Wave propagation and geometrical optics

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We discuss formally rigorous method of constructing solutions of the ordinary differential equations in the problems of wave propagation through stratified inhomogeneous media. The method is based on transformation of a homogeneous differential equation to a formally inhomogeneous one, the operator of which admits exact geometrical optics approximation solutions. The inhomogeneous differential equation is conventionally reduced to Volterra integral equation, which is transformed to the canonical set of two ordinary differential equations of the first-order. We propose, for the obtained system, a rigorous method for constructing a sequence of approximations to the exact solution of the initial differential equation. The scheme proposed for constructing a sequence of approximations in solving an ordinary differential equation can be used in problems of wave propagation through stratified inhomogeneous media. The method is applicable in the presence of losses and has no restrictions on the scales of inhomogeneities. The validation of the method does not use asymptotic considerations.

Introduction

The geometrical optics approximation is widely used in solving various physical problems related to wave propagation (see, for example, Ref. 1). It is especially important in problems on wave propagation through smoothly inhomogeneous media. A formally rigorous approach to analyzing wave equations is presented in this paper by analyzing propagation of acoustic waves, as an example, in a stratified inhomogeneous medium; the approach is closely related to the geometrical optics approximation and is free of restrictions on the medium properties.

As known,[2] even small variations of the medium parameters can essentially influence its reflectance. Hence, a method for constructing solutions of wave equations is desirable, which allows one to successively approach the exact solution. Mathematically, the geometrical optics can be considered as the limiting case of the wave theory if assuming the wavelength to vanish. This allows the asymptotic methods to be used in validation of the geometrical optics and construction of a sequence of approximations.

In this paper, the problem of constructing a sequence of approximations to the solution of an acoustic wave equation for a stratified inhomogeneous medium without the use of asymptotic methods is considered. The basic form of the geometrical optics approximation forms the basis for the method considered below, without the use of any asymptotic considerations (see, for example, Ref. 3).

Basic set of equations

Let the dependences of the parameters of a medium, in the case of acoustic waves, is set by the real functions $\rho(x)$ and $c(x)$, i.e., by the density of aqueous medium and by the speed of sound in it. Also assume the aqueous medium parameters to tend to constant values ρ_-, c_- and ρ_+, c_+ when $x \to -\infty$ and $x \to \infty$, correspondingly.[2] Within this model, the problem on emission of acoustic waves can be reduced to the equation for Green's function

$$\frac{\mathrm{d}^2 G(x,k)}{\mathrm{d}x^2} + k^2 n^2(x) G(x,k) = \delta(x - x_0), \qquad (1)$$

where $k = \omega / c_0$ and $n^2(x) = c_0^2 / c^2(x)$. Note, that the considered problem can be conventionally generalized to the case of slant incidence of the wave on a stratified medium, as considered, e.g., in Ref. 2.

To construct the Green's function $G(x, k)$, it is sufficient to find a pair of linearly independent solutions of the homogeneous equation

$$\frac{\mathrm{d}^2 u(x,k)}{\mathrm{d}x^2} + k^2 n^2(x) u(x,k) = 0 . \qquad (2)$$

It is just this problem that makes the subject of this study. As known,[3] a starting point for seeking a geometrical optics approximation is solution of a steady-state wave equation for a homogeneous medium. Indeed, wave propagation through a medium with slowly varying parameters (at a fixed wavelength) is considered similar to this in the case of a homogeneous medium with solution parameters close to the parameters of inhomogeneous medium for the interval under study. This point of view does not use asymptotic considerations and can equally be considered as a special method.

Based on the analogy with propagation through a homogeneous medium, solutions of Eq. (2) are to be sought using the following scheme. Let us introduce a pair of functions

$$f_\pm(x,k) = \frac{1}{\sqrt{n(x)}} \exp\left(\pm ikn_+ x \mp ik \int_x^\infty \left[n(y) - n_+ \right] \mathrm{d}y \right), \quad (3)$$

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asymptotical behavior of which is specially normalized at infinity. These functions do, formally, satisfy the homogeneous equation

$$\frac{d^2 f_{\pm}(x,k)}{dx^2} + \left\{k^2 n^2(x) + \chi(x)\right\} f_{\pm}(x,k) = 0, \quad (4)$$

where

$$\chi(x) = -\sqrt{n(x)}\frac{d^2}{dx^2}\frac{1}{\sqrt{n(x)}} = -\frac{3\left(n'(x)\right)^2}{4n^2(x)} + \frac{n''(x)}{2n(x)}. \quad (5)$$

The function $\chi(x)$ is called Schwarzian (see Ref. 4, Vol. 5). The Schwarzian is invariant with respect to fractional linear transformations of the defining function $n(x)$.

