RECONSTRUCTION OF WAVE FRONT SET MODES FROM IMAGE FUNCTIONALS

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We propose a method for reconstruction of the wave front modes from the functionals of point spread function on a present set.

Imaging properties of an optical system (OS) are characterized by an aberration function $\Phi(\xi, \eta)$ of the wave front (WF) at the exit pupil Ω. The wave function of a field[1] from a point source at the recording plane (z = const) of an OS with the aberration function $\Phi(\xi, \eta)$ is described, accurate to a constant factor by the function

$$g(x, y, z, \Phi) = \iint_{\Omega} e^{-iz(\xi^2 + \eta^2)/2}\, e^{-(x\xi + y\eta) + k\Phi(\xi,\eta)}\, d\xi\, d\eta, \qquad (1)$$

where $k = 2\pi/\lambda$ is the wave number. The field intensity at (x, y, z) point is $h(x, y, z, \Phi) = |g(x, y, z, \Phi)|^2$. The measuring device adds noise to this intensity $I(x, y, z, \Phi) = h(x, y, z, \Phi) + \varepsilon(x, y)$. Let us assume that the aberration function $\bar{\Phi}(\xi, \eta)$ and the intensity $I(x, y, z, \bar{\Phi})$ on the set ω on the recording plane correspond to an actual realized WF, whereas the intensity $h(x, y, z, \Phi)$ calculated with the help of integral (1) corresponds to an arbitrary function $\Phi(\xi, \eta)$. Then problem on the WF reconstruction using a physical model of image formation reduces to the determination of the function $\Phi(\xi, \eta)$ from the equation

$$I(x, y, z, \bar{\Phi}) = h(x, y, z, \Phi) + \varepsilon(x, y),\ (x, y) \in \omega \qquad (2)$$

with known left-hand side and probability parameters of the noise ε.

Equation (2) makes the basis of different indirect methods of the aberration function determination. One way to solve equation (2) consists in the function $\Phi(\xi, \eta)/\lambda$ representation with a finite segment of a series over some basis functions

$$\Phi/\lambda = \sum_{s=1}^{N} \zeta_s\, \Phi_s(\xi, \eta). \qquad (3)$$

The initial problem reduces to the determination of coefficients vector (modes) $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_N)$ from the equation (2) written in the form

$$I(x, y, z, \bar{\zeta}) = h(x, y, z, \zeta) + \varepsilon(x, y),\ (x, y) \in \omega. \qquad (4)$$

Reconstruction of the function Φ from the equation (4) has been first proposed by Sautwel.[2] He solved it by the minimization method of weighted quadratic discrepancy $S(z, \zeta)$ between I and h functions. Numerical modeling in Ref. 2 provides for a reliable evaluation of the solution $\bar{\zeta}$ only at very small values and few modes.

In Ref. 3 the generalized discrepancy $S(\zeta) = \sum_q S(z_q, \zeta)$ is proposed that takes into account measurements in several planes. The numerical modeling based on the generalized discrepancy gave reliable estimation of the mode vector in a number of cases, when the method from Ref. 2 didn't provide such an estimation. The interest in the equation (4) is due to the fact that, being successfully solved, it gives a simple WF reconstruction method.

In this paper solution of equation (4) is considered using a modified iterative method by Newton[4] using equations:

$$\zeta_0 = 0, \quad I(x, y, z, \bar{\zeta}) - h(x, y, z, \zeta_k) = \frac{\partial h(x, y, z, 0)}{\partial \zeta}(\zeta_{k+1} - \zeta_k) + \varepsilon(x, y),$$
$$k = 0, 1, 2, \dots . \qquad (5)$$

The choice of the initial approximation $\zeta_0 = 0$ is not occasional. First, according to the problem conditions, the modes often cannot be large. Second, at $\zeta = 0$ the analysis of the partial derivatives vector-string $dh/d\zeta$ is simplified.. Third, if the OS is adaptive, then the WF correction leads to $\bar{\zeta} \to 0$. In adaptive systems the correction can be made at every iteration based on the modes estimations in the first approximation. Such an approach was considered in Ref. 5 and was called the instrumental iterative method.

