GENERALIZED MIE SOLUTION FOR A FOCUSED BEAM

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The problem of scattering of a focused incident beam by a spherical particle is solved. The specific cases which admit of analytic solution are considered. The numerical examples illustrating a dependence of scattering phase function on the incident beam geometry are given.

1. INTRODUCTION

A solution to the problem of scattering by elementary scattering volume containing independent randomly oriented scattering particles was given in Ref. 1 for the case of an axisymmetric focused beam, with the property of additivity of the Stokes parameters and Müller matrices of an individual particle, which is a consequence of incoherence of the radiation scattered by particles, being used.

Such an approach cannot be used for an individual particle.[1] However, the necessity of solving this problem is inspired by the fact that proper allowance must be made for the geometry of experiment, for instance, with the use of optical particle counters with the different geometry of an incident beam.

In the present paper the generalization of the classical Mie solution to the case of a focused incident beam is considered. The influence of the incident beam geometry on the spatial distribution of scattered radiation is investigated.

We assume that an incident beam represents the superposition of local beams (plane electromagnetic waves) propagating inside the cone with angle at its apex $2\upsilon_0$, oriented along the Z axis. The result of interaction between incident beam and the particle is superposition of the results of interaction of each local beam and the particle, with scattered waves being coherent.

2. LP– AND CP–REPRESENTATIONS OF AN ELECTRIC FIELD

We use a right–handed coordinate system, whose origin is located on the particle center, to describe the scattering of a plane electromagnetic wave by a spherical particle. The propagation direction of a local beam is specified by the unit vector $\mathbf{n} = (\theta, \varphi)$, where θ and φ are the polar and azimuth angles, respectively, in the spherical system of coordinates. The components of an electric field strength are referred to the meridional reference plane containing the Z axis and the propagation direction of a local beam

$$\mathbf{E} = E_1\,\boldsymbol{\theta} + E_2\,\boldsymbol{\varphi}\,, \quad (1)$$

where $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$ are unit vectors parallel and perpendicular to the reference plane. We note that $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$, and $\mathbf{n}$ are the unit vectors of the right–handed coordinate system ($\boldsymbol{\theta}\times\boldsymbol{\varphi} = \mathbf{n}$), whose clockwise rotation through the angle α from the direction $\mathbf{n}$ is specified in the new coordinate system by new components of the electric field[2]

$$\begin{bmatrix} E'_1 \\ E'_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \quad (2)$$

The transformation described by Eq. (2) has eigenvalues $\exp(i\,\alpha)$, $\exp(-\,i\,\alpha)$ and corresponding normalized eigenvectors $2^{-1/2}(1, i)$, $2^{-1/2}(1, -i)$. The unitary transformation[2,3]

$$\begin{bmatrix} E_{+1} \\ E_{-1} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \quad (3)$$

can be interpreted as the change of the basis of two linearly polarized states (1, 0), (0, 1) (LP–representation) to the basis of two circularly polarized states $2^{-1/2}(1, i)$, $2^{-1/2}(1, -i)$ (CP–representation) corresponding to counterclockwise and clockwise polarized electromagnetic radiation of unit intensity. The rotation effect in the CP–representation has simpler form than in the LP–representation and is described by the diagonal matrix[2,3]

$$\begin{bmatrix} E'_{+1} \\ E'_{-1} \end{bmatrix} = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix} \begin{bmatrix} E_{+1} \\ E_{-1} \end{bmatrix}. \quad (4)$$

3. AMPLITUDE SCATTERING MATRIX

In so–called far zone ($r > 1$) the components of an incident plane electromagnetic wave and scattered spherical wave are connected via the relationship[4]

$$\begin{bmatrix} E^s_1 \\ E^s_2 \end{bmatrix} = \frac{\exp(i\,k\,r)}{-\,i\,k\,r}\,\mathbf{S}(\mathbf{n}_s;\,\mathbf{n}_i) \begin{bmatrix} E^i_1 \\ E^i_2 \end{bmatrix}, \quad (5)$$

where S is the amplitude scattering matrix, $k = 2\pi\lambda^{-1}$, and λ is the wavelength of incident radiation.

