Modal isoplanatism of phase fluctuations

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The concept of isoplanatism is considered as applied to problems of adaptive correction of atmospheric distortions. Wave aberrations presented in the form of series expansion in terms of Zernike polynomials are used to calculate the angular correlation of modal components of phase fluctuations of optical radiation propagating in the turbulent atmosphere. The size of the isoplanatic area in an adaptive optical system is determined. The influence of the model of vertical profile of the structure parameter of atmospheric refractive index fluctuations, the outer scale of atmospheric turbulence, and the size of receiving aperture of a telescopic system are analyzed.

Introduction

In the general imaging theory, the strict or global isoplanatism of any imaging device consists in the invariance to a shift of the imaging operator.[1] The imaging operator for linear devices, as is well-known, can be expressed through the point spread function (PSF), which in the case of strict isoplanatism is independent of generalized coordinates on an object and an image. Since the strict global isoplanatism is very rare in practice, the device operation is considered in isoplanatic zones,[2] the field of view is divided into.

The condition of isoplanatism is that the derivative of PSF with respect to the object displacement is zero. The norm of this derivative can serve a measure of anisoplanatism. The goal of the theory of isoplanatism in optical systems is just to estimate this parameter. The point spread function of optical systems is fully determined by aberrations, as well as the shape, size, and transmittance of its pupil.[3] If the last factors are considered to remain constant accurate to the second order of smallness upon displacement of an object point, then anisoplanatism is determined by changes of aberrations. This leads to the use of the term "isoplanatic angle" in the case of ordinary optical systems for separation of the angular region in the field of view, in which aberrations of an optical system remain unchanged and, consequently, the optical transfer function (OTF) is constant.[4]

The classical theory of isoplanatism[2,4–6] is developed for the axial zone of centered optical systems with an image at a finite distance, which restrict the application of this concept in designing modern optoelectronic systems, such as atmospheric adaptive optics systems, in which aberrations are caused by the action of the atmospheric turbulence. Since such systems depend strongly on the quality of the information used, the angular anisoplanatism becomes a factor imposing serious restrictions on the field of view of adaptive optical systems and on the operation of such systems as a whole.[7]

Certainly, the concept of isoplanatism, which is already traditional, in application to adaptive optical systems calls for refinement. In the first turn, this is connected with the fact that nearly always we deal with the partial phase adaptive correction. The concept of isoplanatic angle for an atmospheric path was introduced in the early 1980s.[8] In this case, the isoplanatic angle play the role of, for example, the extreme allowed angle of the position of a reference source (several sources), or it determines the maximal angular size of an object, which can be corrected by the system.[9,10]

Isoplanatic angle

The standard definition of the atmospheric isoplanatic angle of the whole atmospheric depth[7] is written as

$$\theta_0 = \left\{2.91k^2 \int_0^\infty d\xi \xi^{5/3} C_n^2(\xi)\right\}^{-3/5}. \quad (1)$$

If we determine this parameter through the coherence radius of a plane wave[11]

$$r_0^{\text{pl}} = \left\{\frac{2.82}{6.88} k^2 \int_0^x d\xi C_n^2(\xi)\right\}^{-3/5}, \quad (2)$$

then obtain

$$\theta_0 = 0.31 r_0^{\text{pl}} / h_\xi, \quad (3)$$

where $k = 2\pi/\lambda$ is the wave number of radiation; ξ is altitude; h_ξ has a dimension of length and determines the thickness of some effective atmospheric layer:

$$h_\xi = \int_0^\infty d\xi \xi C_n^2(\xi) \Bigg/ \int_0^\infty d\xi C_n^2(\xi) = \mu_1/\mu_0, \quad (4)$$

$\mu_0 = \int_0^\infty d\xi C_n^2(\xi)$ and $\mu_1 = \int_0^\infty d\xi \xi C_n^2(\xi)$ are the zero moment and the first-order moment of the vertical profile of the structure parameter of refractive index fluctuations. Consequently, the isoplanatic angle can be defined as an angle, at which the coherence radius is seen within the effective atmospheric layer from the distance equal to the thickness of this layer.

