OPTIMIZATION OF THE EFFICIENCY OF COMPENSATION FOR PHASE DISTORTIONS BY ADAPTIVE OPTICAL SYSTEMS

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An approach is proposed to describe an adaptive optical system (AOS) in the form of matrices of coefficients of series expansion of the wave–front sensor and corrector characteristics over a common basis of orthogonal polynomials. Calculational relations are obtained for estimating the quality and potential efficiency of compensation for the phase distortions by an arbitrary AOS. These relations are used to estimate the rms error of correction by the criterion of minimum over the aperture. The efficiency of a 7–channel AOS in compensating for the atmospheric turbulence is calculated.

1. INTRODUCTION

The problem of the efficiency of adaptive optical systems (AOS) compensating for distortions of light beams caused by the turbulence when the beams propagate through the atmosphere or an optical channel is the matter of great concern in a number of papers (see, for example, Refs. 1–4). But, as a rule, in these papers the efficiency of an AOS is analyzed either from the point of view of the accuracy of wave–front approximation by a corrector,[2] or the quality of its estimate at the sensor output.[3] In Ref. 4 the efficiency of an algorithm of compensation for the phase distortions is analyzed. The relations obtained there make it possible to estimate the system quality by Strehl number but this requires rather cumbersome calculations. At the same time, the problem of calculating the quality and efficiency of an AOS can sufficiently simply be formalized by the matrix description of its elements in a common basis of orthogonal polynomials and by representation of the wave–front distortions in the same basis. Relations obtained in this case make it possible to estimate the potential efficiency of the AOS for arbitrary criteria of the system quality.

2. EFFICIENCY OF COMPENSATION FOR PHASE DISTORTIONS

Let us represent a distorted wave front, within the limits of the receiving aperture of a recorder, $\varphi(\mathbf{r})$ by a series expansion over the orthogonal system of functions $\{\varphi_n(\mathbf{r})\}$

$$\varphi(\mathbf{r}) = \sum_{n=1}^{\infty} p_n \, \varphi_n(\mathbf{r}) \tag{1}$$

and describe it by the vector column $\mathbf{P}^{\mathrm{T}} = p_1, p_2, \ldots$ (T is the transposition sign). Let each mth channel of an M–channel sensor of the wave front make some linear transformation $L_m\{\varphi(\mathbf{r})\}$ of the wave front arriving at the recorder aperture, and at its output the following signal be formed:

$$u_m = L_m\{\varphi(\mathbf{r})\} .$$

Such a sensor can be of the Hartmann type which produces output signals proportional to the wave–front gradients averaged over the subapertures. In this case an interaction of radiation with the sensor can be described by the relation

$$\mathbf{u} = G\mathbf{p} ,$$

where $\mathbf{u}^{\mathrm{T}} = u_1, u_2, \ldots, u_m$ and G is the matrix composed of the elements

$$g_{mn} = L_m\{\varphi_n(\mathbf{r})\} . \tag{2}$$

Let us describe the interaction between the corrector and radiation. When activated with the control signal the corrector introduces the phase shift

$$\varphi(\mathbf{r}) = \sum_{k=1}^{K} \vartheta_k \, \Psi_k(\mathbf{r}) , \tag{3}$$

where $\Psi_k(\mathbf{r})$ and t_k are the response functions and control signals for the corrector channels: $K = 1, \ldots, k$ is the number of channels. In the basis $\{\varphi_n(\mathbf{r})\}$ the response functions take the form

$$\Psi_k(\mathbf{r}) = \sum_{n=1}^{\infty} q_{nk} \, \varphi_n(\mathbf{r}) , \tag{4}$$

where

$$q_{nk} = \int_{\Omega} \varphi_n(\mathbf{r}) \, \Psi_k(\mathbf{r}) \, d^2\mathbf{r} ,$$

and the integration is carried out over the corrector aperture. By substituting Eq. (4) into Eq. (3) and representing the phase shift introduced by the corrector in terms of series expansion (1) we obtain

$$\mathbf{p} = Q\mathbf{v} ,$$

where $\vartheta^{\mathrm{T}} = \vartheta_1, \vartheta_2, \ldots, \vartheta_k$ and the matrix Q is composed of the elements q_{nk}.

