THE INVERSE PROBLEM METHOD IN THE POLARIZATION SOUNDING OF DISPERSED MEDIA

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The theory of polarization sounding of dispersed scattering media as it applies to optical monitoring of aerosol atmospheric pollutants is presented. The proposed methods of interpretation are based on integral operators of the mutual transformation of the elements of the Mueller matrix for a polydispersed system of spherical particles. Operator equations for determining the refractive index of the material of the particles under study are constructed. The information possibilities of the operator approach are illustrated in the paper for the example of the solution of a complex atmospheric-optical problem concerning the separation of molecular and aerosol scattering matrices based on polarization measurements.

Optical sounding of dispersed scattering media with polarized radiation is based on the analysis of the scattering matrix $\hat{D} = \{D_{ij}\}$, which transforms the Stokes vector of the scattered wave $\vec{I}^{(0)}$ into the vector of the incident wave $I^{(s)}$. The collection of elements {D^} contains the physical information about the medium under study; this information can in principle be extracted from polarization measurements. The exact theory of polarization sounding is based on the principle of combined interpretation implemented with the help of special mutual transformation operators, of the entire aggregate of elements of the matrix $\hat{D}$. This theory is presented in a monograph written by the author.[1]

This paper is devoted to further development and extension of the theory of polarization sounding of dispersed media and, in particular, the extension of the operator approach not only to combined inversion of the elements of the Muller matrix, but also to their interpretation in an experiment with a deficiency of measuring information. It should be noted that in atmospheric-optical studies the method of polarization sounding is implemented technically with the help of nephelometers based on the ground of airborne and bistatic lidars. These optical system^ can play an important role in the organization of optical monitoring of atmospheric pollutants. The theory of interpretation presented below enables the development of practical methods for studying real aerosol systems of both natural and human origin using the indicated devices.

THE LIGHT-SCATTERING MATRIX OF A POLYDISPERSED SYSTEM OF PARTICLES AND THE MATRIX OF OPERATORS OF THE MUTUAL TRANSFORMATION OF ITS ELEMENTS

The fact that the elements of the scattering matrix can be regarded as a collection of mutually coupled functions of the scattering angle ϑ can be clearly illustrated for the example of a polydispersed system of spherical particles. Indeed, in the latter case any element of the corresponding scattering matrix as a function of the angle ϑ and the wavelength λ can be represented by a one-dimensional parametric integral of the form

$$D_{ij}(\vartheta,\lambda) = \int_{R_1}^{R_2} Q_{ij}(\bar{m}, r, \vartheta, \lambda)\pi r^2 n(r)dr. \quad (1)$$

The factors $Q_{ij}(\bar{m}, r, \vartheta, \lambda)$ are calculated using the formulas of Mie's theory with a fixed refractive index m of the constituent material of the particles. In the integral representation (1) the function $n(r)$ characterizes the particle size spectrum in the interval $R = [R_1, R_2]$. It is not difficult to see that the variables ϑ and λ play the role of parameters; this explains the use of the term "parametric integral". In what follows we shall write intergral of the type (1) in a more compact form, introducing the integral operators Q. Then expression (1) will be equivalent to the notation $D(\vartheta) = (Qs)(\vartheta)$ where $s(r)$ denotes the function $\pi r^2 n(r)$. The introduction of the operators Q is also justified from the computational viewpoint because they can be very simply replaced by matrix operators $\hat{Q}$ when discrete measurement are processed. It should be stressed that the assumption that the scattering particles are spherical determines not only the structure of the matrix $\{D_{ij}\}$ but it also gives a method for calculating its elements numerically.

We shall now examine together the pair of elements $D_{11}(\vartheta)$ and $D_{11}(\vartheta)$ of the Mueller matrix for a system of spherical scattering particles. Each of these functions is represented by the indicated integral with a corresponding kernel. In the first case it is given by the expression $[i_1(x, \vartheta) + i_2(x, \vartheta)]/2x^2$, where i_1 and i_2 are functions of the dimensionless intensity and $x = 2\pi r/\lambda$, while in the second case it is given by the expression $[i_1(x, \vartheta) - i_2(x, \vartheta)]/2x^2$. From the analytical viewpoint the optical characteristics $D_{11}(\vartheta)$ and $D_{12}(\vartheta)$ cannot be regarded as fully independent, at least because they have, as parametric integrals of the form (1), a common weighting function $s(r)$. This last fact can be constructively employed to construct a functional (operator) equation relating the indicated characteristics. Indeed, assuming, for example, that the function $D_{11}(\vartheta)$ is known and starting from its representation $D_{11}(\vartheta) = (Q_1 s)(\vartheta)$ it is possible to construct the inverse regularizing operator $Q_{1\alpha}^{-1}$ with the help of completely standard computational procedures. This operator gives a well-grounded estimate $s_\alpha(r) = (Q_{1\alpha}^{-1} D_1)(r)$ of the weighting function $s(r)$. The function $s_\alpha(r)$ approaches the exact distribution $s_0(r)$ as $\alpha \to 0$, if the error σ in $D_{11}(\vartheta)$ also approaches zero. With the help of this approach it is possible to construct the function