At every iteration the solution of the linear equality (5) is performed relative to $\zeta_{k+1} - \zeta_k$ difference which, due to the noise and linearization error, reduces to the compromise projection of the left-hand side of equation (5) onto the linear subspace L_N, defined by partial derivatives $dh/d\zeta_s$ on the set ω. Therefore it is important that these partial derivatives are linearly independent . The linear independence can

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be provided by changing the measurement scheme and OS parameters. Among these are the z-coordinate of the measurement plane, the intensity measurement area ω, and so on.

The problem of projection of the equation (5) left-hand side on L_N can be reduced to solution of a system of linear algebraic equations

$$I_j(z, \bar{\zeta}) - F_j(h(z, \zeta_k)) = \\ = \sum_{s=1}^{N} F_j(\partial h(z, 0)/\partial\zeta_s)\,(\zeta_{k+1} - \zeta_k) + \varepsilon_j,\ j = 1, ..., N, \quad (6)$$

where F_j are continuous linear functionals of the function $h(z, \zeta) = h(x, y, z, \zeta)$ where x and y are variables while z and ζ, being parameters. $I_j(z, \zeta)$ is the variant of the $F_j\,(h(z, \zeta))$ functionals distorted by ε_j random noise components. Let us call F_j functionals the image functionals.

The problem is to choose the functionals F_j in a form that provides the matrix $A(z, 0) = (F_j\,(\mathrm{d}h(z, 0)/\mathrm{d}\zeta_s))$ to be well-posed, and the iterative method to be rapidly convergent.

The first method of image functionals choice is obvious. An example of this is a biorthogonal system of functionals $\{F_j\}$ corresponding to the system of functions $\{\mathrm{d}h(x, y, z, 0)/\mathrm{d}\zeta_s\}$. Then $A(z, 0) = E$ is a unit matrix. In this case the left-hand side of (6) immediately gives the difference $\Delta\zeta = \zeta_{k+1} - \zeta_k$ with accuracy ε_j.

Biorthogonal system of functionals is derived from the linear equalities

$$F_j\,(\partial h(z, 0)/\partial\zeta_s) = \delta_{sj}, \quad s = \overline{1, N}\,, \quad (7)$$

where δ_{js} are the Kronecker symbols. At a fixed j the problem of F_j determination from (7) is called the finite-dimensional moments problem, which is well studied. If one considers the $\mathrm{d}h/\mathrm{d}\zeta_s$ derivatives as elements in Hilbert space, then the linear functional is given by a scalar product $F(\mathrm{d}h/\mathrm{d}\zeta_s) = (F, \mathrm{d}h/\mathrm{d}\zeta_s)$, where F is the element of that same space. The functional of a minimum norm which solves the problem (7) has the form

$$F_j = \sum_{k=1}^{N} \frac{\partial h}{\partial\zeta_k}\gamma_{kj} = \frac{\partial h}{\partial\zeta}\gamma_j\,.$$

Substitution of this expression into (7) leads to the system of equations for the γ_{kj} coefficients

$$\sum_{k=1}^{N} (\partial h/\partial\zeta_s, \partial h/\partial\zeta_k)\,\gamma_{kj} = \delta_j$$

or, in a matrix form, $\Gamma\gamma_j = E_j$, where E_j is jth column of the unit matrix. The coefficient vector γ_j is thus the jth column of the inverse matrix Γ^{-1}. The solution $\Delta\zeta$ obtained with the help of biorthogonal functionals

$$\Delta\zeta_s = F_s(\Delta h),\ \Delta h = h(x, y, z, \bar{\zeta}) - h(x, y, z, \zeta_n), \quad (8)$$

corresponds to ζ determined by the least squares method.

$$\min_{\zeta} \left\| \Delta h - \frac{\partial h}{\partial\zeta}\zeta \right\|^2.$$

The necessary condition of the extremum leads to the matrix equation

$$\Gamma_\zeta = (\Delta h, \partial h/\partial\zeta)^T,$$

from which it follows that $\zeta_j = (\Delta h, \partial h/\partial\zeta)\,\gamma_j = F_j(\Delta h)$.