Transform the homogeneous Eq. (2) to the inhomogeneous equation of the form

$$\frac{d^2 u(x,k)}{dx^2} + \left\{k^2 n^2(x) + \chi(x)\right\} u(x,k) = \chi(x) u(x,k). \quad (6)$$

It is convenient to consider Eq. (2) in the form of Eq. (6), as the functions (3) are solutions of the homogeneous equation for the functional in the left part of Eq. (6). Thus, the problem of constructing solutions of Eq. (2) is reduced to constructing solutions of the inhomogeneous Eq. (6). Note, that the performed transformations are rigorous and are not based on asymptotic considerations. Moreover, they do not require any smoothness of the medium properties. In the case when the function $n(x)$ can vanish the function $\chi(x)$ has singularities; but this case is not considered in this paper.

Following a conventional way, transform the inhomogeneous Eq. (6) to an integral equation establishing the corresponding conditions at infinity. For this purpose, construct the Green's function of the following equation:

$$\frac{d^2 G(x,k)}{dx^2} + \left\{k^2 n^2(x) + \chi(x)\right\} G(x,k) = \delta(x - x_0). \quad (7)$$

Since the complete set of solutions of the homogeneous equation defined by the functional in the left-hand part of Eq. (7) is known, the Green's function is to be defined as a linear combination of these solutions at $x < x_0$:

$$G(x,x_0) = C_+ f_+(x,k) + C_- f_-(x,k). \quad (8)$$

Consider the Green's function as zero at $x > x_0$. There are two conditions for determining the coefficients of this linear combination, i.e., continuity of the Green's function at the point $x = x_0$ and the constraint to the difference of its derivatives at this point. The linear set of equations follows from these conditions, by solving which we obtain

$$C_+ = f_-(x_0,k)/\Delta, \quad C_- = -f_+(x_0,k)/\Delta, \quad (9)$$

where

$$\Delta = f_+(x,k) f_-^{(1)}(x,k) - f_+^{(1)}(x,k) f_-(x,k). \quad (10)$$

Since the equation under study does not contain the first derivative, the function $\Delta(x, k) = \Delta(k)$ and its value (k and $n(x)$ are real) can be calculated, e.g., at $x \to \infty$. It is easy to show that this value equals to $\Delta = -2ik$. Again, after convenient transformations, we obtain the Green's function in the following form:

$$G(x,x_0,k) = \frac{\theta(x_0 - x)}{2ik\sqrt{n(x)n(x_0)}} \times$$

$$\times\left\{\exp\left[ik\int_x^{x_0} n(y)dy\right] - \exp\left[-ik\int_x^{x_0} n(y)dy\right]\right\}, \quad (11)$$

which is identically zero at $x > x_0$. Note, that the Green's function (11) plays an auxiliary role, so there is no need to formulate conditions for the radiation at infinity. Such conditions are meaningful only for the Green's function that corresponds to the initial problem on the emission of waves.

With the help of Green's function (11) Eq. (6) can be reduced to Volterra integral equation

$$u_+(x) = f_+(x) + \frac{1}{2ik}\int_x^{\infty}\frac{\chi(y)}{\sqrt{n(y)n(x)}}\left\{\exp\left[ik\int_x^y n(z)dz\right] - \right.$$

$$\left. - \exp\left[-ik\int_x^y n(z)dz\right]\right\} u_+(y)dy. \quad (12)$$

The same equation can be written in the form

$$u_+(x) = f_+(x) -$$

$$-\frac{1}{2ik}\int_x^{\infty}\chi(y)\left\{f_+(x)f_-(y) - f_-(x)f_+(y)\right\} u_+(y)dy. \quad (13)$$