$$D_{12}^{(\alpha)}(\vartheta) = (Q_2 Q_{1\alpha}^{-1} D_{11})(\vartheta) = (W_{21}^{(\alpha)} D_{11})(\vartheta), \tag{2}$$

which gives an estimate of the element $D_{12}(\vartheta)$. Thus, relying on methods for inverting parametric integrals, we constructed an operator $W_{21}^{(\alpha)}$ which relates the two elements D_{11} and D_{12} of the scattering matrix. By studying in pairs the remaining elements of Mueller's matrix for a system of spherical particles a corresponding matrix of operators $\{W_{ij}^{(\alpha)}\}$ can be constructed. The methods for constructing such operators, including operators arising in the theory of multifrequency optical sounding of dispersed media, Eire presented in Refs. 1, 2, and 3. We note only that the elements and obviously have a greater analytic similarity than that which formed the basis of the construction of the operator $W_{21}^{(\alpha)}$. To prove this, it is sufficient to study the analytical form of the kernels Q_{11} and Q_{12}, and their dependence on the functions i_1 and i_2. Unfortunately, it is not yet possible to give a simple method for applying this feature in practice in the problems of interpretations, if one starts from the computational formulas given in Mie's theory.[4]

OPERATOR EQUATIONS FOR DETERMINING THE ELEMENTS OF THE SCATTERING MATRIX FROM POLARIZATION MEASUREMENTS

In optical experiments on the scattering of polarized light by real dispersed media it is not the components of the Stokes vector themselves that are measured but rather some quantities $P_j^{(s)}$ ($j = 1, 2, 3, 4$). The relation between the vectors $\mathbf{I}^{(s)}$ and $\mathbf{P}^{(s)}$ in the single-scattering approximation is quite simple:

$$\vec{P}^{(s)} = B\vec{I}^{(s)}, \tag{3}$$

where B is some function of the distance z from the receiver to the scattering volume (for a bistatic polarization lidar), the angle ϑ, and the parameters of the receiving measuring channel. The collection $\{P_j^{(s)}\}$ (the same as the vector $\mathbf{P}^{(s)}$) can be denoted by a single term "optical signal". We shall now study the functional equations which relate the components of the vectors $\mathbf{I}^{(s)}$ and $\mathbf{I}^{(0)}$ with the elements of the scattering matrix. The structure of the matrix $\hat{D}$, which transforms the vector $\mathbf{I}^{(0)}$ into $\mathbf{I}^{(s)}$, for polydispersed systems of spherical particles permits writing out separately two equations for $I_2^{(s)}$and $I_1^{(0)}$ and two equations for the next pair of components $I_3^{(s)}$and $I_4^{(s)}$. The first pair has the form:

$$\left.\begin{aligned} I_1^{(s)} &= D_{11} I_1^{(0)} + D_{12} I_2^{(0)}, \\ I_2^{(s)} &= D_{12} I_1^{(0)} + D_{11} I_2^{(0)}, \end{aligned}\right\} \tag{4}$$

and the second pair has the form

$$\left.\begin{aligned} I_3^{(s)} &= D_{33} I_3^{(0)} + D_{34} I_4^{(0)}, \\ I_4^{(s)} &= D_{34} I_3^{(0)} + D_{33} I_4^{(0)}, \end{aligned}\right\}. \tag{5}$$