Results of numerical calculations under conditions of the vertical propagation for different models of the vertical profile of $C_n^2(\xi)$ [Refs. 11–13] are generalized in Table ($\lambda = 0.5$ μm). Let us analyze the influence of the telescope height on these parameters (Fig. 1).

Model of the vertical profile of turbulence	$r^{\rm pl}$, m	θ_0, μrad	h_ξ, m
Model for Cerro Paranal Observatory, Chili	0.1595	7.64546	665
Model for Mauna Kea Observatory, Hawaii	0.1349	11.3797	1558
Model for AMOS Observatory, Maui, Hawaii	0.1802	17.5750	932
Hufnagel-Valley 5/7 model	0.0502	6.89989	989
Greenwood model	0.1292	13.7097	1687

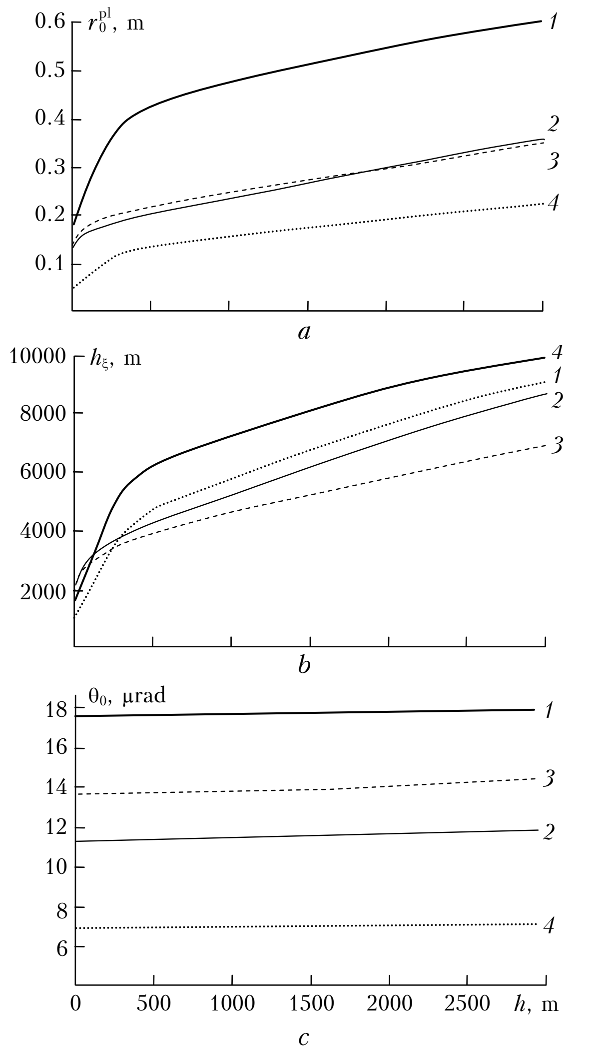

Fig. 1. Coherence radius of a plane wave (*a*), effective thickness of the turbulent atmosphere (*b*), and isoplanatic angle (*c*) as functions of the telescope height h for different models of the turbulence profile: AMOS model (*1*), Mauna Kea model (*2*), Greenwood model (*3*), and HV 5/7 model (*4*).

One can see that the increase of the telescope height leads to the increase in the coherence radius and the effective height, but does not influence the isoplanatic angle. The obtained result indicates that the traditional determination of the isoplanatic angle for adaptive optical systems does not reflect the actual pattern of isoplanatism and calls for refinement. In particular, because of the difference in geometric characteristics (path length in this case), differences in the optical beam propagation should exist. It is obvious that at the partial correction, the statistics of residual wavefront aberrations is different, because the power spectrum decreases, and the isoplanatic angle should increase. In addition, this definition ignores the aperture size and the influence of the outer scale of turbulence, which, as was shown in numerous papers, causes residual distortions in adaptive optical systems.