For a comprehensive description of the system it is necessary to introduce the function which relates the corrector control signals to the data obtained from the wave–front sensor. Let us restrict ourselves by consideration of the linear function of the form

$$\mathbf{v} = R\mathbf{u} ,$$

where R is the control matrix. Taking into account the above–defined matrices and introducing the vector

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$\boldsymbol{\gamma}^{\mathrm{T}} = \gamma_1, \gamma_2, ..., \gamma_M$, which describes the error caused by the wave–front sensor, we derive the relation for the vector of a residual correction error

$$\Delta\mathbf{p} = (E - QRG)\mathbf{p} + QR\boldsymbol{\gamma} , \qquad (5)$$

where E is the unit matrix. Let us define the criterion of the system quality I in the form of a functional of the residual correction error $J(\Delta\mathbf{p})$ averaged over an ensemble of realizations of the wave front. Then, with Eq. (5) taken into account, we obtain

$$I = \langle J((E - QRG)\mathbf{p} + QR\gamma)\rangle , \qquad (6)$$

where <...> is the sign of averaging over the ensemble of realizations. By substituting matrices determining individual AOS into Eq. (6) and averaging one can estimate the quality of such systems according to a chosen criterion. Hence it follows that for the given sensor, wave–front corrector, and statistics of the vectors $\mathbf{p}$ and $\boldsymbol{\gamma}$ the quality of the system is the function of R and is limited by the value I which is maximum with respect to R. As shown in Ref. 1, the efficiency of an AOS can be characterized by the ratio of functionals of quality for the cases of the AOS presence and its absence. Therefore, we can state that the efficiency of an AOS is limited by the value

$$\eta = \frac{\langle J(\mathbf{p})\rangle}{\min\limits_R \langle J((E - QRG)\mathbf{p} + QR\mathbf{g})\rangle} , \qquad (7)$$

where the matrix R is determined by the methods of calculus of variations. Thus, Eqs. (6) and (7) determine the quality and potential efficiency of the AOS composed of separate elements. By comparing the efficiency of an actual AOS with that calculated by Eq. (7) one can estimate the optimal way of development of such systems according to the chosen criterion of quality.

3. EFFICIENCY OF AN AOS BY THE CRITERION OF MINIMUM RMS ERROR OF CORRECTION AVERAGED OVER THE APERTURE

Let us obtain an estimate of the maximum efficiency of an AOS for an important particular case of the criterion of minimum of the rms correction error $\overline{\varepsilon^2}$ averaged over the aperture. It is easy to see that in this case the quality of an AOS can be estimated by the relation

$$\overline{\varepsilon^2} = \langle[(E - QRG)\mathbf{p} + QR\gamma]^{\mathrm{T}}[(E - QRG)\mathbf{p} + QR\gamma]\rangle . \qquad (8)$$

Let the errors of sampling reads at the sensor outputs be statistically independent and have the same statistics, then after averaging we obtain

$$\overline{\varepsilon^2} = \mathrm{Tr}\,[(E - QRG)\,B\,(E - QRG)^{\mathrm{T}}] + \sigma^2 \mathrm{Tr}\, QRR^{\mathrm{T}}Q^{\mathrm{T}}, \qquad (9)$$

$$\eta = \mathrm{Tr}\,B/\,\overline{\varepsilon^2} , \qquad (10)$$

where the sign Tr means the sum of diagonal elements of the matrix and B is the matrix of the elements

$$b_{ij} = \langle p_i\, p_j\rangle = \int\limits_{\Omega}\int B_{\varphi}(\mathbf{r}', \mathbf{r}'')\, \varphi_i(\mathbf{r}')\, \varphi_j(\mathbf{r}'')\, \mathrm{d}^2\mathbf{r}'\, \mathrm{d}^2\mathbf{r}'' ,$$