If the polarization of the sounding radiation is chosen so that $I_2^{(0)} = 0$, then the first element of the scattering matrix D_{11} usually called the coefficient of directed light scattering is expressed very simply in terms of $I_1^{(0)}$and $I_1^{(s)}$, namely, $I_1^{(s)} = D_{11} I_1^{(0)}$. Introducing the ratio $a_1^{(s)} = I_1^{(s)} / I_1^{(0)}$ into this equality we find

$$D_{11} = a_1^{(s)}. \tag{6}$$

Since the function $a_1^{(s)}(\vartheta)$, where ϑ is the scattering angle, is determined directly in an experiment we can talk about direct measurement of the optical characteristic $D_{11}(\vartheta)$. If $a_1^{(s)}(\vartheta)$ is known with an error not exceeding 5–10% in the range of angles $(0, \pi)$, then inversion of the characteristic $D_{11}(\vartheta)$ gives a completely consistent estimate of the particle-size spectrum.[3] The corresponding inverse problem requires that the refractive index of the aerosol material be known. At the same time, if the optical constants of the material $\overline{m}'$ and $\overline{m}''$ are known *a priori* for the system of spherical particles, then the matrix of operators $\{W_{ij}\}$, which was discussed above, is also determined in equal measure. To simplify the writing of the transfer operators, we shall drop below the upper index "α".

If the operator W_{21} is introduced into the first equation of the system (4) by replacing D_{12} by $W_{21}D_{11}$, then we obtain a new functional (operator) equation for D_{11}

$$c_2^{(0)} W_{21} D_{11} + D_{11} = a_1^{(s)}, \qquad (7)$$

where $c_2^{(0)} = I_2^{(0)} / I_1^{(0)}$. Since operators of the type W are integral operators, Eq. (7) is an integral equation of the second kind and therefore the problem of solving it is a properly posed mathematical problem. Equation (7), unlike Eq. (6), permits finding the element D_{11} in the case of arbitrary polarization of the incident radiation. Of course, this requires completely determined a priori information about the dispersed medium under study, since the computational formulas of Mie's theory must be employed to calculate the matrix analog $\hat{W}_{21}$ for the operator W. Thus Eq. (7) permits finding the function $D_{11}(\vartheta)$ from measurements of only one component $I_1^{(s)}$ with an arbitrary vector $I^{(0)}$. Since in the case of arbitrary polarization the first equation of the system (4) is not determined in and of itself because it contains two unknown functions $D_{11}(\vartheta)$ and $D_{12}(\vartheta)$, it can be asserted that introducing the operator W_{21} is essentially equivalent to reducing the volume of measuring information. In the approach presented the elements $D_{11}(\vartheta)$ and $D_{12}(\vartheta)$ are determined from $\mathbf{I}_1^{(s)}$ using the information contained in Mie's theory, i.e., in the theory describing the scattering of an optical wave by a spherical particle of size r with a fixed refractive index. The example studied clearly illustrates the possibilities of the operators W_{ij} and their role in the described theory of polarization sounding of dispersed media. They permit further determining the equations which in and of ‘ themselves are not determined owing to the lack of the corresponding measurements.

Let us return to Eq. (7). If the numerical solution of this equation is constructed based on the method of successive approximations, i.e., in accordance with an iteration scheme of the form

$$c_{2j}^{(0)} \sum_{1=1}^{n} \omega_{1j} D_1^{(p-1)} + D_j^{(p)} = a_{1j}^{(s)}, \quad j = 1, \ldots, n, \qquad (8)$$

where p is the number of the approximation; $c_{2j}^{(0)} = c_2^{(0)}(\vartheta_j)$; $D_1 = D_{11}(\vartheta_1)$ and $\{w_{ij}\} = \hat{W}_{21}$, then, as is well known, the condition

$$\| c_2^{(0)} \hat{W}_{21} \| < 1. \qquad (9)$$

must be satisfied.

Since the equality

$$I_1^{(0)2} = I_2^{(0)} + I_3^{(0)2} + I_4^{(0)2} \qquad (10)$$

holds for the incident polarized radiation, $c_2^{(0)}(\vartheta) \le 1$ for all angles ϑ in the range $(0, \pi)$. In this connection it remains to show that

$$\| \hat{W}_{21} \| < 1. \qquad (11)$$

The operators satisfying this condition are customarily called compression operators.

If one starts from the equality (10), the condition

$$I_1^{2(s)} \ge I_2^{(s)2} + I_3^{(s)2} + I_4^{(s)2} \qquad (12)$$

and the expressions for D_{ij} obtained from Eqs. (4) and (5), then it is not difficult to show that the following inequality holds:

$$D_{11}^2 \ge D_{12}^2 + D_{33}^2 + D_{34}^2. \qquad (13)$$

The relation (13) is sufficient to prove (11) rigorously. Similar inequalities for the operators W_{ij} play a very important role in the construction of iteration schemes for processing optical data.

