ANALYTICAL FORMULA FOR PHASE RECONSTRUCTION FROM LIGHT FIELD INTENSITY

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A formula expressing the phase profile of a two-dimensional light field in terms of its intensity distribution for a monochromatic coherent light beam is derived based on the Radon transform of generalized functions.

The problem of wavefront reconstruction based on intensity distributions is known in optics as the phase problem.[1] Most of the published works on this problem concern iteration methods.[2] There also exist other methods for solving this problem.[3] In Ref. 4 a formula is derived for reconstructing the Wigner function, in which the intensity distribution in an infinite medium is employed. It is obvious that this approach, which employs excess information for determining the phase of the wave, is not optimal and requires that the volume of data necessary for reconstruction be reduced. In addition, information about the phase, contained in the Wigner function, is not convenient for analysis and comparison of different reconstruction algorithms. In this paper, an analytical formula for reconstructing the phase is derived based on the relation established in Ref. 4 between the projection of the Wigner function and the intensity of the wave field. The derivation is made for a field that depends on one transverse coordinate.

In the parabolic-equation approximation the distribution of the field $u(x, z)$ of a two-dimensional light beam is described by the following expression:

$$u(x,\ y) = \frac{1}{1+i}\sqrt{\frac{k}{\pi z}}\ \times$$
$$\times \int_{-\infty}^{\infty} u_0(x')\cdot\exp\left\{i\cdot\frac{k}{2z}\cdot(x-x')^2\right\} dx',\quad z>0, \tag{1}$$

where $u_0(x) = u(x,\ 0)$ is the field in the plane $z = 0$. Then the intensity of the field

$$I(x,\ z) = u(x,\ z)u^*(x,\ z) = \frac{k}{2\pi z}\ \times$$
$$\times\left|\int_{-\infty}^{\infty} u_0(x')\cdot\exp\left\{i\cdot\frac{k}{2z}\cdot(x-x')^2\right\} dx'\right|^2 \tag{2}$$

Introducing the Fourier transform of $u(x_1,\ z)u^*(x_2,\ z)$ with respect to the difference variable $\xi = x_1 - x_2$, $2x = x_1 + x_2$, we obtain the Wigner function

$$W(x,\ p) = \int_{-\infty}^{\infty} u_0(x+\xi/2)\ u_0^*(x-\xi/2)\ \times$$
$$\times\ \exp(-ip\xi)d\xi. \tag{3}$$

We now separate in the representation for the field $u_0(x)$ the real phase of the wave $S_0(x)$

$$u_0(x) = I_0^{1/2}(x)\cdot\exp\left\{iS_0(x)\right\}. \tag{4}$$

Using Eq. (4) it is easy to obtain

$$I_0(x)\frac{d}{dx}S_0(x) = \mathrm{Im}\frac{d}{d\xi}u_0(x+\xi/2)\ u_0^*(x-\xi/2)\Big|_{\xi=0}. \tag{5}$$

Substituting into Eq. (5) the representation $u_0(x+\xi/2)\ u_0^*(x-\xi/2)$, in terms of the Wigner function we obtain

$$\frac{d}{dx}S_0(x) = \frac{\int_{-\infty}^{\infty} W(x,\ p)p dp}{\int_{-\infty}^{\infty} W(x,\ p) dp}. \tag{6}$$

The relation (6) makes it possible to reconstruct the local slope of the wavefront based on the moments of the Wigner function and to reconstruct the phase itself to within a constant component. Thus

$$S_0(x) = S_0(0) + \int_0^x dx' \frac{\int_{-\infty}^{\infty} W(x',\ p)p dp}{\int_{-\infty}^{\infty} W(x',\ p) dp}. \tag{7}$$

