Diffraction field from sector areas in the Fresnel zone

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The paper describes the results of analytical solutions of the Fresnel diffraction problem obtained for the general case of a sector aperture and, as a consequence, for the case of the angular area representing the part of the infinite plane. The results of the calculations of diffraction field spatial structure, carried out with the use of the found solutions are presented for some practically important situations.

Introduction

The Fresnel diffraction, described by the square approximation of the general Kirchhoff formula and confirmed by numerous experiments refers to the classical problems.[1] As is known the diffraction field in the Fresnel zone is reduced to the integral over the aperture area with a fast-varying integrand. Depending on the aperture shape, such integrals are rarely calculated explicitly. At the same time, at large values of phase parameter, the direct numerical methods become here useless. In such cases their asymptotic estimate is used. However, the asymptotic methods have their limits as well. Therefore, a search for analytical solutions for the Fresnel diffraction from the areas of various configurations is of great interest.

In the field of optics such solutions are necessary for obtaining three-dimensional light distribution near the focus, that is especially important for estimating the magnitude of allowance in the required position of plane of systems, forming the image. In the microwave part of radiowaves they are also necessary for analysis of spatial field structure from different diffraction systems (minimizing screens, complex terrain of area: mountains, hills, artificial buildings, and so on).

In spite of the fact that the diffraction problem in the Fresnel formulation of the problem has a long-term history, its analytical solutions were found only for apertures of two configurations: rectangular and round. Both solutions were obtained by Lommel in 1885 and 1886 (Refs. 2 and 3). Apertures of the sector shape have not deserved such attention, although the interest to them in the microwave part of radiowaves became much greater.

Sector apertures are of interest also due to the analytical generality of the solution of the problem, which can serve a basis for constructing a series of solutions for diffraction from other angular and segment areas, where the analytical solutions are lacking.

Diffraction field from sector aperture

In the case of a spherical wave incident from a point source on the aperture plane of radius R and the angle φ it is reasonable to select the polar coordinates ρ, φ with the center coinciding with the sector aperture center (Fig. 1).

In fact, the problem is in deriving an analytical expression for a multiplier of diffraction extinction $\Phi = W/W_0$, where W is the diffraction field of the aperture, and $W_0 = \exp[(i2\pi r_0)/\lambda]/r_0$ is the field of a direct unperturbed wave at an observation point behind the aperture at a distance r_0 from the source; λ is the wavelength.

The sought function Φ, depending on whether the beam passes through the aperture or beyond it, is found as a sum or difference of results of integration over two adjacent constituent sectors when the beam passes along their common edge (Fig. 1):

$$\Phi(\rho_0,\varphi_0) = \Phi(\rho_0,\varphi_1) \pm \Phi(\rho_0,\varphi_2), \quad (1)$$

where

$$\Phi(\rho_0,\varphi_j) = -\frac{i}{2}\pi n \exp(i\pi n_0)\int_0^1 \exp(i\pi n u^2)u \times \left[\frac{2}{\pi}\int_0^{\varphi_j} \exp(-i2\pi\sqrt{nn_0}u\cos\varphi_j)\,\mathrm{d}\varphi\right]\mathrm{d}u. \quad (2)$$

Here $\sqrt{n} = R/b_1$, $\sqrt{n_0} = \rho_0/b_1$ are relative values of the sector radii and the point of the beam passage in the Fresnel zones, where $b_1 = \sqrt{[\lambda r_1(r_0 - r_1)]/r_0}$ is the radius of the first Fresnel zone in the plane of the aperture, r_1 is the distance from the aperture to the source; $u = \rho/R$ is the integration variable relative to the sector radius magnitude.

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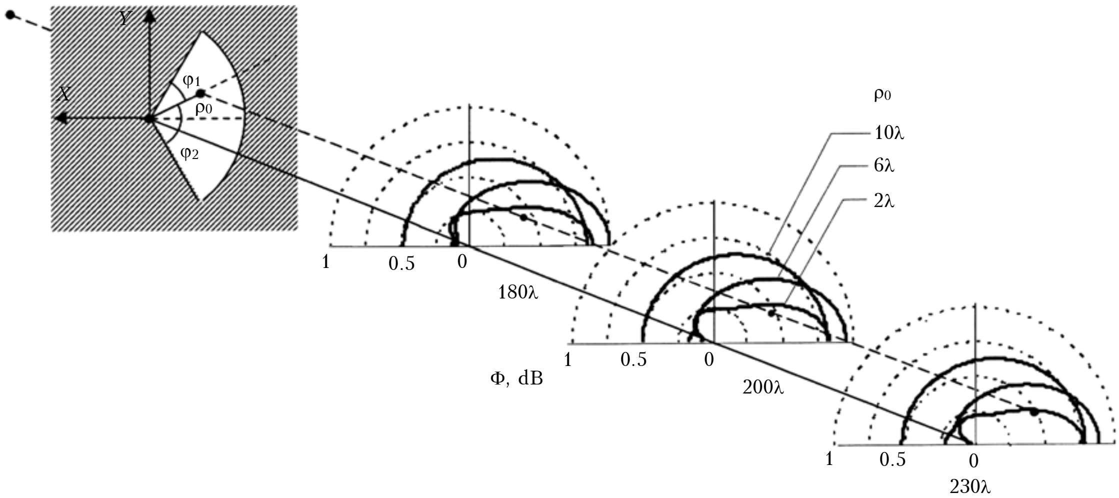