Now introduce a pair of functions defined by the equations

$$\begin{cases} u_+^+(x,k) = f_+(x,k)\left\{1 - \dfrac{1}{2ik}\displaystyle\int_x^{\infty}\chi(y)f_-(y,k)u_+(y)dy\right\}, \\ u_+^-(x,k) = f_-(x,k)\dfrac{1}{2ik}\displaystyle\int_x^{\infty}\chi(y)f_+(y,k)u_+(y)dy. \end{cases} \quad (14)$$

Differentiating these functions, we obtain the set of differential equations

$$\begin{cases} \dfrac{du_+^+(x)}{dx} = \dfrac{f_+^{(1)}(x)}{f_+(x)}u_+^+(x) + \dfrac{\chi(x)}{2ikn(x)}u_+(x), \\ \dfrac{du_+^-(x)}{dx} = \dfrac{f_-^{(1)}(x)}{f_-(x)}u_+^-(x) - \dfrac{\chi(x)}{2ikn(x)}u_+(x). \end{cases} \quad (15)$$

The second set of differential equations defining the second linearly independent solution of the initial wave equation $u_-(x, k)$ can be obtained in a similar way. If one forms the first column of the matrix $Z(x, k)$ from $u_+^{\pm}(x,k)$ components of the solution $u_+(x, k)$ and the second column from the $u_-^{\pm}(x,k)$ components, then two sets of introduced equations take the following form, in the matrix representation

$$\frac{dZ(x,k)}{dx} = ikn(x)I_3 Z(x,k) -$$

$$-\frac{n'(x)}{2n(x)} I_0 Z(x,k) + \frac{\chi(x)}{ikn(x)} I_+ Z(x,k), \quad (16)$$

where I_α are the Pauli matrices, $I_+ = (I_3 + iI_2)/2$. In passing from system (15) and from the second one (analogous to it) system, to the matrix equation (16) the following equations were used:

$$f_+^{(1)}(x) = \left\{ikn(x) - \frac{n'(x)}{2n(x)}\right\} f_+(x),$$
$$f_-^{(1)}(x) = \left\{-ikn(x) - \frac{n'(x)}{2n(x)}\right\} f_-(x). \quad (17)$$

In passing to the new function by use of the substitution $Z(x,k) \Rightarrow Z_1^{[0]}(x,k)/\sqrt{n(x)}$, Eq. (16) takes the form

$$\frac{dZ_1^{[0]}(x,k)}{dx} = ikn(x) I_3 Z_1^{[0]}(x,k) + \frac{\chi(x)}{ikn(x)} I_+ Z_1^{[0]}(x,k). \quad (18)$$

It follows from Eq. (18) that the reflection ceases to decrease with the increase of frequency if the Schwarzian is limited, which is the case with acoustic waves.

Equation (18) can be written as

$$\frac{dZ_1^{[0]}(x,k)}{dx} = ikn(x) I_3 \left[1 - \frac{\chi(x)}{2k^2 n^2(x)}\right] Z_1^{[0]}(x,k) +$$
$$+ \frac{\chi(x)}{2kn(x)} I_2 Z_1^{[0]}(x,k). \quad (19)$$

If one introduces the designations that

$$\alpha_3^{[0]}(x) = kn(x)\left[1 - \frac{\chi(x)}{2k^2 n^2(x)}\right]$$

and (20)

$$\alpha_2^{[0]}(x) = \frac{\chi(x)}{2kn(x)},$$

then Eq. (19) takes the form

$$\frac{dZ_1^{[0]}(x,k)}{dx} = i\alpha_3^{[0]}(x) I_3 Z_1^{[0]}(x,k) +$$
$$+ \alpha_2^{[0]}(x) I_2 Z_1^{[0]}(x,k). \quad (21)$$

The subscript of the coefficients shows that this is the coefficient of the Pauli matrix with the same subscript, while the superscript indicates the coefficient at the corresponding iteration step. The subscript of the target matrix points to a missing item with the corresponding Pauli matrix while the superscript shows the cycle number of the three successive eliminations of the equation terms. This is considered in detail below.