In finishing our analysis of the questions pertaining to the determination of the coefficient of directed light scattering $D_{11}(\vartheta)$ for an aerosol system of particles based on data from polarization sounding, we must call attention to one other important application of the operators W_{ij}. In accordance with Eq. (6) the function $D_{11}(\vartheta)$ can also be found in the case of sounding of the dispersed medium under study with unpolarized light for which $\vec{I}^{(0)} = \{\vec{I}_1^{(0)}, 0, 0, 0\}$. If in so doing the operators W_{ij}, can be calculated in a well-sounded manner, then it is also possible to determine all of the remaining elements of D_{ij}. As a result it is possible to solve a very important optical problem, namely, to predict the "response" of an aerosol system to an incident polarized wave from data obtained by sounding the system with unpolarized light. The possibility of solving such prediction problems is another advantage of the method of optical operators or, more precisely, the method of the inverse problem of the theory of light scattering by polydispersed systems. The enumerated possibilities are realized with the help of corresponding program complexes for optical data processing and interpretation[6]. It should also be noted that the . information possibilities of the optical operators W_{ij} in the analysis and interpretation of the light-scattering data can be fully determined in the solution of more complicated optical problems than the ones discussed above. The point is that the structure of the matrix $\{D_{ij}\}$ for a system of spherical particles is very simple, and no special difficulties arise in the solution of the starting system of functional equations from the theory of polarization sounding. The other problem arises when structurally more complicated scattering matrices are encountered in atmospheric-optics studies. An example of this is a polydispersed system of nonspherical particles oriented randomly in the illuminated volume. Such a- system of particles corresponds better to real aerosol systems than an ensemble of spherical particles. For a system of nonspherical particles oriented randomly in the scattered volume, the system of four equations associated with the transformation $\mathbf{I}^{(s)} = \hat{D}\mathbf{I}^{(0)}$

contains six unknown functions $D_{ij}(\vartheta)$. It is obvious that it is not determined and it is necessary to increase, in an appropriate manner, the volume of experimental measurements by means of sounding with radiation with different polarization. Because of a technical difficulty such experiments can hardly be performed under the conditions of the real atmosphere. Currently existing data, as a rule, were obtained under laboratory conditions.[7]

An alternative solution of this problem can be obtained on the optical operators W_{ij}, supplementing the system $\mathbf{I}^{(s)} = \hat{D}\mathbf{I}^{(0)}$ with operator equations relating the unknown functions D_{ij}. In using computational methods significant, but fully surmountable, difficulties, associated with the calculation of the light-scattering characteristics of nonspherical particles, arise.[7] The possibility of determining the matrices from the corresponding experimental data, obtained by means of polarization sounding on model aerosol systems under laboratory conditions, also should not be overlooked.[2,3]

DETERMINATION OF THE REFRACTIVE INDEX OF THE CONSTITUENT MATERIAL OF THE SCATTERING PARTICLES FROM POLARIZATION MEASUREMENTS

If the first problem of the theory of polarization sounding of dispersed media is to determine the elements of the scattering matrix $\{D_{ij}\}$, then the problem of determining the real and imaginary parts of the complex, refractive index of the constituent material of the particles must be regarded as the second problem. In constructing the corresponding computational procedures we shall assume that the coefficient of directed light scattering $D_{11}(\vartheta)$ has already been determined previously by one of the methods studied above. To find the refractive index from the components of the vector $\vec{I}^{(s)}$ it is now no longer necessary to know their absolute values or the ratios $a_1^{(s)} = I_1^{(s)} / I_1^{(0)}$. We can thus transfer to the polarization vector $\vec{c}^{(s)} = \{1, c_2^{(s)}, c_3^{(s)}, c_4^{(s)}\}$, introducing its components into the corresponding computational formulas. We rewrite the system (1) using the operator W_{21}:

$$\left.\begin{aligned} I_1^{(s)} &= I_1^{(0)} D_{11} + I_2^{(0)} W_{21} D_{11}, \\ I_2^{(s)} &= I_1^{(0)} W_{21} D_{11} + I_2^{(0)} D_{11}. \end{aligned}\right\} \quad (14)$$

