OPTIMAL PHASE CORRECTION OF FOCUSED BEAMS IN A RANDOMLY NONUNIFORM MEDIUM

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Received September 15, 1990

An algorithm for selecting the optimal phase when correcting turbulent fluctuations of a focused Gaussian beam is studied. The calculations are performed by the method of functional integration.

It is shown that in order to optimize the average on-axis intensity of a focused Gaussian beam a phase-conjugation algorithm must be used for the correction, and the phase of a spherical wave, calculated in the approximation of the method of smooth perturbations, is employed as the correcting phase.

In this paper we calculate the average intensity of a focused Gaussian beam, formed in a turbulent atmosphere with the help of an adaptive optical system that operates based on the phase-conjugation algorithm. A possible variant of the choice of the optimal correcting phase for focusing the beam of radiation is studied.

It is well known that the scattering of light by nonuniformities of the index of refraction of a medium results in broadening of optical beams, the appearance of fluctuations in the intensity of the radiation, and a decrease in the average intensity near the axis, all of which ultimately substantially degrade the power characteristics of optical systems in the atmosphere. Different types of adaptive systems that are used to reduce the influence of these atmospheric effects.[1,2] Adaptive optical systems in turn are divided into systems that maximize some functionals of the transmitted field and systems that are based on the realizability of the principle of reversibility.[1,2]

For optical systems the correction algorithm that realizes the principle of reversibility is phase conjugation (PC).[3] It should be noted that the phase conjugation algorithm assumes conjugation of the total phase of the wave and reproduction of the amplitude. Adaptive phase correction algorithms are employed as alternatives to the method of phase conjugation. In these systems of adaptive optics only the phase of the wave is controlled, while the distribution of the radiation intensity over the initial cross section of the beam is unchanged. It is well known[4] that for linear media the field of a wave that has passed through a slab of randomly nonuniform medium can be represented as a superposition of distribution of the field on the transmitting aperture

$$U(x_1,\ \rho) = \iint d^2\rho_1 U_0(\rho_1) G(x_1,\ \rho;\ x_0,\ \rho_1), \tag{1}$$

where $U_0(\rho_1)$ is the initial distribution of the field on the transmitting aperture and $G(x_1,\ \rho;\ x_0,\ \rho_1)$ is the Green's function for a randomly nonuniform medium in the slab between the planes $x = x_0$ and $x = x_1$. ' We shall write the Initial distribution of the field $U_0(\rho_1)$ in the form

$$U_0(\rho_1) = A(\rho_1)\exp(i\varphi(\rho_1)) , \tag{2}$$

where $A(\rho_1)$ and $\varphi(\rho_1)$ are the amplitude and phase of the initial distribution. The phase $\varphi(\rho_1)$ is controlled with the help of an adaptive device. If the phase approximation[5] of the form

$$G(x_1,\ \rho;\ x_0,\ \rho_1) = G_0 \exp\left\{iS(x_1,\ \rho;\ x_0,\ \rho_1)\right\} , \tag{3}$$

is used for Green's function, where G_0 is the Green's function for free space while the phase $S(x_1,\ \rho;\ x_0,\ \rho_1)$ characterizes the fluctuations along the path (assuming that in a spherical wave the fluctuations are purely phase fluctuations), we obtain, substituting Eqs. (2) and (3) into Eq. (1),

$$U(x_1,\ \rho) = \iint d^2\rho_1 A(\rho_1) G_0(x_1,\ \rho;\ x_0,\ \rho_1) \times$$
$$\times \exp\left\{i\varphi(\rho_1) + iS(x_1,\ \rho;\ x_0,\ \rho_1)\right\}. \tag{4}$$

If the purpose of the correction is[7] to maximize the intensity functionals in the plane x_1, then we obtain the following condition for optimal phase correction

$$\varphi(\rho_1) = -S(x_1,\ \rho;\ x_0,\ \rho_1). \tag{5}$$

The condition (5) depends on the observation point ρ_1. To maximize Strehl's parameter

$$I(0) = \iint d^2\rho_{1,2} A(\rho_1) A^*(\rho_2) G_0(x_1,\ 0;\ x_0,\ \rho_1) \times$$
$$\times G_0^*(x_1,\ 0;\ x_0,\ \rho_2)$$

the condition (5) becomes

$$\varphi(\rho_1) = -S(x_1,\ 0;\ x_0,\ \rho_1), \tag{6}$$

and therefore the initial phase of the distribution (2) must be the conjugate of the phase of a point source placed at the origin of coordinates.[7] The measured phase of a wave from a reference point-like beacon can be used to construct an adaptive system for correcting distortions.

