Solution of dispersion equations of elliptic ferrite-filled waveguide at longitudinal magnetization

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Dispersion equations are derived in analytical form for ordinary and extraordinary waves in an elliptic ferrite-filled waveguide at longitudinal magnetization. The obtained expressions allow one to determine the critical wavelength, plot the dependences of the propagation constant on the geometrical parameters of ferrite-filled waveguide, and define a single-mode operation of elliptic waveguides of the microwave and optical ranges. Limiting transition to isotropic filling is shown.

Plane and corrugated elliptic waveguides are widely used in modern microwave devices owing to their well-known properties (see, e.g., Refs. 1 and 2). At present, the properties of anisotropic optical waveguides are of interest, as well as devices on their basis for processing and transfer data.[3] Optical waveguides, as having imperfectly round cross-section, can be related to elliptic ones.

A key operation parameter of any waveguide is the critical frequency (critical wavelength), at which electromagnetic waves (EMW) stop to propagate. The value of critical frequency depends on a mode, waveguide geometrical sizes, and parameters of the filling medium.

This work is devoted to derivation of the dispersion equation for ordinary and extraordinary waves in a regular elliptic ferrite-filled waveguide at longitudinal magnetization. In the analysis, the losses in waveguide wall and ferrite are taken negligible.

The shape of waveguide cross-section determines the choice of coordinate system (Fig. 1).

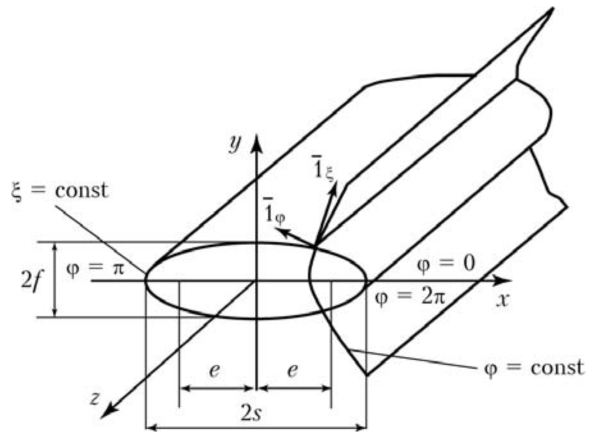

Fig. 1. Coordinate system of elliptic cylinder: e is the focal length, s is the semimajor axis; f is the semiminor axis; $\bar{1}_\xi, \bar{1}_\varphi$ are the orts.

The ferrite permeability tensor at longitudinal magnetization has the form[4]

$$\|\mu\| = \begin{bmatrix} \mu & jk & 0 \\ -jk & \mu & 0 \\ 0 & 0 & \mu_\| \end{bmatrix}, \tag{1}$$

where μ and k are the tensor components, which are functions of frequency ω, magnetization M_0, and impressed magnetic field H_0; $\mu_\|$ is independent of magnetic field strength.

Decomposing[5] the Maxwell equation to longitudinal and transversal parts relative to $\mathrm{rot}\bar{E}$ and $\mathrm{rot}\bar{H}$, two wave equations for EH and HE wave are obtained in the form

$$\begin{cases} \dfrac{\partial^2 E_z}{\partial \xi^2} + \dfrac{\partial^2 E_z}{\partial \varphi^2} + \dfrac{g_+^2 g_-^2}{a^2} e^2 (\cosh 2\xi - \cos 2\varphi) E_z = 0, \\ \dfrac{\partial^2 H_z}{\partial \xi^2} + \dfrac{\partial^2 H_z}{\partial \varphi^2} = 0; \end{cases} \tag{2}$$

$$\begin{cases} \dfrac{\partial^2 H_z}{\partial \xi^2} + \dfrac{\partial^2 H_z}{\partial \varphi^2} + \dfrac{\mu_\|}{\mu} \dfrac{g_+^2 g_-^2}{c^2} e^2 (\cosh 2\xi - \cos 2\varphi) H_z = 0, \\ \dfrac{\partial^2 E_z}{\partial \xi^2} + \dfrac{\partial^2 E_z}{\partial \varphi^2} = 0, \end{cases} \tag{3}$$

where ξ and φ are the transverse and z is the longitudinal coordinate curves of the elliptic coordinate system (see Fig. 1);

$$g_\pm^2 = \omega^2 \varepsilon (\mu \pm k) - \gamma^2; \quad a^2 = \omega^2 \varepsilon \mu - \gamma^2;$$

$$c^2 = w^2 \varepsilon \frac{\mu^2 - k^2}{\mu} - \gamma^2,$$

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$\gamma = \beta + j\alpha$ is the propagation constant; β is the phase coefficient; α is the attenuation coefficient; ω is the frequency; e is the focal length; E_z and H_z are the longitudinal components of electric and magnetic fields, respectively.

Applying the variable separation method to the first equation of system (2), obtain ordinary (relative to E_φ) and modified (relative to E_ξ) canonical Mathieu equations with real b and q:

$$\begin{cases} \dfrac{\mathrm{d}^2E_\varphi}{\mathrm{d}\varphi^2} + (b - 2q\cos2\varphi)E_\varphi = 0, \\ \dfrac{\mathrm{d}^2E_\xi}{\mathrm{d}\xi^2} - (b - 2q\,\mathrm{ch}2\xi)E_\xi = 0, \end{cases} \tag{4}$$

where b is the separation constant; $q = \dfrac{g_+^2g_-^2}{4a^2}e^2$ is the root of the modified Mathieu equation of integer order.