Fig. 1. Geometry of the problem and spatial structure of the diffraction field (multiplier of the diffraction extinction Φ) for the sector aperture with the radius $R = 10.95\lambda$ and the angle $\varphi = 2\pi/3$. Distances from the aperture to the source $r_1 = 300\lambda$, up to the observation planes $r_2 = 180\lambda$, 200λ, and 230λ. Every curve corresponds to a definite distance from the central axis: $\rho_0 = 10\lambda$, 6λ, and 2λ.

As a result of labor-consuming mathematical procedures with the use of properties of incomplete cylindrical functions,[4] which were not given here in detail, for Eq. (2) the following analytical expression was derived

$$\Phi(\rho_0,\varphi) = -\frac{\exp\left[i\pi(n_0+n)\right]}{4}\left[V_0^-(\varphi,p) - iV_1^-(\varphi,p)\right] -$$

$$-\frac{i}{2}F(v_3)\left[F(v_2) - F(v_1)\right] + \frac{\exp(i\pi n_0)}{4}\times$$

$$\times\left[\sum_{k=0}^{\infty}(-1)^k\frac{(2\pi n_0)^{2k}}{A_{2k}}C_{2k} - i\sum_{k=0}^{\infty}(-1)^k\frac{(2\pi n_0)^{2k+1}}{A_{2k+1}}C_{2k+1}\right]. \quad (3)$$

Here, by analogy with the Lommel task on the circle,[3] the functions are introduced having an external similarity with the Lommel functions[6]:

$$V_S^-(\varphi,p) = \sum_{k=0}^{\infty}(-1)^k\left(\sqrt{\frac{n_0}{n}}\right)^{2k+S}E_{2k+S}^-(\varphi,p), \quad (4)$$

where $E_\nu^-(\varphi,p)$ are incomplete cylindrical functions in the Poisson form[4,5] having the integral representation

$$E_\nu^-(\varphi,p) = \frac{2p^\nu}{A_\nu}\int_0^\varphi \exp(-ip\cos t)\sin^{2\nu}t\,\mathrm{d}t \quad (5)$$

and the representation in the form of power series relative to p

$$E_\nu^-(\varphi,p) = \frac{p^\nu}{A_\nu}\sum_{k=0}^{\infty}C_{k,\nu}(\varphi)\frac{(-ip)^k}{k!},$$

where

$$C_{k,\nu}(\varphi) = 2\int_0^\varphi \cos^k t\sin^{2\nu}t\,\mathrm{d}t.$$

Besides, $F(v)$ in Eq. (3) is the known Fresnel integral[6] with parameters

$$v_1 = -\sqrt{2n_0}\cos\varphi,\quad v_2 = v_1 + \sqrt{2n},\quad v_3 = \sqrt{2n_0}\sin\varphi;$$

$$C_\nu = 2\int_0^\varphi \sin^{2\nu}t\,\mathrm{d}t;\quad A_\nu = 2^\nu\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right),$$

where $\Gamma(\nu + 1/2)$ is the gamma-function with the natural values ν; $p = 2\pi\sqrt{n_0}\sqrt{n}$.

All the series, appearing in Eq. (3), are absolutely convergent, but suitable for calculations at $n_0 \le n$, i.e., in the observation plane within the limits of the circle near the axis with the radius of the diffraction sector hole. The obtained solution is valid for arbitrary values of the radius and the sector aperture angle.