Denoting the transformation given by Eq. (3) by $\mathbf{U}$ and using Eqs. (3) and (5), we obtain the expression for the amplitude scattering matrix in the CP–representation

$$\mathbf{C}(\mathbf{n}_s;\,\mathbf{n}_i) = \mathbf{U}\,\mathbf{S}(\mathbf{n}_s;\,\mathbf{n}_i)\,\mathbf{U}^{-1}\,, \quad (6)$$

$$\mathbf{C} = \begin{vmatrix} C_{+1+1} & C_{+1-1} \\ C_{-1+1} & C_{-1-1} \end{vmatrix} = \frac{1}{2} \begin{vmatrix} S_{11} + i\,S_{12} - i\,S_{21} + S_{22} & S_{11} - i\,S_{12} - i\,S_{21} - S_{22} \\ S_{11} + i\,S_{12} + i\,S_{21} - S_{22} & S_{11} - i\,S_{12} + i\,S_{21} + S_{22} \end{vmatrix}. \quad (7)$$

Using the expression for the amplitude scattering matrix S (see Ref. 5, p. 636) and Eq. (7), after simple but cumbersome manipulation we obtain the expression for the elements of the amplitude scattering matrix[5,6]

$$C_{+1+1} = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} (2\,n + 1) \exp\left[\text{im}\,(\varphi_s - \varphi_i) \times \right.$$

$$\left. \times\, d^{\,n}_{1m}(\theta_s)\, d^{\,n}_{1m}(\theta_i)\,(b_n + a_n)\right] ,$$

$$C_{+1-1} = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} (2\,n + 1) \exp\left[\text{im}\,(\varphi_s - \varphi_i) \times \right.$$

$$\left. \times\, d^{\,n}_{1m}(\theta_s)\, d^{\,n}_{-1m}(\theta_i)\,(b_n - a_n)\right] , \qquad (8)$$

$$C_{-1+1} = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} (2\,n + 1) \exp\left[\text{im}\,(\varphi_s - \varphi_i) \times \right.$$

$$\left. \times\, d^{\,n}_{-1m}(\theta_s)\, d^{\,n}_{1m}(\theta_i)\,(b_n - a_n)\right] ,$$

$$C_{-1-1} = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} (2\,n + 1) \exp\left[\text{im}\,(\varphi_s - \varphi_i) \times \right.$$

$$\left. \times\, d^{\,n}_{-1m}(\theta_s)\, d^{\,n}_{-1m}(\theta_i)\,(b_n + a_n)\right] ,$$

where $d^{\,n}_{qm}(\text{q})$ are the Wigner functions[7] and a_n, b_n are the known Mie coefficients.[4]

4. SCATTERED FIELD FOR THE CASE OF FOCUSED INCIDENT BEAM

Changing the reference plane for an incident plane electromagnetic wave to meridional one with $\varphi_i = 0$ on account of Eq. (4), we obtain the following expression to within a constant factor:

$$E^{\,s}_{+1} = C_{+1+1} \exp(i\,\varphi_i)\, E^{\,i}_{+1} + C_{+1-1} \exp(-\,i\,\varphi_i)\, E^{\,i}_{-1} ,$$
$$E^{\,s}_{-1} = C_{-1+1} \exp(i\,\varphi_i)\, E^{\,i}_{+1} + C_{-1-1} \exp(-\,i\,\varphi_i)\, E^{\,i}_{-1} . \qquad (9)$$

Taking into account the above–mentioned assumptions and the coherence of local scattered beams, the components of the scattered field for the case of the focused beam propagating in directions confined to the conic solid angle Ω have the form:

$$<E^{\,s}_{\pm 1}> = \int_{\Omega} E^{\,s}_{\pm 1}\, d\omega \Big/ \int_{\Omega} d\omega\ . \qquad (10)$$

Let us consider some special cases in which Eq. (10) admits of analytic solution.

4.1. Incident beam homogeneous in intensity and polarization ($E^{\,i}_{+1}(\theta_i, \varphi_i) = \text{const}$). The local beams are considered to have unit intensity without loss of generality.

4.1.1. Counterclockwise polarized light ($E^{\,i}_{+1} = 1$, $E^{\,i}_{-1} = 0$). After substitution of Eq. (9) into Eq. (10) and integration we obtain

$$<E^{\,s}_{\pm 1}> = \frac{1}{2} \sum_{n=1}^{\infty} (2\,n+1)\, \text{e}^{iz_s}\, d^{\,n}_{\pm 11}(\theta_s) <d^{\,n}_{11}(\upsilon_0)> (b_n \pm a_n), \qquad (11)$$

where

$$<d^{\,n}_{11}(\upsilon_0)> = \int_0^{\upsilon_0} d^{\,n}_{11}(\theta) \sin\theta\, d\theta \,/\, (1 - \cos\upsilon_0)\ . \qquad (12)$$

The analytic expression of integral (A11) and the main properties of the Wigner functions are given in Appendix.