Using the technique for deriving the dispersion equation for isotropic waveguide[4] in the case of gyrotropic one, from Eqs. (2) and (4) obtain

$$K_\xi^2 + K_\varphi^2 = \frac{e^2d^2}{a^2}g_+^2g_-^2, \tag{5}$$

where

$$K_\xi^2 = -b + 2q\cosh2\xi,\ K_\varphi^2 = b - 2q\cos2\varphi;$$

$$a^2 = w^2\varepsilon\mu - \gamma^2;\ d^2 = \frac{1}{2}(\cosh2\xi - \cos2\varphi).$$

From Eqs. (4) and (5) derive the equation for the constant of EH-wave propagation at longitudinal magnetization:

$$\gamma_{1,2} = \sqrt{w^2\varepsilon\mu - \frac{K_\xi^2 + K_\varphi^2}{2e^2d^2} \pm \sqrt{w^4e^2k^2 + \frac{\left(K_\xi^2 + K_\varphi^2\right)^2}{4e^4d^4}}}. \tag{6}$$

In particular, for isotropic case ($k = 0$):

$$\begin{cases} \gamma_1 = w\sqrt{\varepsilon\mu}, \\ \gamma_2 = \sqrt{w^2\varepsilon\mu - \dfrac{K_\xi^2 + K_\varphi^2}{e^2d^2}} = \sqrt{w^2\varepsilon\mu - K^2}, \end{cases} \tag{7}$$

where $K^2 = \dfrac{K_\xi^2 + K_\varphi^2}{e^2d^2}$ is the transversal wave number.

The phase coefficient β is connected with the transversal wave number K^2 by the equation[4]

$$\beta^2 = w^2\varepsilon\mu - K^2, \tag{8}$$

where $w^2\varepsilon\mu = K_0^2$ is the propagation constant in unrestricted space.

The wave exists in a waveguide if β is the real number, i.e.

$$w^2\varepsilon\mu > K^2. \tag{9}$$

It follows from the first equation of system (7) that the transversal wave number $K^2 = 0$, hence, EMW does not propagate in the waveguide at $\gamma_1 = w\sqrt{\varepsilon\mu}$. Substituting the above-obtained equations for $K^2, K_\xi^2, K_\varphi^2, d^2$ in the second equation of system (7) and taking into account Eq. (8), obtain the following dispersion equation (without attenuation):

$$\gamma_2 = \beta_2 = \sqrt{w^2\varepsilon\mu - K^2} =$$

$$= \sqrt{w^2\varepsilon\mu - \frac{K_\xi^2 + K_\varphi^2}{e^2d^2}} = \sqrt{\left(\frac{2\pi}{\lambda}\right)^2 - \frac{4q}{e^2}}. \tag{10}$$

Define the critical wavelength from Eq. (10) (at $\beta = 0$):

$$\lambda_{\mathrm{cr}} = \frac{\pi e}{\sqrt{q}} = \frac{\pi sl}{\sqrt{q}}, \tag{11}$$

where s is the semimajor axis; l is the eccentricity.

Note, that Eq. (11) completely coincides with the well-known equation[1] for elliptic isotropic air-filled waveguide.

Apply the variable separation method both for extraordinary HE and ordinary EH waves. Again, obtain the ordinary and modified Mathieu equations from the first equation of system (3):

$$\begin{cases} \dfrac{\mathrm{d}^2H_\varphi}{\mathrm{d}\varphi^2} + (b - 2q\cos2\varphi)H_\varphi = 0, \\ \dfrac{\mathrm{d}^2H_\xi}{\mathrm{d}\xi^2} - (b - 2q\cosh2\xi)H_\xi = 0, \end{cases} \tag{12}$$

where $q = \dfrac{\mu_\parallel}{\mu}\dfrac{g_+^2g_-^2}{4c^2}e^2$.

As in the previous case, obtain the HE-wave propagation constant at longitudinal magnetization from Eqs. (3) and (12) using the variable separation method:

$$\gamma_{1,2} = \sqrt{w^2\varepsilon\mu - \frac{\mu\left(K_\xi^2 + K_\varphi^2\right)}{2\mu_\parallel e^2d^2} \pm \sqrt{w^4\varepsilon^2k^2 + \frac{\mu^2\left(K_\xi^2 + K_\varphi^2\right)^2}{4\mu_\parallel^2e^4d^4} - \frac{w^2\varepsilon k^2}{\mu_\parallel e^2d^2}\left(K_\xi^2 + K_\varphi^2\right)}}. \tag{13}$$

In particular, for isotropic case ($k = 0$, $\mu = \mu_{\|}$)

$$\begin{cases} \gamma_1 = w\sqrt{\varepsilon\mu}, \\ \gamma_2 = \sqrt{w^2\varepsilon\mu - \dfrac{K_\xi^2 + K_\varphi^2}{e^2 d^2}} = \sqrt{w^2\varepsilon\mu - K^2}, \end{cases} \quad (14)$$

which coincides with Eq. (7) and, hence, Eq. (11).

It follows from Eqs. (6) and (13) that the phase velocities of *EH* and *HE* waves differ and depend on the external magnetic field strength, ferrite magnetization, and frequency, as the components of the ferrite permeability tensor depend on these parameters. The critical wavelength and dependence of the propagation constant γ on the waveguide geometric parameters are defined from these equations, as well as single-mode operation of elliptic waveguides of the microwave and optical ranges. It is also important that the obtained equations are true for isotropic case as well.

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