Phase unwrapping from its principal value

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The iteration procedure for elimination of the smoothing error occurring when phase unwrapping from its principal value by the least-square method is described. As is shown by the example of a synthesized aperture radar interferogram, even two applications of the least-square method are sufficient for total elimination of the error.

Introduction

Phase unwrapping from the digital phase data limited by its principal value (modulo 2π) is the last step of many algorithms of the image numerical analysis. It is also the integral part of the majority of interferometric methods of optical coherent metrology and profilometry, as well as radar monitoring of the underlying surface topography. As a rule, the result of interferometric measurements is a map of interferometric bands, i.e., two-dimensional cosine distribution of the phase modulated by the measured physical quantity, where the phase is defined only within the limits of its principal value.

The phase unwrapping is carried out from the phase gradient $\mathbf{g}(\mathbf{r})$, determined from the interferogram within the limits of the principal value:

$$\mathbf{g}(\mathbf{r}) = P[\nabla P[\psi(\mathbf{r})]], \qquad (1)$$

where the operator $P[...]$ means the reduction of the parameter in parenthesis to the principal phase value interval $[-\pi, \pi)$; $\mathbf{r} = \{x, y\}$ is the two-dimensional vector; $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}$.

To determine $\psi(\mathbf{r})$, one can integrate Eq. (1):

$$\psi(\mathbf{r}) = \psi_0 + \int_{L_0}^{L(\mathbf{r})} \mathrm{d}l\,\mathbf{g}(\mathbf{r});\ \psi_0 = \psi_0(L_0). \qquad (2)$$

Here $L(\mathbf{r})$ is the integration path over the $\mathbf{g}(\mathbf{r})$ gradient domain. Therefore, with the known gradient, the phase ψ can be determined accurate to a constant. If amplitude fluctuations of interferometric signals are small, then the principal phase value gradient, limited by its own principal value, is free of the solenoidal component ($\nabla \times \mathbf{g} = 0$) and coincides with the true phase gradient. In this case, the integral (2) is independent of the integration path, and we obtain a unique solution for the phase (accurate to a constant). In the presence of noises (the appearance of the speckle-structure of the interferometric signal field) or a sharp altitude difference on the reflecting surface, the limits of phase gradient variations can exceed 2π, therefore, the persistence of the phase gradient vector field is broken down ($\nabla \times \mathbf{g} \neq 0$), there appear branch points, and the phase surface becomes discontinuous. This leads to the integration path dependence of integral (2) and to an ambiguity in the phase solution. The ignoring of the violation of the $\mathbf{g}(\mathbf{r})$ vector potentiality due to noises can result in large ($\sim 2\pi$) errors in the phase determination, which are accumulated while integrating in Eq. (2).

There are two methods allowing one to avoid the ambiguous solution when determining the phase from its gradient. The first one consists in bypass of the domain $\nabla \times \mathbf{g} \neq 0$ while integrating. The second one is based on determination of the phase ψ from the solution of Eq. (1) and consists in minimization of the quadratic form of the gradient difference between wrapped and unwrapped phases. The branch cut technique[1,2] belongs to the first type, while the least-square method (LSM)[2,3] – to the second one. The branch cut technique assumes that the phase branch points appear pairwise and are phase-cut coupled. Thus, a tree of cuts is constructed so that to connect a positive point by a line with the nearest negative one. After all cuts have been made, the integration is carried out along the path nonintersecting the constructed tree. A large number of algorithms for phase unwrapping[2,4–9] realize one or another modification of the above methods.

Both methods have disadvantages. The branch cut technique requires finding paired branch points, which is not always possible. The LSM, though suppresses noises, caused mainly by fluctuations of interfering signals, reconstructs the phase nearby the phase-surface discontinuities between the paired points with errors. The both methods lead to errors resulting in the loss of the so-called "hidden phase,"[10] when phase jumps greater than 2π are caused by properties of the object under study rather than by signal and noise fluctuations.

The algorithm of the "improved" least-square method was suggested and approved in Refs. 10–12. The method determines the hidden phase nearby the phase-surface discontinuities correctly in the case of the proper identification of all branch points and their pairness. But this is not always possible.

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Besides, the common LSM disadvantage inheres in the algorithm: it leads to the phase unwrapping error nearby the phase discontinuities even in the case of exact determination of the hidden phase due to smoothing properties of LSM. A way to reduce the smoothing error is suggested in this paper.

Phase unwrapping

Consider the procedure of phase unwrapping from its principal value by the example of an interferogram obtained with a synthesized aperture radar over the New Caledonia (France) afforded by the European Space Agency (Copyright by CNES/ESA). Figure 1 shows an interferogram fragment of 128 × 128 points.