4.1.2. Clockwise polarized light ($E^{\,i}_{+1} = 0$, $E^{\,i}_{-1} = 1$). Using Eq. (A6), we obtain

$$<E^{s}_{\pm 1}> = \frac{1}{2} \sum_{n=1}^{\infty} (2\,n + 1)\, \text{e}^{-i\varphi_s}\, d^{\,n}_{\pm 1-1}(\theta_s) <d^{\,n}_{11}(\upsilon_0)> (b_n \pm a_n). \qquad (13)$$

4.1.3. Elliptically polarized light. A plane electromagnetic wave with arbitrary polarization can be represented as a linear combination of basis states in the *CP*–representation, i.e., in this case the components of the scattered field are linear combination of Eqs. (11) and (13).

We note that Eqs. (11) and (13) are simplified to known expressions for the amplitudes of scattered field[8] using formulas

$$\lim_{\upsilon_0 \to 0} <d^{\,n}_{11}(\upsilon_0)> = 1\ ,\quad n = 1, 2, \ldots \qquad (14)$$

for the case of a parallel incident beam (Ω = 0).

4.2. Incident beam inhomogeneous in intensity [$E^{\,i}_{\pm 1}(\theta_i\,, \varphi_i) = E^{i}_{\pm}(\theta_i)$]. Let us assume that the functions $E^{i}_{\pm\text{ю}}(\theta_i)$ satisfy the conditions of expansion in the Wigner functions $d^{\,n}_{00}(\theta_i)$

$$E^{\,i}_{\pm}(\theta_i) = \sum_{n=0}^{\infty} C^{\pm}_{n}\, d^{\,n}_{00}(\theta_i)\ . \qquad (15)$$

Hereafter we also use the formula[7]

$$d^{\,n}_{qm}(\theta)\, d^{\,n'}_{q'm'}(\theta) = \sum_{n''=|n-n'|}^{n+n'} C^{\,n''q+q'}_{nq\,n'q'}\, C^{\,n''m+m'}_{nm\,n'm'}\, d^{\,n''}_{q+q'\ m+m'}(\theta)\ , \qquad (16)$$

where $C^{\,nm}_{n_1 m_1 n_2 m_2}$ are the Clebsh–Gordan coefficients.[7]

We represent the incident beam as a sum of two coherent beams and find the amplitude of scattered field separately for each of them.

4.2.1.

$$E^{\,i}_{+1}(\theta_i, \varphi_i) = E^{i}_{+}(\theta_i)\ ,\quad E^{\,i}_{-1}(\theta_i, \varphi_i) = 0\ .$$

$$<E^{\,s}_{\pm 1}> = \frac{1}{2} \sum_{n=1}^{\infty} (2\,n + 1)\, \text{e}^{i\varphi_s}\, d^{\,n}_{\pm 11}(\theta_s)\,(b_n \pm a_n) \times$$

$$\times \sum_{n'=0}^{\infty} C^{+}_{n'} \sum_{n''=|n-n'|}^{n+n'} \left[C^{\,n''1}_{n1n'0}\right]^2 <d^{\,n''}_{11}(\upsilon_0)>. \qquad (17)$$

4.2.2.

$$E^{\,i}_{+1}(\theta_i, \varphi_i) = 0\ ,\quad E^{i}_{-1}(\theta_i, \varphi_i) = E^{i}_{-}(\theta_i)\ .$$

$$<E^{\,s}_{\pm 1}> = \frac{1}{2} \sum_{n=1}^{\infty} (2\,n + 1)\, \text{e}^{-i\varphi_s}\, d^{\,n}_{\pm 1-1}(\theta_s)\,(b_n \pm a_n) \times$$

$$\times \sum_{n'=0}^{\infty} C^{-}_{n'} \sum_{n''=|n-n'|}^{n+n'} \left[C^{\,n''1}_{n1n'0}\right]^2 <d^{\,n''}_{11}(\upsilon_0)>. \qquad (18)$$

5. STOKES VECTOR FOR SCATTERED RADIATION

Stokes vector parameters in the *CP*–representation have the form[3]

$$I_2 = E_{-1} E^*_{+1} = \frac{1}{2}(Q - i\,U)\ ,\quad I_0 = E_{+1} E^*_{+1} = \frac{1}{2}(I - V)\ ,\qquad (19)$$

$$I_{-0} = E_{-1} E^*_{-1} = \frac{1}{2}(I + V)\ ,\quad I_{-2} = E_{+1} E^*_{-1} = \frac{1}{2}(Q + i\,U)\ ,$$

where asterisk means the complex conjugate value and I, Q, V, and U are the Stokes parameters in the LP–representation.[4] The intensity of scattered radiation is defined as

$$I^s = (I^s_0 + I^s_{-0})\ . \qquad (20)$$

Let us consider the particle scattering characteristics for the case discussed in paragraphs 4.1.1 and 4.1.2.