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The nonuniqueness of the reconstruction is connected with the loss of information about the constant phase increase, owing to the multiplication of complex conjugate fields. We introduce the Radon transformation[5] of the Wigner function

$$\hat{W}(q, \underline{n}) = \iint_{-\infty}^{\infty} W(x, p)\cdot\delta(xn_1 + pn_2 - q)dxdp, \tag{8}$$

where $n = \{n_1, n_2\}$ and δ is the Dirac delta function. Substituting Eq. (3) into Eq. (8) and comparing the result of the subsequent integration with Eq. (2), we obtain a relation between the Radon projection of the Wigner function $\hat{W}(q, n)$ and the intensity

$$\hat{W}(q, n) = \frac{2\pi}{n_1} I\left[\frac{q}{n_1}, \frac{n_2}{n_1} k\right]\cdot\theta\left[k\frac{n_2}{n_1}\right], \tag{9}$$

where

$$\theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

The relation (9) can be used to reconstruct the Wigner function itself with the help of the inverse Radon transform[5]

$$W(x, p) = -\frac{1}{k\pi}\int_{-\infty}^{\infty} dx' \int_{0}^{\infty} \frac{I(x', z')}{(x - x' - pz'/k)^2} dz'. \tag{10}$$

This formula was first derived in Ref. 4. It should be noted, however, that in Ref. 4 it contains, incorrectly, an integration over the negative region of the z axis.

The inversion (10) can be directly used to reconstruct the phase (7). We shall attempt, however, to derive an inversion formula that does not contain the step in which the Wigner function is reconstructed. For this reason, we shall now turn to the mathematical apparatus of Radon transforms of generalized functions.[5]

According to Eq. (7), in order to reconstruct the phase it is necessary to calculate the zeroth and first moments of the Wigner function. For this we shall employ an analog of the Plancherel formula for the Radon transform. Thus, for two functions, one of which $f(x, y)$ is infinitely differentiable and rapidly decreasing together with all its derivatives, we have

$$\iint_{-\infty}^{\infty} f(x, y)\cdot g(x, y)dxdy = -\frac{1}{2\pi^2}\iint_{-\infty}^{\infty} dp_1 dp_2 \times$$
$$\times \int_{-1}^{1} dn_2\, \hat{f}\left[p_2, \sqrt{1-n_2^2}, n_2\right]\cdot q\left[p_1, \sqrt{1-n_2^2}, n_2\right] \times$$
$$\times \frac{1}{(p_1 - p_2)^2}\cdot\frac{dn_2}{\sqrt{1-n_2^2}}, \tag{11}$$

where $\hat{f}(p; n_1, n_2)$ and $\hat{g}(p; n_1, n_2)$ are the Radon transforms of the functions f and g. Because we are interested in $\int f(x, y)ydy$ we assume that in Eq. (11) $g(x, y) = y\delta(x_1 - x)$. Then in accordance with Ref. 5 the Radon transform of the function $g(x, y)$ is

$$\hat{g}(p, n_1, n_2) = (p - x)\cdot n_2^2\cdot \mathrm{sgn} n_2 +$$
$$+ 2\delta'(n_2)(p - x)\ln|p - x|, \tag{12}$$

The first term in Eq. (12) is an imaginary function,[5] so that we obtain

$$\int_{-\infty}^{\infty} f(x, y)ydx = -\frac{1}{\pi^2}\iint_{-\infty}^{\infty} dp_1 dp_2\cdot(p_1 - x) \times$$
$$\times \ln|p_1 - x| \frac{1}{(p_1 - p_2)^2}\frac{\partial}{\partial n}\hat{f}(p_2, 1, n)\Big| . \tag{13}$$

Correspondingly, for the zeroth moment of $f(x, y)$ we have

$$\int f(x, y)\, dx = \frac{1}{\pi^2}\iint dp_1 dp_2 \times$$
$$\times \ln|p_1 - x| \frac{1}{(p_1 - p_2)^2}\hat{f}(p_2, 1, 0). \tag{14}$$

