The method of consequent decomposition in the theory of lidar sensing of dense media

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For a function that determines the ratio of the contribution from multiple scattering within the frames of small-angle approximation of the transfer theory to a singly scattered signal, an asymptotic series by powers of a small parameter is constructed, i.e., the viewing angle of the receiver. A connection between the series' coefficients and microstructural parameters of the media is established. Boundaries of applicability of the asymptotic approximation are estimated numerically.

Introduction

The problem of determining the size of cloud droplets by methods of lidar sensing attracts attention of specialists in atmospheric optics for a long time. Promising tools for solving this problem are lidars with a variable viewing angle (MFOV-lidars).[1–6] The signal component that is stipulated by multiple scattering (MS) depends on particles' size. Information about the size can be obtained by solving the inverse problem for the lidar equation. The existing mathematical descriptions of lidar signals were obtained from the theory of radiation transfer[4,6] and they are not quite appropriate for practical solving of the problem due to complicated analytical expressions and laborious calculations. In such cases, mathematicians replace the considered function by a simpler one obtained by decomposition the former into a functional series.

In descriptions of MFOV-lidar signals the quantitative MS measure is often assigned to the value of a singly scattered signal. Asymptotic decompositions are efficient approximation methods for the above-mentioned function, which is given as an integral. This approach was developed in the theory of laser sensing at multiple scattering on the base of analysis of asymptotic behavior of lidar signals at large viewing angles.[6–8] The opposite case is considered in this paper: the viewing angle of the receiver is taken as a small parameter to construct an asymptotic series by powers of this parameter. The relation between coefficients of the series and microstructural parameters of the media have been determined. The effect of the number of taken into account decomposition terms (such as the small-angle scattering phase function and optical thickness) on the accuracy of the asymptotic approximation is estimated numerically.

1. Formulation of the problem and a method for its solution

1.1. Initial relationships

We consider the following function (correcting factor):

$$m(\gamma_r) = (z\gamma_r)\int_0^\infty J_1(z\gamma_r\nu)F(\nu)\,d\nu, \qquad (1)$$

which determines the relative contribution of multiple scattering in a lidar signal when sensing media with strong scattering anisotropy, such as clouds or a suspension of particles in sea water. The designations in Eq. 1 are as follows: γ_r is the viewing angle of the lidar's receiving system; z is the distance from the lidar to the area, in which the sensing pulse is backscattered; $J_1(.)$ is the Bessel function of the first kind of the first order; ν is the spatial frequency;

$$F(\nu) = \exp[g(\nu)] - 1; \qquad (2)$$

$$g(\nu) = 2\int_0^z \sigma(z-s)\tilde{x}(\nu s)\,ds, \qquad (3)$$

$\sigma(z)$ is the scattering coefficient at the wavelength of sensing radiation λ; $\tilde{x}(p)$ is the Hankel transform of the small-angle scattering phase function $x(\gamma)$ satisfying the normalizing condition $2\pi\int_0^\infty x(\gamma)\gamma\,d\gamma = 1$.

The formula (1) follows from the theory of radiation transfer within the frames of the wide-spread model.[6,9,10] According to this model, multiple scattering is taken into account in the small-angle approximation, and the large-angle scattering is

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accounted in the single approximation. Some additional assumptions are as follows:

– the radiation source is point-like, monodirectional, and sends a $\delta(t)$-pulse along the positive direction of the Oz axis;

– the source and receiver of radiation are placed in the plane $z = 0$, and their optical axes are aligned;

– the function of the receiver's sensitivity to the angular coordinate has a circular symmetry and a step form;

– spatial variability of a lidar signal in the plane of receiving aperture is not taken into account;

– variation of the scattering phase function can be neglected in the neighborhood of the backward direction.

These conditions and simplifications are not of principle for the sequel. They do not restrict generality of the results below and can be easily taken into account within the framework of the model considered.

1.2. Asymptotical decomposition of the correcting factor

The problem is to construct asymptotic series for the function $m(\gamma_r)$ (1) at $\gamma_r \to 0$. The solution of the problem is based on asymptotic properties of the function $F(\nu)$ at $\nu \to \infty$. Namely, it is supposed that the asymptotic decomposition

$$F(\nu) \sim \sum_{n=1}^{\infty} a_n \nu^{-n}, \quad a_n \neq 0\,, \tag{4}$$

takes place at $\nu \to \infty$.

This property imposes certain requirements upon the behavior of the Hankel transform of the phase function $\tilde{x}(p)$. As a rule, these requirements are fulfilled for typical models describing the scattering in the small-angle area. For instance, in the case of a scattering phase function in the approximation of Fraunhofer diffraction, the function $g(\nu)$ (3) can be represented in the following form[6] for sufficiently large ν:

$$g(\nu) = a/\nu. \tag{5}$$

For a homogeneous layer of thickness L in scattering by spherical particles, the relation (5) is fulfilled as

$$\nu \geq \nu_{\max} = 2kR/L \tag{6}$$

and the coefficient at ν is

$$a = \frac{16}{3\pi}\sigma k\bar{r}\,, \tag{7}$$

where $k = 2\pi/\lambda$; R is the maximum radius of particles; σ is the scattering coefficient; $\bar{r} = \langle r^3\rangle/\langle r^2\rangle$, the parentheses $\langle.\rangle$ denote averaging by the ensemble of particles.

Substituting the expression for $g(\nu)$ (5) into Taylor's series for $F(\nu)$ (2), we obtain an exact equality in Eq. 4 for $\nu \geq \nu_{\max}$. A similar result can be obtained for the part of the small-angle scattering phase function that is formed by laws of geometrical optics.

