(z, \mathbf{R})$ is the coherence area. For a linear regular medium this was shown in Ref. 8. For a nonlinear medium this condition will be satisfied if the induced fluctuations of the dielectric constant have a negligible effect on the field statistics, i.e., under the same assumptions as for Eq. (1). The fulfilment of condition (9) means that in the process of radiation propagation the degree of the coherence between any pair of diffraction rays remains unchanged.

Then we can write

$$\exp\left( -\frac{x^2}{4a_x^2} - \frac{y^2}{4a_y^2} + bxy \right) = \exp\left( -\frac{x_0^2 + y_0^2}{4a_c^2} \right) , \quad (10)$$

where $a_c$ is the initial coherence radius of the radiation and $\rho(z = 0\} = \rho_0 = \{x_0, y_0\}$.

Then we assume that the initial coherence radius is much smaller than the beam radius. In this case, if we determine the diffraction–ray path from Eq. (4) the paths of other rays located within the area of the ray coherence, can be determined as variations of the initial path and are described by the equation following from Eq. (4)

$$\frac{d^2\delta\mathbf{R}(z)}{dz^2} = \frac{1}{2} (\delta\mathbf{R}(z) \cdot \nabla_{\mathbf{R}})\, C_{\mathbf{R}}\, \varepsilon_{\rm eff}(z, \mathbf{R}(z)) , \quad (11)$$

where $\varepsilon_{\rm eff}$ is the effective dielectric constant introduced in analogy with Eq. (8) and $\mathbf{R}(z)$ is the solution of Eq. (4) with the initial conditions $\mathbf{R}(z = 0) = \mathbf{R}_0$ and $d\mathbf{R}(z = 0)/dz = \theta_0$. For the focused ray we have $\theta_0 = \mathbf{R}_0 / F^{-}$($F$ is the focal length).

Equation (11) is the linear vector equation of the second order. Assuming $\delta\mathbf{R}(z) = \rho(z)$ and prescribing the initial conditions $\delta\mathbf{R}(z = 0) = \rho_0$ and $d\delta\mathbf{R}(z = 0)/dz = \rho_0 / F$, we derive from Eq. (11)

$$x(z) = \nu_{1x}(z)\, x_0 + \nu_{2x}(z)\, y_0 ,$$

$$y(z) = \nu_{1y}(z)\, x_0 + \nu_{2y}(z)\, y_0 , \quad (12)$$

where $\mathbf{v}_1 = \{\nu_{1x}, \nu_{1y}\}$, $\mathbf{v}_2 = \{\nu_{2x}, \nu_{2y}\}$ is the fundamental system of solutions of Eq. (11) obtained with the following initial conditions:

$$\mathbf{v}_1(z = 0) = \{1, 0\},\ d\mathbf{v}_1(z = 0)/dz = \{F^{-1}, 0\} ;$$

$$\mathbf{v}_2(z = 0) = \{0, 1\},\ d\mathbf{v}_2(z = 0)/dz = \{0, F^{-1}\} .$$

By solving system (12) for $x_0$ and $y_0$ and substituting these values into Eq. (10), we derive

$$a_x^2 = \frac{\Delta^2}{\nu_{1y}^2 + \nu_{2y}^2}\,,\ a_y^2 = \frac{\Delta^2}{\nu_{1x}^2 + \nu_{2x}^2}\,,$$

$$b = \frac{\nu_{1x}\,\nu_{1y} + \nu_{2x}\,\nu_{2y}}{\Delta^2}\,, \qquad \Delta = \nu_{1x}\,\nu_{2y} - \nu_{2x}\,\nu_{1y}\,. \tag{13}$$

It is easy to verify that substituting the coherence function having the Gaussian distribution in the difference coordinates, whose parameters satisfy Eq. (13), into Eq. (9), we derive

$$\int\!\!\int_{-\infty}^{\infty} |\Gamma(z,\, \mathbf{R},\, \rho)|\ \mathrm{d}\rho = 4\pi a_\kappa^2\, \Delta W(z,\, \mathbf{R})\,. \tag{14}$$

Taking into account the fact that the following relations

$$\left|\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}\mathbf{R}_0}\right| = \left|\frac{\mathrm{d}\delta\mathbf{R}}{\mathrm{d}\delta\mathbf{R}_0}\right| = \begin{vmatrix} \nu_{1x} & \nu_{1y} \\ \nu_{2x} & \nu_{2y} \end{vmatrix} = \Delta$$

are valid along the diffraction rays, from Eq. (6) we obtain

$$W(z,\, \mathbf{R}(z)) = W(z = 0,\, \mathbf{R}_0)/\Delta\,, \tag{15}$$

and then the right side of Eq. (14) is equal to constant.