One of special cases, confirming the validation of the obtained solution, is the problem of diffraction from a circular aperture. In this case at $\varphi = 2\pi$ the incomplete cylindrical functions in the Poisson form (5) become the Bessel functions $J_\nu(p)$ (Ref. 6), the introduced functions $V_S^-(\varphi,p)$ (4) are transformed in the Lommel functions $V_S(n, n_0)$ (Ref. 6), the second term in Eq. (3) vanishes because of $v_3 = 0$, and the third term becomes a unity, because the multiplier in square brackets at $\varphi = 2\pi$ is simply a series expansion of the function $\exp(-i\pi n_0)$. Hence, we come to the known Lommel solution[1,3]:

$$\Phi(\rho_0,2\pi) = 1 - \exp\left[i\pi(n_0+n)\right]\left[V_0(n,n_0) - iV_1(n,n_0)\right],$$

where

$$V_S(n,n_0) = \sum_{k=0}^{\infty}(-1)^k\left(\sqrt{\frac{n_0}{n}}\right)^{2k+S}J_{2k+S}(p)$$

is the Lommel function.

Diffraction field from the angular area

The solution (3) due to its generality can be used for constructing a series of solutions for diffraction from other angular and segment regions where the analytical solutions are lacking. One of such regions is the part of an infinite plane limited by a definite angle. It is evident that such a region is a special case of the sector with infinitely high value of its radius. Therefore, to find the analytical expression for the multiplier of the diffraction extinction Φ, the limiting transition $n \to \infty$ was used. It should be noted that when executing the analytical procedures at $n \to \infty$, the general term appears:

$$\left[e^{i\pi n u^2} u^{-\nu} E_\nu^-(\varphi, pu) \right]_0^\infty, \qquad (6)$$

which, at the corresponding upper limit converts the initial integral (2) into a divergent one. To overcome this analytical difficulty, we used the known method, which is often employed in the problems of radiowaves propagation.[7,8] The main point of the method lies in the fact that a small positive part (i.e., $k + i\alpha$) is assigned to the wave number $k = 2\pi/\lambda$, which corresponds to the presence of insignificant conductance of the medium. The quantity $k + i\alpha$, entering implicitly into the phase part of the first multiplier of Eq. (6), leads to the occurrence of the decreasing amplitude part, which vanishes at putting the upper limit in Eq. (6). By making subsequent analytical operations for the multiplier of the diffraction extinction, the following expression was obtained:

$$\Phi = 1 + \frac{i}{2} F(v_3)\left[F(v_2 = \infty) - F(v_1)\right] - \frac{1}{4} e^{i(\pi/2)v_0^2} \times$$

$$\times \left[\sum_{k=0}^{\infty} (-1)^k \frac{(\pi v_0^2)^{2k}}{A_{2k}} C_{2k} - i \sum_{k=0}^{\infty} (-1)^k \frac{(\pi v_0^2)^{2k+1}}{A_{2k+1}} C_{2k+1} \right], \quad (7)$$

where $v_0 = \sqrt{2}(\rho_0/b_1)$, all other parameters have the same values that in the case of the sector aperture. Figure 2 shows the axes of relative coordinates $\alpha_x = x/b_1$ and $\alpha_y = y/b_1$, which later will be necessary for presenting the results of calculations.

The obtained solution differs from Eq. (3): the first term disappears (contributions of rounded parts of sector region and some triangular regions). The formula (7) is valid in the entire plane of observation, and, in contrast to Eq. (3), is convenient for calculations without limitations throughout the plane. The calculation of the diffraction field, as well as of the sector screen, is a result of integration over two adjacent angular regions (Fig. 2).

Equation (7) is derived for the angular region with an arbitrary value of the aperture, therefore at $\varphi = \pi$ it results in the known solution for a half-plane[7,9]:

$$\Phi = \frac{1}{2}\left[1 - \sqrt{2} e^{-i\frac{\pi}{4}} \int_0^{\sqrt{2n_0}} e^{i\frac{\pi}{2}v^2} dv \right].$$

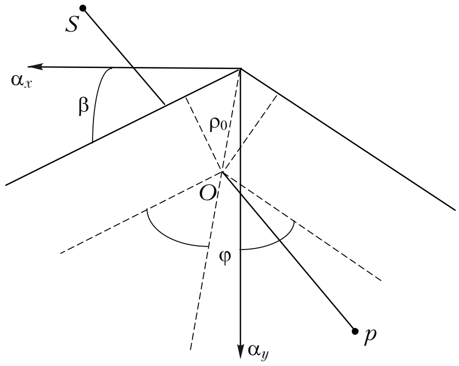

Fig. 2. Geometry of the problem for the angular region.