Angular correlation of phase fluctuations

Definition of isoplanatism in the classical theory as uniformity of aberrations of an optical system over the field leads to the necessity to describe it with the aid of the aberration function as a function of ray coordinates. In geometric optics, it is either wave aberration or eikonal.[5]

In atmospheric adaptive optical systems, this role is played by wave aberrations caused by the atmospheric turbulence. This means that the size of the isoplanatic area in such a system is determined by the existence of correlation between phase distortions of the radiation wave front in the turbulent atmosphere. That is why we calculate the angular correlation of modal components of phase fluctuations, since nearly always we deal with a partial correction, and consider the practical application of our results to the operation of adaptive optical systems.

For this purpose, we represent the function of wave aberration in the form of series expansion in terms of Zernike polynomials, which contain the information on spatial properties of phase fluctuations. Let the optical radiation from two extraterrestrial sources be incident on the telescope aperture with the diameter D at the zero angle and at the angle θ. The phase represented as series expansion in terms of orthogonal Zernike polynomials for the wave front incident at a zero angle has the form[14]

$$S(\rho) = \sum_{j=1}^{\infty} a_j Z_j\left(\frac{2\rho_1}{D}\right), \tag{5}$$

while the wave front at the angular distance θ is

$$S(\rho,\theta) = \sum_{j=1}^{\infty} a_j(\theta) Z_j\left(\frac{2\rho_2}{D}\right). \tag{6}$$

Using the property of orthogonality of polynomials at a circle, that is,

$$\iint \mathrm{d}^2\rho Z_j\left(\frac{2\rho_1}{D}\right) Z_{j'}\left(\frac{2\rho_2}{D}\right) = c_j \delta_{jj'},$$

where ρ_1 and ρ_2 characterize the position in the plane, R is the radius of the telescope aperture, and $\delta_{jj'}$ is the Kronecker delta, we can write for the expansion coefficients

$$a_j = \int_R d^2\rho W(\rho) S(\rho,\theta) Z_j\left(\frac{2\rho}{D}\right). \quad (7)$$

Then the correlation function takes the form

$$B = \langle a_j(\rho_1,0) a_j(\rho_2,\theta)\rangle =$$

$$= \int\int d^2\rho_1 d^2\rho_2 W(\rho_1) W(\rho_2) \langle S(\rho_1,\theta) S(\rho_2)\rangle Z_j\left(\frac{2\rho_1}{D}\right) Z_j\left(\frac{2\rho_2}{D}\right) =$$

$$= \left(\frac{1}{\pi R^2}\right)^2 \int_R d^2\rho_1 d^2\rho_2 \langle S(\rho_1,\theta) S(\rho_2)\rangle Z_j\left(\frac{2\rho_1}{D}\right) Z_j\left(\frac{2\rho_2}{D}\right). \quad (8)$$

Using the representation of a random process in the form of the Fourier–Stieltjes integral, taking into account the delta correlation of spectral components, and integrating over the angular coordinate,[8] we obtain the following equation for the correlation function:

$$B(\theta) = 8\pi \int_0^\infty \chi d\chi F(\chi) \frac{J_2^2(\chi)}{\chi^2}\left[J_0\left(\frac{2\theta h_\xi}{D}\chi\right) \mp J_2\left(\frac{2\theta h_\xi}{D}\chi\right)\right]. \quad (9)$$

We use the following model of the turbulent spectrum:

$$\Phi_n(\chi) = 0.033 C_n^2(\xi)\chi^{-11/3}\left(1 - \exp\left[-\chi^2/\chi_0^2\right]\right), \quad (10)$$

where $C_n^2(\xi)$ is the vertical profile of the structure parameter of the refractive index; χ is the wave number for turbulent inhomogeneities.

As a result, we obtain the following equation for the coefficient of angular correlation $b_{x,y}(\theta) = B(\theta)/B(0)$ of random wavefront tilts, which manifest themselves as an image jitter in astronomic observations[15]:

$$b_{x,y}(\theta) = \left\{\int_0^\infty d\xi C_n^2(\xi) \int_0^\infty d\chi \chi^{-11/3}\left[1 - \exp\left(-\frac{\chi^2}{\chi_0^2}\right)\right] \times \right.$$