Thus the phase from a reference point source, employed for correction using the phase conjugation algorithm, corresponds completely to the phase that maximizes functionals of the intensity (when the field is calculated in the phase approximation[5]) for a point corresponding to the position of the point reference source. This can serve as a basis for constructing an experimental correction algorithm when working with reference sources. At the same time, in order to evaluate the effectiveness of adaptive optical correction systems one must know how to calculate the residual phase distortions in the system. Here the approximation employed for the function $S(x_1,\ \rho;\ x_0,\ \rho_1)$ is important.

In Ref. 6 it was shown that the phase (3) in the geometric optics approximation is effective for calculating the characteristics of the corrected field. Here we shall try to find the optimal phase from the standpoint of maximizing, for example, the average intensity. In addition, we shall describe the propagation of radiation with the help of the functional representation of the field, corresponding to the parabolic equation approximation, describing the propagation of a wave in a randomly nonuniform medium.

Consider the problem of adaptive focusing of a Gaussian beam in a turbulent atmosphere. We give the initial distribution of the field in the plane $x_0 = 0$ in the form

$$U_0(\rho) = A(\rho)\ \exp\left\{-i\varphi(\rho) - i\ \frac{k\rho^2}{2L}\right\}, \tag{7}$$

where $\varphi(\rho)$ is the controlled phase of the beam, while the quadratic phase term in Eq. (7) gives focusing of radiation in a uniform medium at a fixed range $x_1 = L$. Assume that the optimal phase is the phase of a spherical wave and depends linearly on the field of fluctuations of the dielectric constant $\varepsilon(x,\ \rho)$ in the volume bounded by the planes $x = 0$ and $x = L$. Then we write $\varphi(\rho)$ in the form

$$\varphi(\rho) = \frac{k}{2}\int_0^L d\xi \iint d^2r\ \varepsilon(\xi,\ r)M(\xi,\ \rho,\ r). \tag{8}$$

Here $M(\xi,\ \rho,\ r)$ is an unknown function, which filters the contributions of nonuniformities of the medium to phase fluctuations on the propagation path. The function $M(\xi,\ \rho,\ r)$ thereby determines the approximation for the phase $\varphi(\rho)$ in Eq. (4).

In the approximation of fluctuations of the dielectric constant which are Gaussian and δ-correlated along the direction of propagation of the radiation, using the functional representation of the field the expression for the average intensity on the axis of a Gaussian beam with the field distribution (7) can be written in the form

$$\langle I(L,\ 0)\rangle = \iint d^2\rho\, d^2\rho' Dv\ Dv' A^*(\rho)A^*(\rho') \times \exp\left\{i\ \frac{k}{2}\int_0^L \left[v^2(\xi) - v'^2(\xi)\right]d\xi\right\}\exp\left\{-\frac{1}{2} \times D(\rho,\ \rho';\ v,\ v')\right\}\delta\left[\int_0^L v(\xi)d\xi\right]\delta\left[\int_0^L v'(\xi)d\xi\right], \tag{8}$$

where

$$Dv = \prod_{\xi=0}^{L}\delta v(\xi)\left\{\int\ldots\int\prod_{\xi=0}^{L}dv(\xi)\ \exp\left[i\ \frac{k}{2}\int_0^L v^2(\xi)d\xi\right]\right\},$$

$$D(\rho,\ \rho';\ v,\ v') = \frac{\pi k^2}{2}\int_0^L d\xi\Bigg\{H\left[\xi,(\rho - \rho')(1 - \xi/L) + \int_\xi^L d\zeta\left[v(\zeta) - v'(\zeta)\right]\right] - \frac{1}{2}\iint d^4r_{1,2}H(\xi,\ r_1 - r_2) \times \left[M(\xi,\ r_1,\ \rho) - M(\xi,\ r_1,\ \rho')\right]\cdot\left[M(\xi,\ r_2,\ \rho) - M(\xi,\ r_2,\ \rho')\right] + \iint d^2r\left[H\left[\xi,\ \rho(1 - \xi/L) + \int_\xi^L d\eta v(\eta) - r\right] - H\left[\xi,\ \rho'(1 - \xi/L) + \int_\xi^L v'(\eta)\,d\eta - r\right]\right]\cdot\left[M(\xi,\ r,\ \rho) - M(\xi,\ r,\ \rho')\right]\Bigg\}, \tag{10}$$

where $H(\xi,\rho) = 2\int\int d^2\kappa\Phi_\varepsilon(\xi,\kappa)[1 - \cos\kappa\rho]$, $\Phi(\xi,\ \kappa)$ is the spatial spectrum of fluctuations of $\varepsilon(\xi,\ \rho)$; v and v' are the functional variables. Using the expression for $\langle I(L,0)\rangle$, we shall study the problem of the optimal choice of $\varphi(\rho)$. For this we calculate the variational derivative of Eq. (5) with respect to the function $M(\xi,\ r,\ \rho)$, and equating it to zero we obtain an equation for the optimal kernel:

$$\frac{\delta\langle I(L,\ 0)\rangle}{\delta M(r,\ R,\ p)} = \frac{\pi k^2}{2}\iint d^2\rho\ Dv\ Dv'\,A(\rho)\,\delta\left[\int_0^L v(\eta)d\eta\right]\times$$

$$\times\ \delta\left[\int_0^L v'(\eta)d\eta\right]\exp\left\{-\frac{1}{2}D(p,\ \rho;\ v,\ v')\right\}\times$$

$$\times\left\{H\left[\tau,\ p(1-\tau/L)+\int_\tau^L v'(\eta)d\eta - R\right]-\right.$$

$$\left.-\ H\left[\tau,\ \rho(1-\tau/L)+\int_\tau^L v(\eta)d\eta - R\right]-\iint d^2r\ \times\right.$$

$$\times\ H(\tau,\ r-R)\left[M(r,\ r,\ p)-M(r,\ r,\ \rho)\right]\times$$

$$\left.\times\ \cos\left\{\frac{k}{2}\int_0^L\left[v^2(\xi)-v'^2(\xi)\right]d\xi\right\}=0.\right. \tag{11}$$

Any attempt , to obtain an exact solution of Fq. (11) encounters serious difficulties, both because of the nonlinear character of the equation with respect to the function sought M and because of the complicated dependence on the functional variables v and v'.

Equation (11) can be solved by the method of perturbations, which in the first approximation is equivalent to neglecting the exponential term. It should be noted that the use of the method of perturbations is more justified for problems connected with the propagation of beams corrected by one or another method than for radiation with fixed characteristics, since the purpose of the correction is to compensate for the perturbations introduced by the medium. The case of "weak perturbations" is more typical for such beams. An analysis based on perturbations permits evaluating the conditions under which adaptive optical systems are effective and to determine the requirements which such systems must satisfy.

The solution of Eq. (11) in the first approximation can be written in the form

$$M_1(\tau,\ R,\ p) = \frac{k}{2\pi\tau(1-\tau/L)}\ \times$$

$$\times\ \sin\frac{k\left[R-p(1-\tau/L)\right]^2}{2\tau(1-\tau/L)}\ . \tag{12}$$

If the propagation of radiation is described in the phase approximation of the Huygens-Kirchhoff method,[5] then an expression analogous to Eq. (9), but differing by the fact that v and v' do not appear in the function $D(\rho,\ \rho';\ v,\ v')$ from the expression (10), is obtained for the average intensity. Next, operating on the expression for $\langle I(L,0)\rangle$, calculated in the phase approximation of the Huygens-Kirchhoff method, like in the case of Eq. (9), we obtain instead of Eq. (12)

$$M_2(\tau,\ R,\ \rho) = \delta\left[R-\rho(1-\tau/L)\right]\ . \tag{13}$$

Substituting Eqs. (12) and (13) into Eq. (8), we obtain an expression for the optimal correcting phase which in the first case is identical to the phase of a spherical wave, calculated in the first approximation of the method of smooth perturbations, and in the second case is identical with the geometric optics expression for the phase of a spherical wave.[10] In other words, the optimal (in the sense that the average on-axis intensity is maximized) phase for correcting distortions of a Gaussian beam is the phase of a point reference source placed at the origin of coordinates, and if the propagation of the radiation is described in the parabolic-equation approximation, then the optimal phase must be calculated in the first approximation of the method of smooth perturbations.

We shall prove that the phase (8) employed in this case is optimal: the deviations of the average intensity from the diffraction intensity are minimum. For this, we compare the residual distortions for the correcting phase in the form (12) and (13). Let the initial amplitude distribution be Gaussian

$$A(\rho) = \exp\left(-\rho^2/2a^2\right).$$

The calculation of the average intensity, normalized to the value in a uniform medium I_0, performed in the first approximation of the perturbation method for the phase in the form (12) and (8) gives the following expression for the relative variation of the average intensity:

$$\Delta_1 = \frac{I_0-\langle I(L,\ 0)\rangle}{I_0} = \frac{\pi k^2}{2}\int_0^L d\xi\iint d^2\kappa\ \Phi_\varepsilon(\xi,\ \kappa)\ \times$$

$$\times\left[1-\exp\left(-\kappa^2a^2(1-\xi/L)^2\right)\right]\sin^2\frac{\kappa^2\xi}{2k}(1-\xi/L), \tag{14}$$

while for the phase in the form (13) and (8)