Introducing the ratio $c_2^{(s)} = I_2^{(s)} / I_1^{(s)} = P_2^{(s)} / P_1^{(s)}$, which does not depend on the instrumental function $B(z, \vartheta)$ whose determination for a bistatic polarization lidar is a very difficult problem the system (14) can be reduced to the following equation:

$$I_1^{(0)} W_{21} D_{11} + I_2^{(0)} D_{11} = c_2^{(s)} (I_1^{(0)} D_{11} + I_1^{(0)} W_{21} D_{11}).$$

This equation can be greatly simplified by denoting the ratio $I_2^{(0)} / I_1^{(0)}$ by $c_2^{(0)}$ and introducing the quantity $g = (c_2^{(0)} - 1)(c_2^{(0)} - c_2^{(s)})$ which, of course, is a function of the scattering angle ϑ. Based on these remarks we obtain finally

$$g W_{21} D_{11} = D_{11}. \quad (15)$$

In this equation the operator is unknown, since the refractive index $\bar{m} = \bar{m}' - i\bar{m}''$ is unknown. It must be especially stressed that the operator W_{21} does not depend on the particle-size distribution function. For this reason if the function $D_{11}(\vartheta)$ has been found in the experiment, then the determination of $\bar{m}'$ and $\bar{m}''$ from Eq. (15) does not require a priori knowledge of the particle-size spectrum. In this respect the method described here for determining $\bar{m}'$ and $\bar{m}''$ which is based on the parametric dependence of the operator $W_{21}(\bar{m}', \bar{m}'')$ differs significantly, with regard to its mathematical rigor and effectiveness, from numerous methods based on the parametric dependence of the optical characteristics on the microstructure and refractive index. A typical example is Ref. 8.

It now remains to employ the pair of equations (5) for D_{33} and D_{34}. Introducing by analogy $c_3^{(s)}$ and $c_4^{(s)}$ we obtain

$$(-c_2^{(0)} c_3^{(s)} W_{21} + c_3^{(0)} W_{31} + c_4^{(0)} W_{41}) D_{11} = c_3^{(s)} D_{11}; \quad (16)$$

$$(-c_2^{(0)} c_4^{(s)} W_{21} + c_4^{(0)} W_{31} - c_3^{(0)} W_{41}) D_{11} = c_4^{(s)} D_{11}. \quad (17)$$

The expressions (15)—(17) form a complete system of operator equations of the theory of polarization sounding which depend on the index $\bar{m}$. In the general case, as already shown above, to determine D_{11} it is necessary to known the operator W_{21}, i.e., the parameters $\bar{m}'$ and $\bar{m}''$, so that the indicated system of three equations, strictly speaking, is not overdetermined. It should also be kept in mind that by virtue of the inequalities (13) the operators W_{ij} in the last three equations cannot be regarded as being completely independent. For this reason, to determine the two quantities $\bar{m}'$ and $\bar{m}''$ we actually have the two operator equations (16) and (17). Since the computational algorithm is usually constructed by the method of least squares it is preferable to employ all three equations together, combining them by the overall optical discrepancy. We note that the starting system of three equations assumes its simplest form in the case when the incident light is linearly polarized when $\vec{c}^{(0)} = \{1, 0, 1, 0\}$, namely,

$$\left.\begin{aligned} W_{21} D_{11} &= c_2^{(s)} D_{11}, \\ W_{43} D_{11} &= c_3^{(s)} D_{11}, \\ W_{41} D_{11} &= -c_4^{(s)} D_{11}. \end{aligned}\right\} \quad (18)$$

The optical discrepancies for these equations can be written in the form

$$\rho_1(\bar{m}', \bar{m}'') = \| W_{11}(\bar{m}', \bar{m}'') D_{11} - c^{(s)}_{\sigma 1} D_{11} \|_{L_2(\Theta)},$$

$$i = 2,\ 3,\ 4. \qquad (19)$$

The vector $\vec{C}^{(S)}_{\sigma}$ is the σ approximation for the starting (exact) vector $\vec{C}^{(S)}_{0}$. The indices $\bar{m}'$ and $\bar{m}''$ are determined by minimizing simultaneously the optical discrepancies. In those cases when the operator W_{21} must be employed to determine D_{11} the general computational scheme is constructed by the method of successive approximations.