$$\left. \times \frac{J_2^2(\chi)}{\chi}\left[J_0\left(\frac{2\theta h_\xi}{D}\chi\right) \mp J_2\left(\frac{2\theta h_\xi}{D}\chi\right)\right]\right\} \Bigg/ \left\{\int_0^\infty d\xi C_n^2(\xi) \int_0^\infty d\chi \chi^{-11/3} \times \right.$$

$$\left. \times \left[1 - \exp\left(-\frac{\chi^2}{\chi_0^2}\right)\right] \frac{J_2^2(\chi)}{\chi}\right\}, \quad (11)$$

where J_0 and J_2 are the Bessel functions. In this equation, the minus sign corresponds to the longitudinal (or parallel) separation angle θ, that is, the tilt along the axis X, while the plus sign corresponds to the tilt along the axis Y.

We use the following model of the vertical profile of $C_n^2(\xi)$:

$$C_n^2(h) = C_{n0}^2 \exp[-h/h_\xi]. \quad (12)$$

In the case of the infinite outer scale, from Eq. (11) after integration we obtain the following equations for wavefront tilts along axes X and Y:

$$b_X(\theta) = -0.1023\left(\frac{h_\xi}{D}\theta\right)^{-2} +$$

$$+0.7211\left(\frac{h_\xi}{D}\theta\right)^{-1/3} {}_pF_q\left[\left\{-\frac{5}{6},\frac{5}{2}\right\},\left\{-\frac{1}{3},3,5\right\},\frac{1}{4}\left(\frac{D}{h_\xi\theta}\right)^2\right] -$$

$$-0.0092\left(\frac{h_\xi}{D}\theta\right)^{-3} {}_pF_q\left[\left\{\frac{1}{2},\frac{23}{6}\right\},\left\{\frac{7}{3},\frac{13}{3},\frac{19}{3}\right\},\frac{1}{4}\left(\frac{D}{h_\xi\theta}\right)^2\right], \quad (13)$$

$$b_Y(\theta) = 0.1023\left(\frac{h_\xi}{D}\theta\right)^{-2} +$$

$$+1.0816\left(\frac{h_\xi}{D}\theta\right)^{-1/3} {}_pF_q\left[\left\{-\frac{5}{6},\frac{5}{2}\right\},\left\{\frac{2}{3},3,5\right\},\frac{1}{4}\left(\frac{D}{h_\xi\theta}\right)^2\right] -$$

$$-0.0363\left(\frac{h_\xi}{D}\theta\right)^{-1} {}_pF_q\left[\left\{-\frac{1}{2},\frac{17}{6}\right\},\left\{\frac{4}{3},\frac{10}{3},\frac{16}{3}\right\},\frac{1}{4}\left(\frac{D}{h_\xi\theta}\right)^2\right], \quad (14)$$

where the parameter $\gamma = \theta h_\xi/D$ plays the role of an argument; ${}_pF_q[\ldots]$ is the generalized hypergeometric function.

Influence of the outer scale of turbulence

The outer scale of the inertion interval of turbulence determines mostly the variance of jitter of the image centroid.[16] Therefore, the consideration of the turbulence outer scale is important in designing the simplest systems for adaptive optics correcting the wavefront tilt. In any case, for modern telescopes the ratio of the aperture diameter to the outer scale of the atmospheric turbulence already cannot be considered as infinitely small.

Then the following parameters represent the scale of the problem:

$$\gamma = \theta h_\xi / D \text{ and } \chi_0 = 2\pi/L_0^*,$$

where L_0^* is the effective outer scale of the turbulence introduced in Ref. 17 as

$$L_0^* = \left\{\int_0^x L_0^{5/3} d\xi C_n^2(\xi) \Big/ \int d\xi C_n^2(\xi)\right\}^{3/5}. \quad (15)$$

It can be substituted for the vertical profile of the turbulence outer scale. One of the causes for introduction of this characteristic is the possibility to simplify considerably mathematical calculations. It should be noted that numerical calculations by Eq. (15) for the considered model profiles of atmospheric turbulence yield the mean value $\chi_0 = 0.1$. The angular correlation of two positions of wavefront tilts along axes X and Y in the case of the turbulence infinite outer scale and with the finiteness of the turbulence outer scale taken into account is shown in Fig. 2.