When employing Tikhonov regularization of the projection Δh on L_N, the vector ζ is a solution of the problem

$$\min_{\zeta} \left\| \Delta h - \frac{\partial h}{\partial\zeta}\zeta \right\|^2 + \alpha\|\zeta\|^2,$$

where α is the regularization parameter, which, in our case, should be so that it provides for the iteration method (6) convergence when there is a noise in the system. The solution of this problem is unique, and it is determined by the same inequality (8) in which the functional $F_j = (\partial h/\partial\zeta)\,\gamma_j$, where γ_j is the jth column of the $(\Gamma + \alpha E)^{-1}$ matrix.

F_j functionals with Tikhonov regularization can be obtained from the solution of the finite-dimensional moments problem

$$F_j(\partial h/\partial\zeta) + \alpha\varepsilon^{\mathrm{T}} E_s = \delta_{sj}, \quad s = \overline{1, N}\,, \quad (9)$$

where ε is the vector characterizing the discrepancy among linear equalities (7).

The left-hand side of Eq. (9) can be considered as a linear functional defined by the (F_j, ε) pair on the direct product $L_2(\omega) \times R^N$ which takes the δ_{sj} values on the elements $(\mathrm{d}h/\mathrm{d}\zeta_s, E_s)$. The functional (F_j, ε) with the minimum norm $(\|F_j\|^2 + \alpha\|\zeta\|^2)^{1/2}$ that gives a solution to the finite-dimensional moments problem, yields, when substituted in Eq. (8), a vector that exactly coincides with that obtained using Tikhonov regularization.

If the linear independence of $\mathrm{d}h/\mathrm{d}\zeta_s$ derivatives on ω is weak, similar to the two non-collinear vectors located on the plane with small angle between them, then biorthogonal functionals can give, using formula (8), unacceptably large values of $\Delta\zeta$ difference. In this case one can look for image functionals using a more general finite-dimensional moments problem (9) where $F_j \in U$, $\varepsilon \in V$. The sets U and V determine the properties of F_j and ε and constraints on them, and, consequently the regularization type.

In conclusion of this section let us note that the method for reconstruction of the set modes successfully used in Ref. 5 may be interpreted as a method for the set modes reconstruction from image functionals, which were taken as sine and cosine Fourier-transformations at discrete frequencies.

The choice image functionals depends on the basis functions. Two bases, often used in optics, are considered below. These are the Zernike polynomials on a circle and piece-wise linear functions on a segmented pupil.

Zernike modes. Let the circular Zernike polynomials serve as the basis functions on the round aperture $\Omega = \{(\xi,\eta)\colon \xi^2 + \eta^2 \le 1\}$.

$$\Phi^m_n(\rho,\theta) = \left(\begin{smallmatrix}\cos m\theta\\ \sin m\theta\end{smallmatrix}\right) R^m_n(\rho),$$

$$m \le M, \quad n = m + 2l \le N,$$

where (ρ,θ) are polar coordinates on Ω; M and N are the limiting numbers of modes. Let us denote the set modes of the basis functions as $\zeta^m_n = \begin{pmatrix} c^m_n \\ s^m_n \end{pmatrix}$.

Let us also show that, by choosing z, one may provide the linear independence of the derivatives $\partial h/\partial\zeta^m_n$ on the circle $\omega = \{(x,y) : x^2 + y^2 \le V\}$, where the radius V, generally speaking, depends on the number of modes. Let (ν,ψ) be the polar coordinates of the (x,y) point. Taking into account the form of the function $g(x,y,z,0)$ and integral representation of the first type Bessel functions[1,6] one obtains

$$g(\nu,\psi,z,0) = 2\pi\, g^0_0(\nu,z);$$

$$\partial g(\nu,\psi,z,0)/\partial\zeta^m_n = 4\pi^2 (i)^{m+1} \left(\begin{smallmatrix}\cos m\psi\\ \sin m\psi\end{smallmatrix}\right) g^m_n(\nu,z),$$

where

$$g^m_n(\nu,z) = \int_1^0 \mathrm{e}^{-iz\rho^2/2} R^m_n(\nu)\, J_m(\nu\rho)\, \rho\, \mathrm{d}\rho;$$

$$\partial h/\partial\zeta^m_n = 16\pi^3 \left(\begin{smallmatrix}\cos m\psi\\ \sin m\psi\end{smallmatrix}\right) r^m_n(\nu,z),$$

where $r^m_n(\nu,z) = \mathrm{Re}\,[i^{m+1} g^0_0(\nu,z)\, g^m_n(\nu,z)]$.