Scheme of constructing the solution

Consider the scheme for constructing a sequence of approximations to the exact solution on the basis of Eq. (21). To do this, demand the following condition to hold

$$\lim_{x\to+\infty} Z_1^{[0]}(x,k) \exp(-ikI_3 x) = I_0, \quad (22)$$

in order to fix the set of solutions sought. Introduce the function $Y_3^{[1]}(x,k)$, which is a solution of the equation

$$\frac{dY_3^{[1]}(x,k)}{dx} = i\alpha_3^{[0]}(x) I_3 Y_3^{[1]}(x,k) \quad (23)$$

with the condition of the type (22) at infinity. In the case of real $n(x)$ and k the function $\alpha_3^{[0]}(x)$ is to be real and the matrix exponent $Y_3^{[1]}(x,k)$ will have the view of the sum of exponents with matrix coefficients. Represent this matrix-function in the form

$$Y_3^{[1]}(x) = \exp\left\{i\beta_3^{[1]}(x) I_3\right\} =$$
$$= I_0 \cos\left[\beta_3^{[1]}(x)\right] + iI_3 \sin\left[\beta_3^{[1]}\right], \quad (24)$$

where

$$\beta_3^{[1]}(x) = \alpha_3^{[0]}(\infty)x - \int_x^\infty \left\{\alpha_3^{[0]}(y) - \alpha_3^{[0]}(\infty)\right\} dy. \quad (25)$$

Such a function form has been chosen to ensure the fulfillment of the condition (22) for the matrix $Y_3^{[1]}(x,k) = Y_3^{[1]}(x)$. Note, that such a form of this (matrix) function is similar to the geometric optics approximation but with somewhat different phase in the exponent.

Let us introduce the designation $Z_1^{[0]}(x,k) = Y_3^{[1]}(x,k) Z_3^{[1]}(x,k)$ that allows Eq. (21) to be reduced to the following equation:

$$\frac{dZ_3^{[1]}(x)}{dx} = \alpha_2^{[0]}(x) \times$$
$$\times \left\{I_2 \cos\left[2\beta_3^{[1]}(x)\right] - I_1 \sin\left[2\beta_3^{[1]}(x)\right]\right\} Z_3^{[1]}(x). \quad (26)$$

Here the Pauli matrix property $I_2 I_3 = iI_1$ is used. Introducing the corresponding notations, Eq. (26) can be written in the standard form

$$\frac{dZ_3^{[1]}(x)}{dx} = \alpha_2^{[1]}(x) I_2 Z_3^{[1]}(x) + \alpha_1^{[1]}(x) I_1 Z_3^{[1]}(z), \quad (27)$$

where

$$\alpha_2^{[1]}(x) = \alpha_2^{[0]}(x) \cos\left[2\beta_3^{[1]}(x)\right],$$
$$\alpha_1^{[1]}(x) = -\alpha_2^{[0]}(x) \sin\left[2\beta_3^{[1]}(x)\right], \quad (28)$$

which have been introduced according to the above-stated principle. The condition for the new matrix at infinity takes the form

$$\lim_{x\to+\infty} Z_3^{[1]}(x,k) = I_0. \quad (29)$$

Now introduce the solution $Y_2^{[2]}(x,k)$ of the equation

$$\frac{\mathrm{d}Y_2^{[2]}(x)}{\mathrm{d}x} = \alpha_2^{[1]}(x)I_2Y_2^{[2]}(x) \qquad (30)$$

with the condition of the type (29) at infinity. This solution has the form

$$Y_2^{[2]}(x) = \exp\left\{-I_2\int_x^\infty \alpha_2^{[1]}(y)\mathrm{d}y\right\} =$$

$$= I_0\cosh\left[\int_x^\infty \alpha_2^{[1]}(y)\mathrm{d}y\right] - I_2\sinh\left[\int_x^\infty \alpha_2^{[1]}(y)\mathrm{d}y\right]. \qquad (31)$$