The scattered radiation flux in the whole conic angle 4π on account of Eq. (A7) has the form

$$\Phi = \frac{2\pi}{k^2} \sum_{n=1}^{\infty} (2n + 1)\,(|a_n|^2 + |b_n|^2)\ \langle d^{\,n}_{11}(\upsilon_0)\rangle^2 \qquad (21)$$

and coincides with the scattering cross section $C_{\rm scat}$ for the case of a plane electromagnetic incident wave.

TABLE I.

θ_s	υ_0 0°	1°	5°	10°
0°	1244.47	1115.16	194.655	63.5497
		1242.77	468.085	74.3988
10°	11.4185	10.6874	11.0932	32.9131
		11.7726	32.4256	45.1948
20°	7.4857	6.9109	5.2298	4.7707
		7.5328	4.3226	2.2676
30°	2.7400	2.6691	2.4212	2.5053
		2.7032	2.6846(–1)	1.6982(–1)
40°	1.3182	1.3262	1.3913	1.3932
		1.2538	3.8716(–2)	1.6307(–1)
50°	5.3715(–1)	5.6772(–1)	6.2692(–1)	6.7099(–1)
		5.0683(–1)	1.8693(–2)	6.6409(–2)
60°	3.1315(–1)	2.9715(–1)	2.7959(–1)	2.9123(–1)
		2.8994(–1)	2.2929(–2)	4.0759(–2)
70°	7.9111(–2)	9.1894(–2)	1.2395(–1)	1.3326(–1)
		7.2457(–2)	1.7701(–2)	1.3283(–2)
80°	9.5849(–2)	8.6848(–2)	6.9054(–2)	6.8797(–2)
		8.9638(–2)	5.8814(–3)	7.0381(–3)
90°	2.4738(–2)	2.6777(–2)	3.1191(–2)	3.3794(–2)
		2.3266(–2)	5.9350(–3)	3.6968(–3)
100°	1.5444(–2)	1.7164(–2)	1.9866(–2)	2.1066(–2)
		1.5086(–2)	7.2455(–3)	4.1076(–3)
110°	5.4005(–3)	9.7217(–3)	2.0476(–2)	2.6723(–2)
		5.7367(–3)	1.8562(–2)	9.6380(–3)
120°	2.9429(–2)	3.8332(–2)	5.0239(–2)	4.4060(–2)
		2.9868(–2)	2.8427(–2)	5.2118(–3)
130°	3.6375(–2)	4.2932(–2)	7.6169(–2)	8.9188(–2)
		3.3305(–2)	6.0338(–2)	1.3136(–2)
140°	2.4655(–1)	2.2934(–1)	2.1549(–1)	1.8985(–1)
		2.3325(–1)	2.7344(–2)	3.4051(–2)
150°	1.2251(–1)	1.2867(–1)	1.6019(–1)	1.7927(–1)
		1.1883(–1)	1.1811(–1)	1.8599(–1)
160°	7.0563(–2)	8.1232(–2)	1.1139(–1)	1.3792(–1)
		6.8005(–2)	7.1432(–2)	4.9046(–2)
170°	1.3890(–1)	1.4444(–1)	1.6606(–1)	1.7916(–1)
		1.3955(–1)	6.1376(–1)	1.5395
180°	2.0613(–1)	1.8439(–1)	3.3481(–1)	2.0715(–1)
		1.9812(–1)	2.5994(–1)	1.5722

The scattering phase function $p(\theta_s) = 4\pi I^s\Phi^{-1}$ satisfies the normalization condition

$$\frac{1}{4\pi} \int\limits_{4\pi} p\, d\,\omega = 1\ . \qquad (22)$$

Table I presents the results of calculation of the scattering phase function for different geometry of an incident beam (values in denominator). The values in numerator are the results of calculation of the scattering phase function of an elementary scattering volume containing independent scatterers.[1,9] Diffraction parameter $\rho = 50$, refractive index of particle $m_p = 1.33$ as well as the geometry and structure of an incident beam were the same for both cases. Using Eqs. (16), (20), and (A7), as well as the properties of the Clebsh–Gordan coefficients,[7] we obtain for the asymmetry $\langle\cos\theta\rangle$ (see Ref. 4)

$$\langle\cos\theta\rangle\,\Phi = \frac{4\pi}{k^2}\,{\rm Re} \sum_{n=1}^{\infty} \frac{n\,(n+2)}{n+1}\,(a_n a^*_{n+1} + b_n b^*_{n+1}) \times$$

$$\times \langle d^{\,n}_{11}(\upsilon_0)\rangle\langle d^{\,n+1}_{11}(\upsilon_0)\rangle + \frac{2n+1}{n\,(n+1)}\,(a_n b^*_n)\ \langle d^{n}_{11}(\upsilon_0)\rangle^2\ . \qquad (23)$$