Thus, with an account of the above–made assumptions, Eq. (4) can be transformed into

$$\frac{\mathrm{d}^2X}{\mathrm{d}z^2} = \frac{1}{2}\frac{\partial\varepsilon_{\mathrm{eff}}}{\partial X} = \frac{1}{2}\frac{\partial\varepsilon(z,\, \mathbf{R})}{\partial X} + $$

$$+ \frac{1}{W(z,\, \mathbf{R})}\left[\frac{\partial}{\partial X}\left(\frac{\partial^2\gamma}{\partial x^2}\right)_{|\rho=0} + \frac{\partial}{\partial Y}\left(\frac{\partial^2\gamma}{\partial x\partial y}\right)_{|\rho=0}\right],$$

$$\frac{\mathrm{d}^2Y}{\mathrm{d}z^2} = \frac{1}{2}\frac{\partial\varepsilon_{\mathrm{eff}}}{\partial Y} = \frac{1}{2}\frac{\partial\varepsilon(z,\, \mathbf{R})}{\partial Y} + $$

$$+ \frac{1}{W(z,\, \mathbf{R})}\left[\frac{\partial}{\partial Y}\left(\frac{\partial^2\gamma}{\partial y^2}\right)_{|\rho=0} + \frac{\partial}{\partial X}\left(\frac{\partial^2\gamma}{\partial x\partial y}\right)_{|\rho=0}\right], \tag{16}$$

where $\mathrm{R} = \{X,\, Y\}$,

$$\left(\frac{\partial^2\gamma}{\partial x^2}\right)_{|\rho=0} = -\frac{W(z,\, \mathbf{R})}{2a_\kappa^2}\frac{\nu_{1y}^2 + \nu_{2y}^2}{\Delta^2}\,;$$

$$\left(\frac{\partial^2\gamma}{\partial y^2}\right)_{|\rho=0} = -\frac{W(z,\, \mathbf{R})}{2a_\kappa^2}\frac{\nu_{1x}^2 + \nu_{2x}^2}{\Delta^2}\,;$$

$$\left(\frac{\partial^2\gamma}{\partial x\partial y}\right)_{|\rho=0} = -\frac{W(z,\, \mathbf{R})}{2a_\kappa^2}\frac{\nu_{1x}\,\nu_{1y} + \nu_{2x}\,\nu_{2y}}{\Delta^2}\,.$$

Equation (11) is analogously transformed into

$$\frac{\mathrm{d}^2\nu_{ix}}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{\partial^2 e_{\mathrm{eff}}}{\partial X^2}\,\nu_{ix} + \frac{\partial^2\varepsilon_{\mathrm{eff}}}{\partial X\,\partial Y}\,\nu_{iy}\right);$$

$$\frac{\mathrm{d}^2\nu_{iy}}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{\partial^2\varepsilon_{\mathrm{eff}}}{\partial Y^2}\,\nu_{iy} + \frac{\partial^2\varepsilon_{\mathrm{eff}}}{\partial X\,\partial Y}\,\nu_{iy}\right); \tag{17}$$

where $i = 1,\, 2$.

The system of Eqs. (15)–(17) completed by the relation for the dielectric constant as a function of the radiation intensity or the spatial coordinates with the abovedefined initial conditions is closed. When describing the diffraction of the Gaussian ray as well as during its propagation through the medium with the parabolic profile of the dielectric constant the system has an analytic solution coinciding with the well–known solutions.[8]

We note that the system of equations is analogous to the system of equations representing the ray approximation for the radiative transfer equation[7] derived in the limiting case of geometric optics (the parameter of nonlinearity approaches to infinity).