Results of calculations

1. Spatial pattern of the diffraction field from the sector aperture

Using the derived solution, we have obtained the patterns of space field structure for sector apertures with arbitrary values of radii and angles. As an example, Fig. 1 shows the spatial field structure for the sector aperture, being symmetric relative to the axis x and having the above-mentioned parameters: $R = 10.95\lambda$ and $\varphi = 2\pi/3$ (angular aperture). The field structure is presented in the form of the distribution Φ in three planes removed from the aperture plane at the distances $r_2 = 180$, 200, and 230λ. The distance from the point source to the aperture plane is equal to $r_1 = 300\lambda$. In every plane the values of Φ, corresponding to some definite value of the angular coordinate φ_0, are plotted radially from the center in units given in the figure. Every curve corresponds to definite value of ρ_0 (i.e., removal from the center). Curves of the distribution Φ have asymmetry relative to the axis y, that is clear, because the aperture itself has the same asymmetry. All values of parameters and proportions between them were selected for the model experiments when developing the protective diffraction screens in the microwave range of radiowaves.

2. Spatial pattern of diffraction field from a weakening screen

In practice of microwave radio range, of interest is the screen in the form of a semicircle with the radius equal to the radius of the first Fresnel zone in the screen plane relative to the points of radiation and observation, i.e., $R = b_1$. Such a screen at the observation point at the central axis, passing through the semicircle center, attenuates the field up to zero. Naturally, of interest is the field distribution near these points. Figure 3 shows for the screen in the form of semicircle with $R = 10.95\lambda$ the results of calculations of the field diffraction pattern close to the point of its minimal level, obtained with the use of the Babinet principle[1,7] and the calculated solution (3) for the sector aperture.

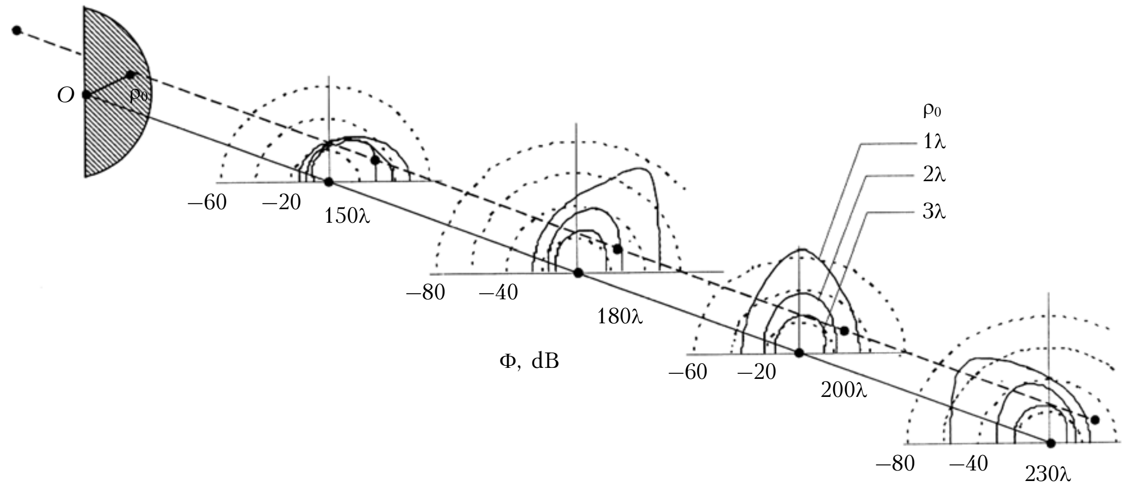

Fig. 3. Spatial patterns of the field (multiplier of the diffraction extinction Φ) for a weakening screen in the form of a semicircle with the radius $R = 10.95\lambda$. Distances from the aperture up to the source $r_1 = 300\lambda$, up to the observation planes $r_2 = 150\lambda$, 180λ, 200λ, and 230λ. Every curve corresponds to the definite distance from the central axis: $\rho_0 = \lambda$; 2λ; 3λ.

For the screen with $R = 10.95\lambda$ the calculated point of the field minimal level on the central axis is at the distance $r_2 = 200\lambda$ from the screen plane, therefore, the plane, in which this point is located, is principal. The field structure is given in the form of the space pattern of Φ in the principal plane at a distance of $r_2 = 200\lambda$ and in three other planes located at distances $r_2 = 150$, 180, and 230λ from the aperture plane. In each plane, the values of Φ, corresponding to some definite value of the angular coordinate φ_0, are plotted radially from the center in units given in the figure. Because the screen is of the weakening type and the field levels reach very small values, the logarithmic units dB were selected in this case. Each curve corresponds to certain remoteness ρ_0 of the observation point from the center.