Eq. (23) coincides with the well–known Debye expression[4] for the case of a plane incident wave ($\upsilon_0 = 0$).

In conclusion we note that formulas analogous to Eqs. (21) and (23) can be easy obtained for the cases discussed in paragraphs 4.1.3 and 4.2.

APPENDIX

The Wigner functions $d^{\,n}_{qm}$(q) are connected with the generalized spherical functions $P^{\,n}_{qm}$(cosq) (see Ref. 10) via the relationship[2]

$$d^{\,n}_{qm}(\theta) = i^{\,m-q}\,P^{\,n}_{qm}(\cos\theta)\ ,\ n \ge \max(|q|,\ |m|) = n_*\ . \qquad (A1)$$

Let us consider the Wigner function properties using the properties of generalized spherical functions and relationship (A1).

The Legendre polynomials and associated Legendre functions are expressed in the form[7,10]

$$d^{\,n}_{00}(\theta) = P^{\,n}_{00}(\cos\theta) = P_n(\cos\theta)\ , \qquad (A2)$$

$$d^{\,n}_{0m}(\theta) = i^{\,m}\,P^{\,n}_{0m}(\cos\theta) = (-1)^m \left[\frac{(n-m)!}{(n+m)!}\right]^{1/2} P^{\,m}_n(\cos\theta), \qquad (A3)$$

where

$$P_n(x) = \frac{1}{2^n\,n!}\,\frac{d^{\,n}}{d\,x^n}\,(x^2 - 1)^n\ , \qquad (A4)$$

$$P^{\,m}_n(x) = (-1)^m\,(1 - x^2)^{m/2}\,\frac{d^{\,m}}{d\,x^m}\,P_n(x)\ . \qquad (A5)$$

The Wigner functions have the following symmetry properties[7,10]:

$$d^{\,n}_{qm}(\theta) = d^{\,n}_{-m-q}(\theta) = (-1)^{m-q}\,d^{\,n}_{mq}(\theta) \qquad (A6)$$

and satisfy the orthogonality condition

$$\int_0^\pi d\theta \sin\theta \, d^{\,n}_{qm}(\theta)\, d^{\,n'}_{qm}(\theta) = \frac{2}{2n+1}\,\delta_{nn'}, \qquad (A7)$$

the recurrent relationship

$$n\sqrt{(n+1)^2-m^2}\sqrt{(n+1)^2-q^2}\, d^{\,n+1}_{qm}(\theta) +$$

$$+ (n+1)\sqrt{n^2-q^2}\sqrt{n^2-m^2}\, d^{\,n-1}_{qm}(\theta) =$$

$$= (2n+1)\,[n(n+1)\cos\theta - mq]\, d^{\,n}_{qm}(\theta) \qquad (A8)$$

with the initial conditions

$$d^{\,n_*}_{qm}(\theta) = \frac{(-1)^{(q-m+|q-m|)/2}}{2^{n_*}}\left[\frac{(2^{n_*})!}{(|q-m|)!\,(|q+m|)!}\right]^{1/2} \times$$

$$\times\,(1-\cos\theta)^{|q-m|/2}\,(1+\cos\theta)^{|q+m|/2}, \qquad (A9)$$

as well as the relation

$$\frac{d}{d\theta}\, d^{\,n}_{qm}(\theta) + \frac{q - m\cos\theta}{\sin\theta}\, d^{\,n}_{qm}(\theta) =$$

$$= -\sqrt{(n-m)(n+m+1)}\, d^{n}_{q\,m+1}(\theta). \qquad (A10)$$

We make use of Eqs. (A3), (A5), (A6), and (A10) to obtain

$$\langle d^{n}_{11}(\upsilon)\rangle = \frac{\sin\upsilon\,[P^{1}_{n}(\cos\upsilon) + P^{-1}_{n}(\cos\upsilon)] + P_n(\cos\upsilon)\,(1-\cos\upsilon)}{n(n+1)(1-\cos\upsilon)}. \qquad (A11)$$

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