The derivatives $\partial h/\partial\zeta^m_n$ will be linearly independent on the circle ω if they represent the $r^m_n(\nu,z)$ functions on $[0, V]$. Let us show that the linear independence of $r^m_n(\nu,z)$ functions can be provided by a proper choice of z. Assume that z coordinate is small enough, so that the functions $r^m_n(\nu,z)$ linearization on z can be performed at the point $z = 0$

$$r^m_n(\nu,z) = r^m_n(\nu,0) + \frac{\partial r^m_n(\nu,0)}{\partial z}\, z.$$

Using the radial polynomials properties one can derive their explicit form

$$g^m_n(\nu,0) = (-1)^{(n-m)/2} J_{n+1}(\nu)/\nu;$$

$$\frac{\partial g^m_n(\nu,0)}{\partial z} = -\frac{i}{2A^m_1}(-1)^{(n-m)/2}\,[J_{n+3}(\nu) -$$

$$- B^m_1 J_{n+1}(z) + D^m_1 J_{n-1}(\nu)]/\nu;$$

with $n > m$ and

$$\frac{\partial g^m_n(\nu,0)}{\partial z} = -i\left[\frac{J_{m+1}(\nu)}{2\nu} - \frac{J_{m+2}(\nu)}{\nu^2}\right],$$

where A^m_1, B^m_1, D^m_1 are the coefficients of the recurrence formula for the radial polynomials.[6]

The functions $g^m_n(\nu,0)$ are real, whereas $\partial g^m_n(\nu,0)/\partial z$ derivatives are imaginary. Therefore, at small z and odd m we have

$$r^m_n(\nu,z) = (-1)^{(m+1)/2} g^0_0(\nu,0)\, g^m_n(\nu,0) =$$

$$= (-1)^{(n+1)/2} J_1(\nu)\, J_{n+1}(\nu)/\nu^2,$$

and for even m

$$r^m_n(\nu,z) =$$

$$= (-1)^{m/2} zi \left(\frac{\partial g^0_0(\nu,0)}{\partial z} g^m_n(\nu,0) + g^0_0(\nu,0)\frac{\partial g^m_n(\nu,0)}{\partial z}\right).$$

Last expressions show that the $r^m_n(\nu,z)$ functions, at different n contain Bessel functions of different orders therefore these functions are linearly independent on any segment $[0, V]$. It is remarkable that the structure of partial derivatives $\partial h/\partial\zeta^m_n$ has the view of basis functions. As a result trigonometric components of the angle θ transform into similar components of the angle ψ, whereas Zernike radial functions transform into the functions proportional to $r^m_n(\nu,z)$. Taking into account this circumstance together with the orthogonality property of trigonometric components of the function, one should seek the determining functionals in the form

$$F^m_n(\nu,\psi) = \left(\begin{smallmatrix}\cos m\psi\\ \sin m\psi\end{smallmatrix}\right) f^m_n(\nu), \quad n = m + 2l \le N.$$

Functions $f^m_n(\nu)$ will be sought, in accordance with the Eq. (9), from the finite-dimensional problem of moments solution

$$16\pi^2 f^m_{m+2p}\,(r^m_{m+2l}) + \alpha\varepsilon^{\mathrm{T}} \mathbf{e}_l = \delta_{lp},\ l = \overline{1, L},$$

where $\mathbf{e}_l$ is the first column of the order matrix, L is the integer part of N-m number, and ε is the discrepancy vector of the length L.