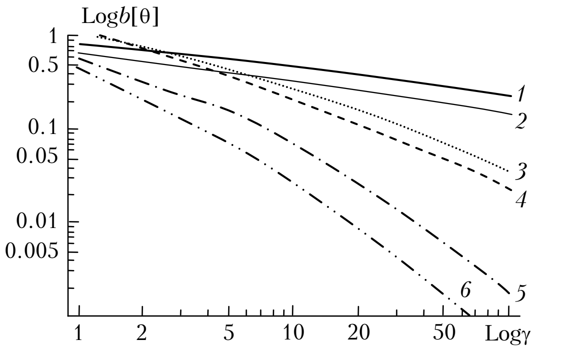

Fig. 2. Angular correlation coefficient for two positions of wavefront tilts at different values of the effective outer scale of turbulence: *Y* (*1*), *X* (*2*), $\chi = 0$; *Y* (*3*), $\chi = 0.1$; *Y* (*4*), $\chi = 0.3$; *X* (*5*), $\chi = 0.1$; *X* (*6*), $\chi = 0.3$.

The results obtained show that the turbulence outer scale strictly affects the angular correlation of lower modal components of phase fluctuations and, consequently, the size of the isoplanatic area of an adaptive optical system. It could be noted that there exist some differences between two tilt positions, and the correlation in the parallel separation angle exceeds that in the longitudinal coordinate, while the angle of the curves can be considered as identical.

The values obtained with the infinite outer scale are much higher than those in the case of the model, dependent on the outer scale. This restricts the useful area of the system's field of view. It should be emphasized that this effect influences the temporal characteristics of the system as well. As a result, it may become necessary to increase significantly (by several times) the working passband.

Correlation of higher modal components of the wave front

Let us consider the angular correlation of higher modal components of phase fluctuations, namely, coma and defocusing. In this case, the analytical equation for the correlation coefficient (11) should be rewritten as

$$b(\theta) = \frac{\int\limits_0^\infty \mathrm{d}\xi C_n^2(\xi) \int\limits_0^\infty \mathrm{d}\chi \chi^{-11/3} 1\left\{-\exp\left[-\frac{\chi^2}{\chi_0^2}\right]\right\} \frac{J_3^2(\chi)}{\chi} J_0\left(\frac{2\theta h_\xi}{D}\chi\right)}{\int\limits_0^\infty \mathrm{d}\xi C_n^2(\xi) \int\limits_0^\infty \mathrm{d}\chi \chi^{-11/3} \left\{1-\exp\left[-\frac{\chi^2}{\chi_0^2}\right]\right\} \frac{J_3^2(\chi)}{\chi}}, \quad (16)$$

$$b(\theta) = \frac{\int\limits_0^\infty \mathrm{d}\xi C_n^2(\xi) \int\limits_0^\infty \mathrm{d}\chi \chi^{-11/3} \left\{1-\exp\left[-\frac{\chi^2}{\chi_0^2}\right]\right\} \frac{J_4^2(\chi)}{\chi} J_0\left(\frac{2\theta h_\xi}{D}\chi\right)}{\int\limits_0^\infty \mathrm{d}\xi C_n^2(\xi) \int\limits_0^\infty \mathrm{d}\chi \chi^{-11/3} \left\{1-\exp\left[-\frac{\chi^2}{\chi_0^2}\right]\right\} \frac{J_4^2(\chi)}{\chi}}. \quad (17)$$

Figure 3 shows the normalized correlation functions for the first three aberrations, namely, two tilt positions, defocusing, and coma in the case of the infinite outer scale of turbulence. One can see that the behavior of the correlation coefficient strictly depends on the order of aberrations and decreases as this order increases. As expected, the lower-order polynomials are correlated much stronger than those of a higher order.