In conclusion it should be noted that any pair of equations in the system (18) formally determines not only the constants $\bar{m}'$ and $\bar{m}''$, but also some functions $\bar{m}'(r)$ and $\bar{m}''(r)$. Indeed, any equation in the system (18) can be put into the form

$$\int_R Q_{11}[\bar{m}'(r),\ \bar{m}''(r),\ r,\ \vartheta]dr = f_1(\vartheta),\ i=2,\ 3,\ 4.$$

For this reason the system (19) must actually be replaced by a system of nonlinear integral equations of the indicated type (Urison's equations[9]). The physical problems for which the dependence of $\bar{m}'$ and $\bar{m}''$ on the particle size is important arise in the study of aerosol systems, interacting, for example with the moisture field, by optical methods.[1]

SEPARATION OF THE MOLECULAR AND AEROSOL SCATTERING COMPONENTS BY THE METHODS OF POLARIZATION SOUNDING

Under the conditions of the real atmosphere light scattering is made up of two factors, namely, scattering by aerosols and air molecules. For this reason, before solving the inverse problems and drawing any conclusions about the physical parameters of the atmosphere it is first necessary to determine the contribution made by each of the indicated components in the optical signals to the scattering. This problem is especially important in the studying of the upper and middle atmosphere by optical methods. Within the framework of the theory of polarization sounding, which was presented above, it is not difficult to construct general functional equations for determining together the optical characteristics of two indicated components. Indeed, since now the overall light-scattering matrix $\hat{D}$, which transforms the vector $\mathbf{I}^{(0)}$ into $\mathbf{I}^{(s)}$, is equal to the sum of two matrices, namely, the aerosol scattering matrix $\hat{D}^{(a)}$ and the molecular scattering matrix $\hat{D}^{(M)}$, by analogy to (4) we have

$$\left.\begin{aligned} I^{(s)}_1 &= I^{(0)}_1 D^{(a)}_{11} + I^{(0)}_2 D^{(a)}_{12} + \beta^{(M)}_{sc}(I^{(0)}_1 M_{11} + I^{(0)}_2 M_{12}), \\ I^{(s)}_2 &= I^{(0)}_1 D^{(a)}_{12} + I^{(0)}_2 D^{(a)}_{11} + \beta^{(M)}_{sc}(I^{(0)}_1 M_{12} + I^{(0)}_2 M_{22}), \end{aligned}\right\}. \qquad (20)$$

The molecular scattering matrix $\hat{D}^{(M)}$ in (20) is represented in the form of the product of the volume coefficient $\beta^{(M)}_{sc}$ and the normalized matrix $\{M_{ij}\}$ .The structure of the latter matrix is close to that of the matrix $\{D^{(a)}_{ij}\}$, the difference being that in the matrix M_{ij} $M_{34} = M_{43} = 0$.

In the system (20) the functions $D^{(a)}_{11}(\vartheta)$ and $D^{(a)}_{12}(\vartheta)$, and the quantity $\beta^{(M)}_{sc}$ are unknown. The values of M_{ij} can be expressed simply in terms of trigonometric functions of the angle ϑ. It is obvious that the system (20) is not determined and it cannot be made determined by choosing the polarization vector $\vec{C}^{(0)}$. Introducing the optical operator W_{21} is the only method for completely determining the system (20) and properly formulating the inverse problem. Indeed, in the latter case the system (20) reduces to the system:

$$c^{(0)}_2 W_{21} D^{(a)}_{11} + D^{(a)}_{11} + \beta^{(M)}_{sc}(M_{11} + c^{(0)}_2 M_{12}) = a^{(s)}_1, \qquad (21)$$
$$W_{21} D^{(a)}_{11} + c^{(0)}_2 D^{(a)}_{11} + \beta^{(M)}_{sc}(M_{12} + c^{(0)}_2 M_{22}) = a^{(s)}_2.$$

This system assumes an especially simple form if we set $\vec{c}^{(0)} = \{1,\ 0,\ 1,\ 0\}$:

$$\left.\begin{aligned} D^{(a)}_{11} + \beta^{(M)}_{sc} M_{11} &= a^{(s)}_1, \\ W_{21} D^{(a)}_{11} + \beta^{(M)}_{sc} M_{12} &= a^{(s)}_2. \end{aligned}\right\} \qquad (22)$$

From the system (22) we are required to find the function $D^{(a)}_{11}(\vartheta)$ and the constant $\beta^{(M)}_{sc}$. At the same time it is obvious that formally the system of two functional equations must determine two unknown functions. For this reason it can be asserted that the system (22) contains more information than is required in the problem at hand in inverting $a^{(s)}_1$ and $a^{(s)}_2$.