By setting $Z_3^{[1]}(x) = Y_2^{[2]}(x)Z_2^{[2]}(x)$, we obtain the equation

$$\frac{\mathrm{d}Z_2^{[2]}(x)}{\mathrm{d}x} = \alpha_1^{[2]}(x)I_1Z_2^{[2]}(x) + i\alpha_3^{[2]}(x)I_3Z_2^{[2]}(z) \qquad (32)$$

for the matrix $Z_2^{[2]}(x)$ from Eq. (27). The coefficients in Eq. (31) are defined by the equations

$$\alpha_1^{[2]}(x) = \alpha_2^{[1]}(x)\cosh\left[2\int_x^\infty \alpha_2^{[1]}(y)\mathrm{d}y\right],$$

$$\alpha_3^{[2]}(x) = -\alpha_2^{[1]}(x)\sinh\left[2\int_x^\infty \alpha_2^{[1]}(y)\mathrm{d}y\right]. \qquad (33)$$

If one introduces now (the third step) the solution of the equation

$$\frac{\mathrm{d}Y_1^{[3]}(x)}{\mathrm{d}x} = \alpha_1^{[2]}(x)I_1Y_1^{[3]}(x), \qquad (34)$$

then, representing the solution as

$$Y_1^{[3]}(x) = \exp\left\{-I_1\int_x^\infty \alpha_1^{[2]}(y)\mathrm{d}y\right\} =$$

$$= I_0\cosh\left[\int_x^\infty \alpha_1^{[2]}(y)\mathrm{d}y\right] - I_1\sinh\left[\int_x^\infty \alpha_1^{[2]}(y)\mathrm{d}y\right], \qquad (35)$$

Eq. (32) can be reduced to the form

$$\frac{\mathrm{d}Z_1^{[3]}(x)}{\mathrm{d}x} = i\alpha_3^{[3]}(x)I_3Z_1^{[3]}(x) + \alpha_2^{[3]}(x)I_2Z_1^{[3]}(z). \qquad (36)$$

where

$$\alpha_3^{[3]}(x) = \alpha_3^{[2]}(x)\cosh\left[2\int_x^\infty \alpha_1^{[2]}(y)\mathrm{d}y\right],$$

$$\alpha_2^{[3]}(x) = -\alpha_3^{[2]}(x)\sinh\left[2\int_x^\infty \alpha_1^{[2]}(y)\mathrm{d}y\right]. \qquad (37)$$

Equation (36) is analogous to Eq. (21) and, hence, its solution can be constructed by the above-described algorithm. The only distinction is that the condition at infinity is to be analogous to Eq. (29) for all the matrices introduced. Emphasize, that the above-considered algorithm allows recurrent construction of the successive approximations within a formally rigorous scheme at weak constrains on the coefficients of the initial equation.

Consider some properties of the equations obtained. If one interprets the functions $u_+^\pm(x,k)$ and $u_-^\pm(x,k)$ introduced as parts of the solution describing the energy transfer along one of two possible directions, then the reflection function can be introduced (by definition) by setting $R_+(x,k) = u_+^-(x,k)/u_+^+(x,k)$. For this function it follows, from Eq. (18), the Riccati-type equation

$$\frac{\mathrm{d}R_+(x,k)}{\mathrm{d}x} = -2ikn(x)R_+(x,k) - -\frac{\chi(x)}{2ikn(x)}\left[1 + R_+(x,k)\right]^2. \qquad (38)$$

Note, that the reflection function obeys the zero conditions at infinity, which corresponds to the absence of reflection from an infinitely distant part of the medium. This is true if the field source is situated to the left of the inhomogeneous part. Analogous equation can be obtained for the second linearly independent solution.

Conclusion

The presented approach to study solutions of Eq. (1) formally resembles the geometrical optics method for the one-dimensional case at its first steps. In fact, this is different approach based on the fact that if the solution of an equation with close, in some sense, coefficients is known, the problem can be set on studying properties of solutions of the target equation using an integral equation obtained from an inhomogeneous differential equation. Which solutions of an auxiliary equation are to be used for integral equation construction is determined by a problem and can be unrelated with the geometrical optics method at all. Thus, the approach used in this work is general.

Besides, the recurrent scheme for construction of the sequence of approximations to the exact solution is proposed, which is formally independent on any supposition on the coefficients of the initial equation. The scheme proposed can also be used to study wave propagation through stratified media. Some changes are possible in the case of reflecting boundaries, e.g., like in problems of vertical sounding of the ionosphere.

Practical efficiency of the scheme proposed can be studied based on mathematical simulation of the propagation processes in various cases of independent interest.

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