$B_{\varphi}(\mathbf{r}', \mathbf{r}'')$ is the covariance function of the phase; σ^2 is the variance of the reading errors at the outputs of the wave–front sensor. To obtain the optimal value of R it is convenient to write Eq. (8) in terms of the components of the matrices

$$\overline{\varepsilon^2} = \sum_{i=1}^{N}\left[\sum_{n=1}^{N}\left(\delta_{in} - \sum_{k=1}^{K} q_{ik} \sum_{m=1}^{M} r_{km}\, g_{mn}\right) \times \right.$$
$$\times \sum_{n'=1}^{N}\left(\delta_{in'} - \sum_{k=1}^{K} q_{ik'} \sum_{m'=1}^{M} r_{k'm'}\, g_{m'n'}\right) B_{nn'} +$$
$$\left. + \sigma^2 \left(\sum_{k=1}^{K} q_{ik} \sum_{m=1}^{M} r_{km}\right)^2\right] ,$$

(here δ_{in}, q_{ik}, r_{km}, and g_{mn} are the components of the unit matrix and Q, R, and G matrices and N is the dimensionality of the matrix B). Let us differentiate the matrix R, equalize the derivatives to zero and return to the matrix form of the representation

$$Q^{\mathrm{T}}QRG(B + B^{\mathrm{T}})G^{\mathrm{T}} - Q^{\mathrm{T}}(B + B^{\mathrm{T}})G^{\mathrm{T}} + \sigma^2 Q^{\mathrm{T}}QR = 0. \qquad (11)$$

Solving Eq. (11) with respect to R we obtain

$$R = (Q^{\mathrm{T}}Q)^{-1}Q^{\mathrm{T}}BG^{\mathrm{T}}(GBG^{\mathrm{T}} + \sigma^2 E)^{-1} , \qquad (12)$$

where E is the N by N unit matrix. Thus, matrix (12) relates the corrector control signals to those of the wave–front sensor in the way which provides the minimum over the aperture rms error of correction (Eq. (9)). By substituting expressions R obtained for an individual AOS and statistics of the wave–front distortions into Eqs. (9) and (10) one can obtain estimates of the maximum achievable in these cases quality and efficiency of a system. Below we illustrate this approach by an example in which the efficiency of an AOS used to compensate for the atmospheric turbulence is estimated.

4. EFFICIENCY OF COMPENSATION FOR THE ATMOSPHERIC TURBULENCE

Let us calculate the potential efficiency of a 7–channel AOS with a wave–front sensor of Hartmann type and deformable controllable mirror. Let the measuring sensor apertures and actuator mechanisms of the corrector be arranged in networks with one and the same step in the form of a ring around central element at the nodes of a hexagonal grid. In this case the functions of the mirror response are equal to each other and are of Gaussian type, while the size of measuring apertures and the widths of the response functions are equal to one half of the distance between the centers of neighboring channels. Let the distortions of the radiation wave front involve the structural function

$$D(\mathbf{r}', \mathbf{r}'') = 6.88(|\mathbf{r}' - \mathbf{r}''|\; r_0)^{5/3} ,$$

where r_0 is the Fried radius. Determine now the matrices describing the sensor, corrector, and covariance of the wave–front distortion fluctuations in the basis of Zernike polynomials. Note also, that it is assumed that each channel of the sensor measures two projections of the wave–front tilts averaged over the corresponding subaperture. Then, the matrix elements can be written in the form

$$g_{mn} = \begin{cases} \dfrac{1}{S} \displaystyle\int_{\Omega_l} \dfrac{\mathrm{d}\varphi_n(\mathbf{r})}{\mathrm{d}x} \, \mathrm{d}^2\mathbf{r} \,, & m = l \,, \\ \dfrac{1}{S} \displaystyle\int_{\Omega_l} \dfrac{\mathrm{d}\varphi_n(\mathbf{r})}{\mathrm{d}y} \, \mathrm{d}^2\mathbf{r} \,, & m = l + 7 \,, \end{cases} \quad (13)$$

where $l = 1, \ldots, 7$ is the number of a sensor subaperture, $m = 1, \ldots, 14$ is the number of a measuring channel, (x, y) are the Cartesian coordinates in the plane of the input aperture of the sensor, S is the subaperture area, and integration is carried out over the sensor subapertures. The elements of the matrix Q are directly obtained from Eq. (4). As to the matrix B, it is known in the given case.[1] Its elements are