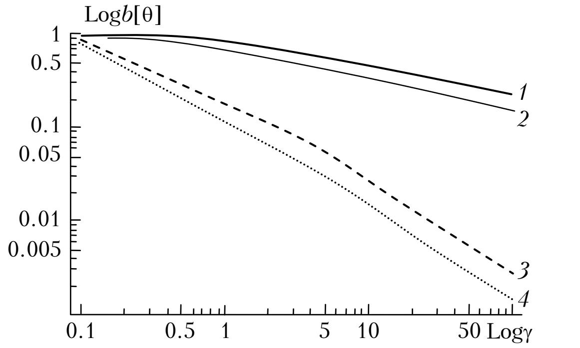

Fig. 3. Angular correlation coefficient for different modal components of phase fluctuations: two positions of wavefront tilts *Y* (*1*) and *X* (*2*), defocusing (*3*), and coma (*4*).

The angle, within which the wavefront tilts remain correlated, is greater than the angle for higher-order aberrations. This favors the increase of "sky coverage" for the operation of simplest systems of adaptive optics, which correct only tilts. In general, it can be concluded that the size of the isoplanatic area is equivalent to the order of the aberration corrected by adaptive optics. This certainly should be taken into account in the modal correction of atmospheric distortions.

Influence of the aperture size

Let us analyze the influence of the aperture size of a telescopic system. For this purpose, we represent h_ξ according to Eq. (3) and obtain the following equation for the parameter of the problem:

$$\gamma = \theta h_\xi / D = 0.31(r_0^{\mathrm{pl}}/D)(\theta/\theta_0). \quad (18)$$

Below, we consider how the dependence of the angular correlation coefficient changes for the fixed ratio D/r_0^{pl}. Then the ratio of the angular separation between the optical beams to the traditional isoplanatic angle becomes the parameter of the problem. Figure 4 illustrates this dependence for the defocusing and the tilt along the axis *X*. The calculation was performed for the ratios $D/r_0^{\mathrm{pl}} = 20$ and 50. It should be noted that in modern telescopes the aperture size tens times exceeds the coherence radius for the visible region.

The aperture size obviously influences the size of the area, in which phase fluctuations are correlated. The angular correlation increases, as the aperture size increases. The correlation is much higher for the tilt, and as the angle between the beams increases, it decreases more slowly than in the case of higher-order aberrations.

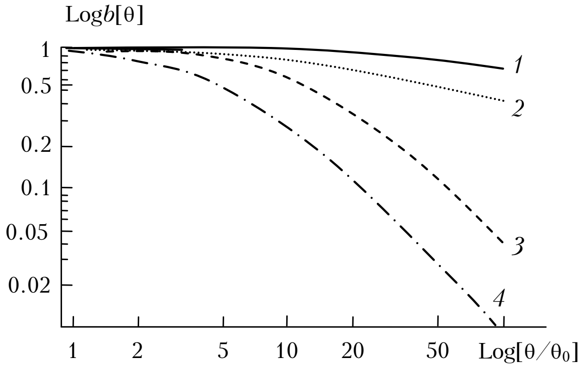

Fig. 4. Angular correlation coefficient for different modal components of phase fluctuations: tilt along the axis X $D/r = 50$ (*1*); $D/r_0^{\text{pl}} = 20$ (*2*) and defocusing $D/r_0^{\text{pl}} = 50$ (*3*); $D/r_0^{\text{pl}} = 20$ (*4*).

Conclusions

The presented attempt to refine the concept of isoplanatism in application to problems of adaptive correction of atmospheric distortions through consideration of the atmospheric turbulence as a random phase aberration in a telescopic system has shown that the size of the isoplanatic area should be better characterized by the allowable angular separation, determined by the size of the area, in which phase distortions of optical radiation, propagating in the turbulent atmosphere, are correlated, rather than the isoplanatic angle traditionally used in adaptive optics.

The results obtained from analytical and numerical studies indicate that this angular separation and, consequently, the size of the isoplanatic area depend directly on the order of phase aberrations of the wave front to be compensated by an adaptive optical system. The size of the isoplanatic area decreases, if the finite outer scale of the atmospheric turbulence is taken into account, and increases, if the aperture size increases.

In conclusion, it should be noted that this approach allows us not only to estimate the allowed angular separation between a reference source and an observed area and to select the optimal angular position of several reference sources, but also to analyze the characteristics, determining the speed of an adaptive optical system.

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