ThE modEs of a sEgmEntEd mirror. Let the exit pupil area be formed by n hexagonal segments Ω whose centers are at (ξ_s, η_s), $s = \overline{1, n}$, points. Let us describe the WF on the aberration segment as a linear function $\alpha_s + \beta_s (\xi - \xi_s) + \gamma_s (\eta - \eta_s)$. Here α_s characterizes the phase deviation of a segment, while the angles β_s and γ_s give the misalignment values. Let us denote the characteristic function of the segment with the center at the origin of the coordinates as $\delta(\xi, \eta)$. Then the basis functions represent orthogonal, on Ω, functions. Let us denote the set modes of basis functions $\Phi_s(\xi, \eta)$ as $\zeta_s = (\alpha_s, \beta_s, \gamma_s)^T$. We suppose that the pupil Ω doesn't contain the central segment. Segments form the belt zones. The first zone consists of 6 segments, the second one from 12, the third one from 18, and so on. In every zone segments can be combined into groups of 6 segments which transform into each other by rotation on an angle multiple of $\pi/3$ relative to the coordinates origin. Let us denote, as $p(s)$, the number of the segment into which the segment s transforms by turn of the pupil area on an angle ω.

Let $F_j = (F_0, F_1, F_2)$ be functionals vector which discriminate the set modes vector ζ_j

$$\int\limits_{\omega} \mathbf{F}_j(x, y) \frac{\partial h(x, y, z, 0)}{\partial \zeta_s} \zeta_s \, dx \, dy = \zeta_s \, \delta_{sj} ,$$

$$s = \overline{1, n} .$$

For $p(j)$ segment of the same group as the segment j, let us consider the functional

$$\int\limits_{\omega} \mathbf{F}_j(x \cos\varphi + y \sin\varphi - x \sin\varphi + y \cos\varphi) \times$$

$$\times \frac{\partial h(x, y, z, 0)}{\partial \zeta_p} \zeta_p \, dx \, dy, \qquad (10)$$

where φ is the angular distance between the segments' k and p centers. Let us turn the coordinate systems Oxy and $O\xi\eta$ at an angle φ. Let us denote the points coordinates in a new coordinate systems by subscript l. From the symmetry of the segments' positions one has

$$g(x, y, z, 0) = g(x_l, y_l, z, 0);$$

$$\frac{\partial g(x, y, z, 0)}{\partial \zeta_p} \zeta_p = \frac{\partial g(x_l, y_l, z, 0)}{\partial \zeta_j} \zeta_{pl} ,$$

where $\zeta_{s1} = (1, \beta_{s1}, \gamma_{s1})^T$ is the set modes vector of the segment s relative to the turned coordinate system, that is

$$\begin{pmatrix} \beta_s \\ \gamma_s \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \sin\varphi \end{pmatrix} \begin{pmatrix} \beta_{s1} \\ \gamma_{s1} \end{pmatrix} .$$

The integral (10) in a new coordinate system equals to

$$\int\limits_{\omega_1 = \omega} \mathbf{F}_j(x_1, y_1) \frac{\partial h(x_1, y_1, z, 0)}{\partial \zeta_s} \zeta_{p(s)1} \, dx_1 \, dy_1 =$$

$$= \zeta_{p(s)1} \, \delta_{sj}.$$

Thus, it is proved that segments of each group the distribution of the function $\mathbf{F}(x, y)$ values coincides accurate to the angle of the turn.

REFERENCES

1. M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1959).
2. D. Freed, ed., *Adaptive optics* [Russian translation] (Mir, Moscow, 1980), 456 pp.
3. G.L. Degtyarev and S.M. Chernyavskii, in: *Adaptive Optics (review)*, (Kaz. Avia. Instutute, Kazan; 1986), pp. 1–7
4. L.I. Kantorovitch and G.P. Akilov, *Functional Analysis* (Nauka, Moscow, 1977), 740 pp.
5. G.L. Degtyarev, A.V. Makhan'ko, and S.M. Chernyavskii, Atmos. Oceanic Opt. **9**, No. 3, 258–260 (1996).
6. D.S. Kuznetzov, *Special Functions* (Vysshaya Shkola, Moscow, 1965), 272 pp.