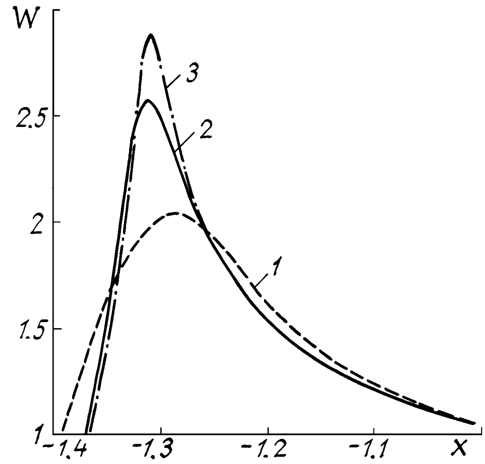


*FIG. 1. The intensity distribution in the region of the aberrational maximum.*

Figure 1 shows the calculated results for the two–dimensional (slit–shaped) beams with the Gaussian initial intensity profiles propagating under conditions of the wind–induced nonlinear refraction. The calculations were performed for the coherent and partially coherent beams with equal initial diffraction divergence (i.e., for the beams with equal Fresnel's numbers). Under conditions of diffraction the intensity profiles of the beams vary vs distance in the same way preserving the Gaussian shape. The nonlinear refraction results in the aberrational distortions of the beams and, as a consequence, in the difference of the intensity distribution. Figure 1 shows the intensity distribution in the region of aberrational maximum at the distance $z = 1.8\ L_\mathrm{r}$ ($L_\mathrm{r}$ is the refraction length) for the coherent (curve *1*) and the partially coherent (curve *2*) beams. Calculations were performed for the parameter of nonlinearity $R_\mathrm{v} = L_d^2/L_r^2 = 10^3$ ($L_d$ is the diffraction length). It can be seen that the intensity of the partially coherent radiation is higher. It is associated with the lower gradient of the diffraction term entering into $\varepsilon_{\mathrm{eff}}$ for the partially coherent radiation in comparison with the coherent one. With increase of the parameter of nonlinearity the magnitude of nonlinear terms decreases inversely proportional to $R_\mathrm{v}$. The contribution of diffraction terms becomes negligible for $R_\mathrm{v} = 10^6$, and the intensity distributions of coherent and partially coherent beams coincide (curve *3*). With increase of the distance the intensity of the abberational maximum and gradients of the intensity distribution also increase. The contribution of diffractional terms increases and for $R_v = 10^6$ the differences appear between the intensity distributions of the coherent and the partially coherent beams. The peak intensities differ nearly by a factor of 2 for $R_v = 10^3$ at the distance $z = 1.9\ L_r$.

These calculations were performed on a grid with a variable step in the plane perpendicular to the propagation axis. The nodes of the grid were the points of intersections of diffraction rays with the given plane. Since the sharp increase in the intensity is caused by crowding of the diffraction rays the automatic adaptation of the grid to the nonlinear distortions of the beam takes place for the collimated and focused beams.

The ray approximation proposed in this paper for the solution of the equation for the coherence function is analogous to the methods of solution of the radiation transfer equation described in Refs. 7 and 9, where the Gaussian shape of the brightness body being the Fourier transform of the coherence function in the difference coordinates was assumed to be unchanged. In spite of the identity of the approximations of the brightness body and coherence function by the Gaussian distributions, they yield different results. The real brightness body is substituted in Refs. 7 and 9 by the Gaussian distribution coinciding with the brightness at the point of maximum. Since the vector perpendicular to the phase front is defined as the weighted mean vector of the brightness body and for the asymmetrical brightness body (this being the case of radiation self–action) it does not coincide with the maximum of the brightness distribution, this approximation leads to the error in determining this vector. The error is also introduced in the determination of brightness at the given point in the form of the integral of the brightness body. When approximating the coherence function by the Gaussian distribution these errors are not introduced. There is one more fundamental difference which is associated with the fact that determinant in Eq. (6) is nonzero, while the corresponding determinants in Refs. 7 and 9 vanish unavoidably for the problems of self–action. For this reason, the calculations of self–action by these methods can be performed either at the distances, for which the determinants are nonzero,[7] or applying more accurate approximations, which require much computation time.[9]

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