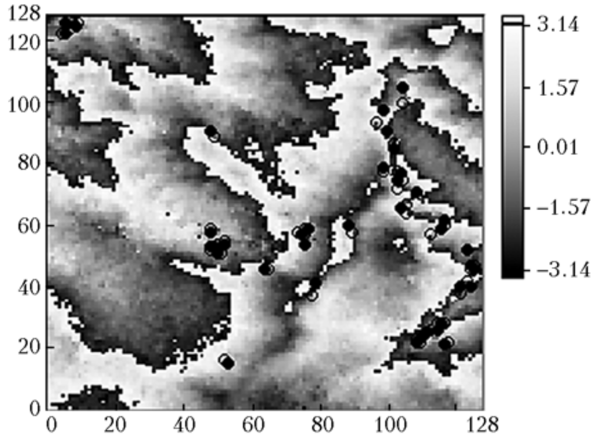

Fig. 1. Interferogram. Signs • and ○ correspond to branch points.

Interferogram values are limited within $[-\pi, \pi)$ range, so, to reconstruct the full phase value (to stitch phase jumps) the least-square method can be used. However, as is shown by the analysis based on the profile summation of phase differences,[10,11,13] the solenoidal part of the phase gradient in the interferogram is not equal to zero everywhere and the interferogram includes the phase branch points (Fig. 1). In this case, the LSM is incapable to unwrap the phase throughout its variation range since it does not "see" the hidden phase determined by phase surface discontinuities passing through paired positive and negative points. Therefore, the improved least-square method (ILSM) was used,[10,11] allowing calculation of the hidden phase component as well. Results of the phase unwrapping by this method are shown in Fig. 2.

Let $\varphi(\mathbf{r})$ denote the phase, which needs to be unwrapped from its principal value, $\varphi_{\text{LSM}}(\mathbf{r})$ is the phase unwrapped by LSM, $\varphi_{\text{hid}}(\mathbf{r})$ is the hidden phase, $\varphi_{\text{ILSM}}(\mathbf{r}) = \varphi_{\text{LSM}}(\mathbf{r}) + \varphi_{\text{hid}}(\mathbf{r})$ is the phase unwrapped by the improved LSM. Then, following Ref. 14, the ILSM phase unwrapping error can be defined as

$$P[\Delta_{\text{sm}}(\mathbf{r})] = P\big[P[\varphi(\mathbf{r})] - P[\varphi_{\text{ILSM}}(\mathbf{r})]\big] = \\ = P\big[P[\varphi(\mathbf{r})] - P[\varphi_{\text{LSM}}(\mathbf{r}) + \varphi_{\text{hid}}(\mathbf{r})]\big]. \quad (3)$$

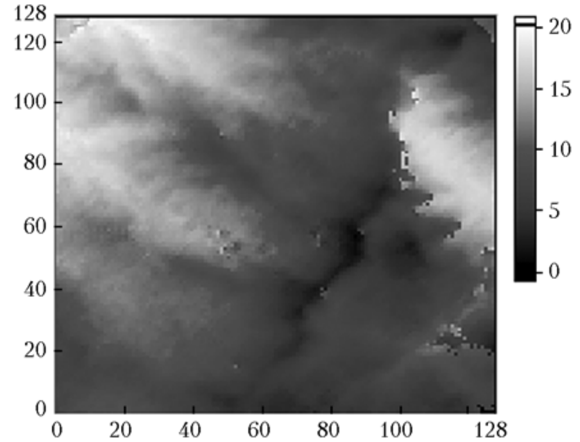

Fig. 2. Phase unwrapping from the interferogram in Fig. 1 by the improved LSM.

The 2D-distribution of the error of ILSM phase unwrapping, limited by its principal value, is shown in Fig. 3.

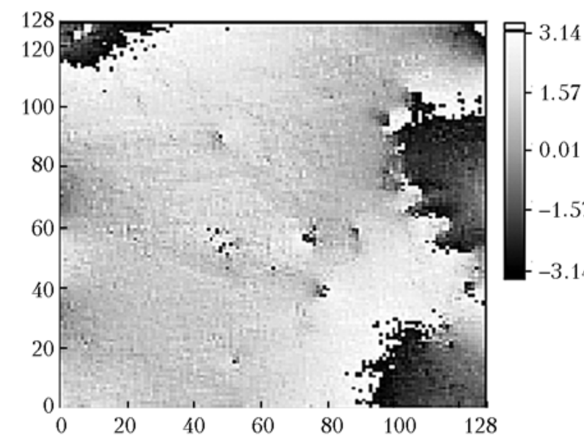

Fig. 3. Convoluted error of the ILSM phase unwrapping.

The error is evidently large and gets values throughout the range from $-\pi$ to π radians, though theoretically it should be close to zero.[10,11] Such a difference between $P[\varphi(\mathbf{r})]$ and $P[\varphi_{\text{ILSM}}(\mathbf{r})]$ is explained by the smoothing effect of the LSM, which interpolates the phase in the region of "unnoticeable" discontinuities from values of the "smooth" phase determined by the potential part of the phase gradient.