As it is seen from the field diffraction patterns, the degree of the field weakening decreases with greater distance from the central point, both along the axis and in the observation plane itself. In the principal plane ($r_2 = 200\lambda$) at a small removal from the center $\rho_0 = \lambda$, the weakening reaches more than −60 dB while at $p_0 = 3\lambda$ the weakening of the order of −20 dB is observed. The same can be observed at shifting along the axis. The diffraction pattern has a certain asymmetry relative to the vertical axis, which becomes stronger at a removal from the principal weakening plane. In this case, the direction of asymmetry depends on the position of the observation point relative to the principal weakening plane.

3. Model of the angular region for calculating the diffraction field at the closed paths

One of the main factors, determining the local level of the field and its space distribution at closed paths, is the influence of transverse profiles of obstacles on the field formation. When calculating radio lines, ground-based obstacles are usually approximated by a half-plane. However, the transverse form of real obstacles often greatly differs from the rectangular form, therefore, the calculations, based on the above approximation, often are not in agreement with the experiment. In the paper by Bachynski and Kingsmill[10] the calculated data are given for the profiles of obstacles in the form of angular regions, which are in good agreement with the experiment for angles close to 180°.

Based on the assumption that the natural obstacles (mountains, hills) by their profiles are close to the angular regions, the obtained analytical expression (7) has made it possible to consider the angular region as an approximating model of natural obstacles.

Figure 4 shows the results of calculations and experimental data on the multiplier of diffraction extinction for the model of angular region. The spatial field distribution is determined by parameters $\alpha_x = x/b_1$ and $\alpha_y = y/b_1$, where b_1 is the radius of the first Fresnel zone.

The plots (Fig. 4*a*) reflect the situation, when the left edge of the obstacle coincides with the horizontal axis. Dashed curve corresponds to the field distribution behind the half-plane. As is seen, the field distributions behind the angular obstacles and the half-plane differ greatly, that results in significant errors introduced in the calculations by the half-plane approximation.

Figure 4*b* shows a comparison of the results with the experiment and with the results by Bachynski and Kingsmill (Ref. 10). The lines correspond to the results of Bachynski and Kingsmill under the conditions: *a* – for angular apertures close to 180°; *b* – for angular apertures, which differ greatly from 180°. Dashed lines relate to the model in the form of half-plane (hp), and solid curves relate to the calculations by Eq. (7). The comparison has shown a marked difference in the results obtained by different methods. At the same time, the experimental data (small circles) are in good agreement with the calculations by Eq. (7).

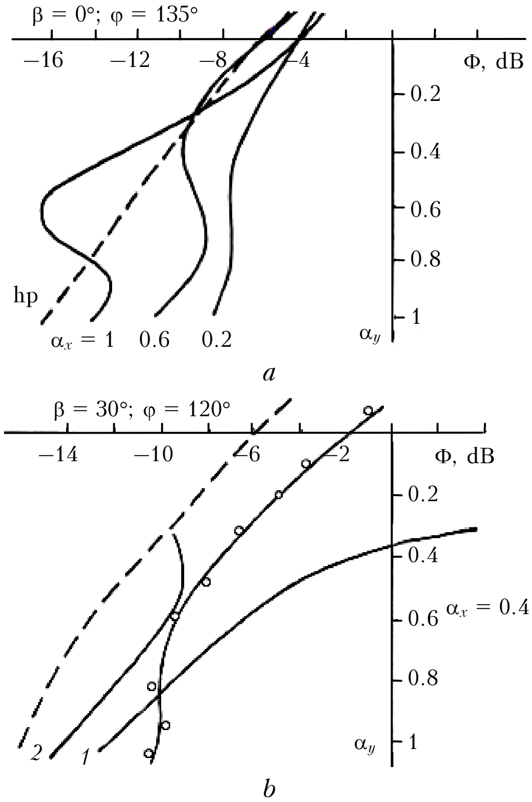

Fig. 4. Spatial distribution of the diffraction field from the angular region as compared with the model of Bachynski and Kingsmill and experimental data.

Conclusion

1. We have obtained the analytical solution of the known problem of the Fresnel diffraction on the sector aperture through incomplete cylindrical functions in the Poisson form, which is valid at all points of observation near the axis passing through the sector center, but is convenient for calculations only in the circular region with radius of a diffracting aperture. The calculation results of spatial structure of diffraction field from the sector aperture (Fig. 1) and from the protective screen in the form of semicircle (Fig. 2) are presented.

2. Due to great generality of the solution of the diffraction problem for the sector, the analytical solution for angular region has also been obtained, which is valid and convenient to be calculated without limitation in the entire observation plane. The results of calculations have shown a strong influence of the obstacle profiles on the spatial field structure and the necessity of more precise accounting for of this influence, for which purpose the model of angular region and obtained solution (7) of the diffraction problem can be proposed.

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