$$\Delta_2 = \frac{I_0-\langle I(L,\ 0)\rangle}{I_0} = \pi k^2\int_0^L d\xi\iint d^2\kappa\ \Phi_\varepsilon(\xi,\ \kappa)\ \times$$

$$\times\left[1-\exp\left(-\kappa^2a^2(1-\xi/L)^2\right)\right]\times$$

$$\times\left[1-\cos\frac{\kappa^2\xi}{2k}(1-\xi/L)\right]. \tag{15}$$

At the same time, for a focused beam with no correction the perturbation method gives

$$\Delta_0 = \frac{I_0-\langle I(L,\ 0)\rangle}{I_0} = \frac{\pi k^2}{2}\int_0^L d\xi\iint d^2\kappa\ \Phi_\varepsilon(\xi,\ \kappa)\ \times$$

$$\times \left[1 - \exp\left(-\kappa^2 a^2 (1 - \xi/L)^2\right)\right] . \qquad (16)$$

The symbols Δ_0, Δ_1, Δ_2 were introduced for the relative variations of the average intensity in a focused beam for a system with no correction, with correction with optimal phase, and with correction based on the geometric-optics phase, respectively. Introducing for the power-law spectrum $\phi_\varepsilon(\xi, \kappa) = 0.033 C_\varepsilon^2(\xi)\kappa^{-11/3}$ the dimensionless parameters $\Omega = ka^2/L$ and $q = k\rho_0^2 / L$, where ρ_0 is the coherence radius of the wave, we transform Eq. 16) as follows:

$$\Delta_0 = 1.1 a^{5/3} k^2 L \int\limits_0^1 dt\ C_\varepsilon^2(Lt)(1 - t)^{5/3} =$$

$$= O\left(\Omega^{5/6} q^{-5/6}\right) . \qquad (17)$$

For $q < \Omega$(even if $q > 1$) the focused beam is strongly broadened.

In wide beams ($\Omega \gg 1$) we obtain from Eqs. (14) and (15), respectively,

$$\Delta_1 \simeq \beta_0^2/4, \qquad \Delta_2 \simeq \beta_0^2/2^{5/6} \qquad (18)$$

where

$$\beta_0^2 = 0.56\ k^{7/6} L^{11/6} \int\limits_0^1 dt\ C_\varepsilon^2(Lt) t^{5/6} (1 - t)^{5/6}$$

$$= 0\left(q^{5/6}\right).$$

At the same time, for narrow beams ($\Omega \ll 1$) we have

$$\Delta_1 \simeq 0.54 a^{5/3} k^2 L \int\limits_0^1 dt\ C_\varepsilon^2(Lt)(1 - t)^{5/3} \simeq 0.49\ \Delta_0, \qquad (19)$$

$$\Delta_2 \simeq 2.2\ a^{5/3} k^2 L \int\limits_0^1 dt\ C_\varepsilon^2(Lt)(1 - t)^{5/3} \simeq 2\Delta_0. \qquad (20)$$

The condition for complete correction of the broadening of the beam and the condition for the method of perturbations to be applicable for calculating the average intensity, together with Eqs. (12) and (13), is that the right parts of Eqs. (18), (19), and (20) must be small. As follows from Eq. (18), in wide beams the use of any method of phase correction Eq. (12) or Eq. (13) results in virtually complete correction of the broadening for $1 < q < \Omega$, when, as one can see from Eq. (17), the uncorrected beam is strongly broadened; in addition, the relative change in the variations of the on-axis intensity owing to correction is

$$\frac{\Delta_{1,2}}{\Delta_0} \sim 0\left(\Omega^{-5/6}\right).$$

For narrow beams ($\Omega \ll 1$), as one can see by comparing Eqs. (17), (19), and (20), correction based on the algorithm (7) and (8) is much less effective. Thus optimal phase correction approximately doubles the average intensity, while the correction (13) even decreases the average on-axis intensity Δ_0. This result is expected, since it follows from Ref. 6 that correction in a narrow beam using a point reference source is ineffective (it is impossible to change significantly, as a result of such phase correction, the values of the moments of the intensity distribution). Even the expansion itself of the field in the form (1) is inapplicable for a narrow beam. It is shown in the same work that in order to correct distortions in narrow beams a wide reference source must be used, and in the limit a reference plane wave must be used.

The results obtained in this work are a systematic justification of correction based on the phase conjugation algorithm. The proposed methodology for optimizing the phase can be developed in the future, but the computational method must be tailored to the characteristic that is being evaluated.

At the same time the results of Ref. 7 are valid only in the region where the phase approximation is applicable, if the results obtained in Ref. 7 are employed for calculating the quality of the correction.

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