This error can be interpreted as the underunwrapped phase part. Let us unwrap the phase function $P[\Delta_{\text{sm}}(\mathbf{r})]$ calculated within the limits of the principal value, using the LSM and add the results to $\varphi_{\text{ILSM}}(\mathbf{r})$. Thus, we obtain the estimation of the phase $\tilde{\varphi}(\mathbf{r}) = \varphi_{\text{ILSM}}(\mathbf{r}) + \Delta_{\text{sm}}(\mathbf{r})$ unwrapped from the principal value. Let us apply the wrapping operator P and find the difference between principal values of phase functions $\varphi(\mathbf{r})$ and $\tilde{\varphi}(\mathbf{r})$ within the limits of the principal value:

$$P[\Delta 1_{\text{sm}}] = P\big[P[\varphi(\mathbf{r})] - P[\tilde{\varphi}(\mathbf{r})]\big], \quad (4)$$

which, similarly to Eq. (3), can be also interpreted as the error of unwrapping or the underunwrapped part of the phase $\varphi(\mathbf{r})$. The LSM-unwrapping allows one to find the estimation

$$\tilde{\varphi}_1(\mathbf{r}) = \tilde{\varphi}(\mathbf{r}) + \Delta 1_{sm} = \varphi_{ILSM} + \Delta_{sm} + \Delta 1_{sm}$$

and then the error

$$P\,[\Delta 2_{sm}] = P\,\{P[\varphi(\mathbf{r})] - P[\tilde{\varphi}_1(\mathbf{r})]\}\,. \qquad (5)$$

The LSM application to Eq. (5) gives the estimation of the unwrapped phase

$$\tilde{\varphi}_2(\mathbf{r}) = \tilde{\varphi}_1(\mathbf{r}) + \Delta 2_{sm} = \tilde{\varphi}(\mathbf{r}) + \Delta 1_{sm} + \Delta 2_{sm} = \varphi_{ILSM} + \Delta_{sm} + \Delta 1_{sm} + \Delta 2_{sm}. \qquad (6)$$

This iteration process can be continued until the required accuracy of the phase estimation is attained. The phase function $\Delta_{sm}(\mathbf{r})$ for the interferogram in Fig. 1 is shown in Fig. 4.

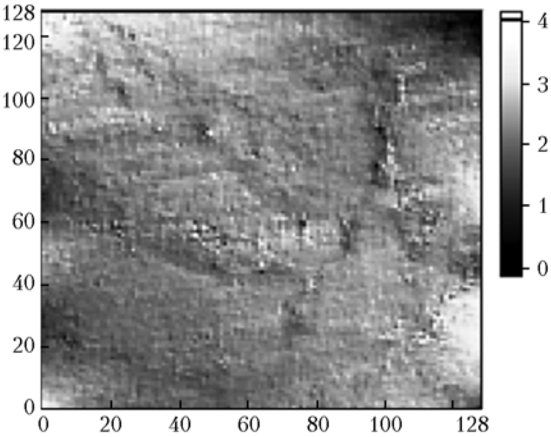

Fig. 4. Error $\Delta_{sm}(\mathbf{r})$.

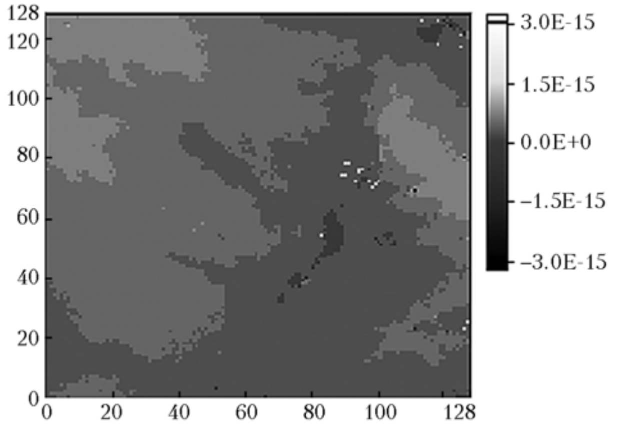

Fig. 5. Phase unwrapping error after the second LSM iteration.

As is seen, the part of the interferogram phase function in Fig. 1, "omitted" by the LSM, can take values grater than π. The error (4) is shown in Fig. 5.

It can be considered as zero, i.e., two iterations are sufficient to eliminate the smoothing error.

Conclusion

The suggested iteration procedure eliminates the smoothing error occurring when phase unwrapping from its principal value by the least-square method. The synthesized aperture radar interferogram exemplifies sufficiency of the double LSM application for total elimination of the error. The use of the improved LSM in the first iteration allows the accounting for the hidden phase. However, if to consider the presence of branch points as the result of noises rather than peculiarities of an object under study, there is no need to account for the hidden phase, and the standard LSM can be used in the first iteration as well.

Acknowledgments

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