$$b_{ij} = a_{ij} (D/r_0)^{5/3} \,,$$

where D is the diameter of the sensor input aperture, a_{ij} are the elements of the matrix A of numerical coefficients. This matrix is almost a diagonal matrix, in addition, its elements decrease in magnitude with increase of indices. The wave phase averaged over the aperture is assumed to be of no importance for the problem under consideration. Therefore, the polynomial describing the invariable component of the phase can be eliminated from the basis. Then, the first diagonal elements are[1] $a_{11} = a_{22} = 4.49 \cdot 10^{-1}$, $a_{33} = a_{44} = a_{55} = 2.32 \cdot 10^{-2}$, and $a_{66} = a_{77} = a_{88} = a_{99} = 6.19 \cdot 10^{-3}$. Let us determine the control matrix R and value of the rms correction error averaged over the aperture. It can be seen that in this case

$$R = (Q^{\mathrm{T}} Q)^{-1} Q^{\mathrm{T}} A G^{\mathrm{T}} (G A G^{\mathrm{T}} + \sigma^2 E)^{-1} \,,$$

$$\overline{\varepsilon^2} = \left(\frac{D}{r_0}\right)^{5/3} \mathrm{Tr}[(E - QRG) A (E - QRG)^{\mathrm{T}}] + \sigma^2 \, \mathrm{Tr} \, Q R R^{\mathrm{T}} Q^{\mathrm{T}}, \quad (14)$$

$$\eta = \left(\frac{D}{r_0}\right)^{5/3} \mathrm{Tr} \, A / \overline{\varepsilon^2} \,.$$

To obtain the numerical values of the efficiency it is necessary to calculate the matrices G and Q by formulas (13) and (4), and then using standard computer programs to calculate the value of R and to substitute the results into Eq. (14). The above–described algorithm was executed on a PC IBM AT–286 in the basis involving 50 Zernike polynomials for different values of q^2. In this case the values of the error variance introduced by the wave–front sensor were correlated with the averaged total power of a signal at the sensor output

$$\langle \mathbf{u}\mathbf{u}^{\mathrm{T}} \rangle = \mathrm{Tr} \, G B G^{\mathrm{T}}$$

and were characterized by the parameter

$$q^2 = \frac{\mathrm{Tr} \, G B G^{\mathrm{T}}}{\sigma^2} \,,$$

which represents the signal–to–noise ratio. Numerical values of the obtained efficiency are shown in the table. As could be expected the correction efficiency increases with q^2 growth and asymptotically approaches the value 3.56. To reach the maximum efficiency it is sufficient to make use of the values $q^2 = 10^2 \ldots 10^3$. The efficiency of other kinds of AOS's can be calculated for different q^2 in a similar way.

TABLE I. Efficiency of compensation for the atmospheric turbulence by a 7–channel adaptive optical system.

q^2	∞	10^4	10^3	10^2	10
η	3.56	3.50	3.46	3.18	2.15

5. CONCLUSION

Description of the AOS elements and the wave–front disturbances in a common basis of orthogonal polynomials enables one to obtain the relations for estimating the quality and potential efficiency of such systems, as well as to relate the corrector control signals to the data obtained from the wave–front sensor that ensures the achievement of the maximum efficiency of an AOS. Comparison of the efficiency of AOS with its limiting value makes it possible to estimate the circuitry design of such systems according to a chosen criterion of the quality.

REFERENCES

1. M.A. Vorontsov and V.I. Shmal'gausen, *Principles of Adaptive Optics* (Nauka, Moscow, 1985).
2. D.A. Bezuglov, Atm. Opt. **4**, No. 12, 900–904 (1991).
3. S.V. Butsev and V.Sh. Khismatullin, Opt. Mekh. Prom–nost', No. 10, 3–5 (1991).
4. V.E. Kirakosyants, V.A. Loginov, V.V. Slonov, et al., Atm. Opt. **4**, No. 7, 550–555 (1991).