Indeed, separating the unknowns in the system (22), we arrive at the two equations

$$W_{21} D^{(a)}_{11} + q D^{(a)}_{11} = f, \qquad (23)$$

where

$$q = -M_{12}/M_{11} \text{ and } f = (M_{11} a^{(s)}_2 - M_{12} a^{(s)}_1,$$

and

$$a^{(s)}_2 - W_{21} a^{(s)}_1 = \beta^{(M)}_{sc}(M_{12} - W_{21} M_{11}). \qquad (24)$$

Equation (23), like also Eq. (7), is an integral equation of the second kind, and there is no need to discuss it in detail. From the viewpoint of information content Eq. (24) is more interesting. It is obvious that it determines some constant $\beta^{(M)}_{sc}$ in the case when the ratio $(a^{(s)}_2 - W_{21} a^{(s)}_1) / M_2 - W_{21} M_{11})$ does not depend on the scattering angle ϑ. To satisfy this condition the experimental data $a_1^{(s)}$ and $a_2^{(s)}$ must be mutually consistent and the a priori employed information presented here by the matrices $\{D_{ij}^{(a)}\}$ (also by the operators W_{ij}) and $\{M_{ij}\}$ must be employed. Thus in the process of interpretation there arises the possibility of determining how well the a priori information employed corresponds to the real situation in the experiment.

The two remaining equations, analogous to the system (20) and referring to the components $I_3^{(s)}$ and $I_4^{(s)}$, can be employed to determine (or refine) the starting approximations $\bar{m}'$ and $\bar{m}''$. Introducing the components of the polarization vector $\vec{C}^{(s)}$, they can be written in the form

$$(c_3^{(s)}I+c_2^{(0)}c_3^{(s)}W_{21}-c_3^{(0)}W_{31}-c_4^{(0)}W_{41})D_{11}^{(a)} = $$
$$=\beta_{sc}^{(M)}(-c_3^{(s)}M_{11}-c_2^{(0)}c_3^{(s)}M_{12}+c_3^{(0)}M_{33}); \quad (25)$$
$$(c_4^{(s)}I+c_2^{(0)}c_4^{(s)}W_{21}-c_4^{(0)}W_{31}+c_3^{(0)}W_{41})D_{11}^{(a)} = $$
$$=\beta_{sc}^{(M)}(-c_4^{(s)}M_{11}-c_2^{(0)}c_4^{(s)}M_{12}+c_4^{(0)}M_{33});$$

The equations of the system (29) simplify appreciably, if the vector $\vec{c}^{(0)} = \{1, 0, 1, 0\}$ is introduced, as done repeatedly above, and then all three operator equations, depending on the optical constants $\bar{m}'$ and $\bar{m}''$, assume the form

$$\left.\begin{aligned}(c_2^{(s)}I-W_{21})D_{11}^{(a)}&=\beta_{sc}^{(M)}(-c_2^{(s)}M_{11}+M_{12}),\\(c_3^{(s)}I-W_{31})D_{11}^{(a)}&=\beta_{sc}^{(M)}(-c_3^{(s)}M_{11}+M_{33}),\\(c_4^{(s)}I+W_{41})D_{11}^{(a)}&=\beta_{sc}^{(M)}(-c_4^{(s)}M_{11}+M_{33}).\end{aligned}\right\} \quad (26)$$

For $\beta_{sc}^{(M)} \ll \beta_{sc}^{(a)}$, i.e., in the case when the contribution of molecular scattering can be neglected the system (26) transforms into the system (18). The operator equations presented above solve completely the problem of separating the aerosol and molecular scattering components based on data from polarization sounding.

In conclusion it should be noted that in the practice of atmospheric-optical studies the components of scattering are usually separated by a. simpler method, namely, by first evaluating data from the temperature and pressure profiles. Of course, this requires appropriate measurements which, incidentally, cannot always give the required spatial resolution. The problem is that the theory studied aboved concerned local volumes of the atmosphere; it was assumed that the corresponding optical information is obtained with the help of polarization nephelometers (airborne optical laboratories[10]) or bistatic lidars.[11] The indicated optical sounding systems make it possible to obtain large volumes of information with high spatial resolution. On the basis of the theory presented above we solved the problem of separating the scattering components by a purely optical method without the help of meteorological measurements and especially without the help of standard models of the molecular atmosphere.

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