It can be shown, with the help of the methods of the theory of complex variables, that

$$\int dp_1(p_1 - x)\cdot\ln|p_1 - x| \frac{1}{(p_1 - p_2)^2} = \frac{\pi^2}{2}\mathrm{sgn}(p_2 - x), \tag{15}$$

$$\int dp_1 \ln|p_1 - x| \frac{1}{(p_1 - p_2)^2} = \pi^2\delta(p_2 - x), \tag{16}$$

where sgn(x) is the sign function

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

Thus the integral of the function $f(x, y)$ over an unbounded region can be expressed In terms of the Radon ray-sums as follows:

$$\int_{-\infty}^{\infty} f(x, y)ydx = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{sgn}(p - x)\frac{\partial}{\partial n} f(p, 1, n)\Big|_{n=0}. \tag{17}$$

$$\int_{-\infty}^{\infty} f(x, y)dy = \hat{f}(x, 1, 0). \tag{18}$$

Correspondingly, for the Integral moments of the Wigner function we obtain

$$\int_{-\infty}^{\infty} W(x,\ p)pdp = k\pi \int_{-\infty}^{\infty} \mathrm{Sgn}(x'-x)\ \frac{\partial I(x',\ z)}{\partial z}\bigg|_{z=0}, \tag{19}$$

$$\int_{-\infty}^{\infty} W(x,\ p)\ dp = 2\pi\cdot I(x,\ 0). \tag{20}$$

In accordance with Eq. (7), using Eqs. (19) and (20) we obtain, finally.

$$S_0(x) = S_0(0) + \frac{k}{2}\int_0^x dx'\left\{\int_{-\infty}^{\infty}\mathrm{sgn}(x''-x')\times \frac{\partial I(x'',\ 0)}{\partial z}dx''\right\}/I(x',\ 0) \tag{21}$$

or

$$S_0(x) = S_0(0) + \frac{k}{2}\int_0^x dx'\left\{\int_{x'}^{\infty}\frac{\partial I(x'',\ 0)}{\partial z}dx'' - \int_{-\infty}^{x'}\frac{\partial I(x'',\ 0)}{\partial z}dx''\right\}/I_0(x'). \tag{22}$$

As an example of the application of the formulas (21) and (22) we shall examine the procedure for reconstructing the phase based on the values of the intensity of a Gaussian beam

$$u(x,\ 0) = \exp\left\{-\frac{1}{2a_t^2}x^2 - \frac{2k}{2\varphi(0)}\cdot x^2\right\}, \tag{23}$$

where a_t is the effective size of the beam and φ_0 is the curvature of the phase front on its axis. Substituting Eq. (23) into Eq. (2) and carrying out the integration for the intensity, we have

$$I(x,\ z) = \frac{a_t^2}{z}\left[1 + \frac{k^2a_t^4}{z^2}\left[1-\frac{z}{\varphi(0)}\right]^2\right]^{-1/2}\times \exp\left\{-\frac{k^2a_t^2x^2}{z^2\left[1+\frac{k^2a_t^4}{z^2}\cdot\left[1-\frac{z}{\varphi(0)}\right]^2\right]}\right\}. \tag{24}$$

Evaluating the derivative of Eq. (24) with respect to z at zero we obtain

$$\frac{\partial I(x,\ 0)}{\partial z} = \frac{1}{\varphi(0)}\exp\left[-x^2/a_t^2\right] - \frac{2}{\varphi(0)}\cdot\frac{x^2}{a_t^2}\cdot\exp\left[-x^2/a_t^2\right]. \tag{25}$$

Substituting Eq. (25) into Eq. (22) and carrying out the integration over the transverse coordinate, taking into account the fact that $I(x,\ 0) = \exp(-x^2/a_t^2)$, we find which is the same as the starting phase distribution.

So, we have derived an analytical formula for reconstructing the phase from the Intensity distribution. It does not contain the excess information present in the inversion formula for the Wigner function (10), which employs the Intensity distribution in an infinite medium. It is obvious that in the numerical implementation of Eq. (22) the reconstruction problem reduces to phase reconstruction based on intensity distributions in two close cross sections.

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