Taking into account Eq. (6), it is convenient in the sequel to pass to the dimensionless variables

$$\eta = \nu/\nu_{\max}, \quad \xi = (z\gamma_r)\,\nu_{\max} \tag{8}$$

in Eq. 1 and write it in the form

$$m(\xi) = \xi I(\xi), \quad I(\xi) = \int_0^{\infty} K(\xi\eta)F(\eta)\,d\eta, \quad K(\eta) = J_1(\eta)\,. \tag{9}$$

With allowance of this, the function $F(\eta)$ for $\eta \geq 1$ can be decomposed into a series by negative powers of η:

$$F(\eta) = \sum_{n=1}^{\infty} a_n \eta^{-n}. \tag{10}$$

The decomposition's coefficients

$$a_n = \frac{1}{n!}\left(\frac{4}{3\pi}\frac{\bar{r}}{R}\tau\right)^n \tag{11}$$

depend on the optical thickness of the layer $\tau = 2\sigma L$ and ratio of the mean radius of particles $\bar{r}$ to maximal R. For particles of similar size, $\bar{r} = R$ and a_n do not depend on particles' radius.

We briefly present the algorithm for construction of the asymptotic series for the integral term $I(\xi)$ in Eq. (9) as $\xi \to 0$ [Ref. 11]. First, let us define the functions $F_{-n}(\eta)$ and $K_n(\eta)$, for which the following recurrence relations take place:

$$F_{-n}(\eta) = \eta[F_{-n+1}(\eta) - a_n/\eta]; \tag{12}$$

$$K_n(\eta) = \eta^{-1}[K_{n-1}(\eta) - K_{n-1}(0)]; \tag{13}$$

$$F_0(\eta) = F(\eta), \quad K_0(\eta) = K(\eta), \quad n = 1, 2, \ldots. \tag{14}$$

Then divide the domain of integration in Eq. 9 into two intervals with the cutpoint at $\eta = 1$. Further, we subtract the first term of the series (10) from $F(\eta)$, and the zero value $K(0)$ of the kernel $K(\eta) = J_1(\xi)$ from the kernel. As a result, rearranging the summands and adding necessary compensating terms, we can obtain the following representation for the integral $I(\xi)$ (9):

$$I(\xi) = b_0 A_0 + a_1 B_0(\xi) + \xi I_1(\xi), \tag{15}$$

where

$$A_n = \int_0^1 F_{-n}(\eta)\,d\eta + \int_1^{\infty} F_{-n-1}(\eta)\eta^{-1}d\eta; \tag{16}$$

$$B_n(\xi) = \int_0^1 K_{n+1}(\eta)\,d\eta + \int_1^{\infty} K_n(\eta)\eta^{-1}d\eta - K_n(0)\ln\xi, \tag{17}$$

$$I_n(\xi) = \int_0^{\infty} K_n(\xi\eta)F_{-n}(\eta)\,d\eta, \tag{18}$$

and $b_n = K_n(0)$. Note that b_n are coefficients of decomposition of the kernel $K(\eta)$ into a power series (in this case, they are coefficients of the series $J_1(\eta) = \sum_{k=0}^{\infty} b_k \eta^k$). A similar procedure is applied to $I_1(\xi)$, which is contained in the right side of Eq. 15. Finally, we come to the following asymptotic decomposition of the integral $I(\xi)$ by powers of the small parameter ξ:

$$I(\xi) \sim \sum_{n=0}^{\infty} D_n(\xi)\xi^n, \quad \xi \to 0; \tag{19}$$

$$D_n(\xi) = b_n A_n + a_{n+1} B_n(\xi). \tag{20}$$

The coefficients a_n and A_n depend on $F(\nu)$, and the coefficients b_n and B_n are defined by the form of the kernel $K(\eta)$. To pass to $m(\gamma_\text{r})$, we should multiply the series (19) by ξ and return to the former variable

$$\gamma_\text{r} = \frac{L}{z}\frac{\xi}{2kR}. \tag{21}$$

Let us consider the first term of the series (19). It corresponds to the expression (15) with a rejected last summand which defines error of this approximation. For the kernel $K(\eta) = J_1(\eta)$ we have $b_0 = 0$, $B_0(\xi) = 1$, $I(\xi) = a_1$, and, as a result, we obtain the following formula:

$$m_1(\gamma_\text{r}) = \left[\frac{8}{3\pi}\frac{z}{L}\tau k\bar{r}\right]\gamma_\text{r}, \tag{22}$$

according to which the correcting factor $m(\gamma_\text{r})$ in the first approximation of the asymptotic decomposition is a linear function of the angle γ_r and increases proportionally to optical thickness of the media τ and mean particles' radius $\bar{r}$.

It is useful to compare the obtained result with that of the theory of lidar sensing with allowance of only one scattering act by small angles (since sounding also takes into account single scattering by small angles, this is an approximation of double scattering). For this approximation, the following expression was obtained in Ref. 12:

$$\hat{m}(\gamma_\text{r}) = 2\pi\tau\left[\int_0^{\tilde{\gamma}} x(\gamma)\gamma\,\mathrm{d}\gamma + \tilde{\gamma}\int_{\tilde{\gamma}}^{\infty} x(\gamma)\,\mathrm{d}\gamma\right]. \tag{23}$$

Here $\tilde{\gamma} = (z/L)\gamma_\text{r}$. Decomposing $\hat{m}(\gamma_\text{r})$ (23) into Taylor's series in the neighborhood of the point $\gamma_\text{r} = 0$ and restricting ourselves by the square approximation, we obtain

$$\hat{m}(\gamma_\text{r}) \approx 2\pi\tau\left[\gamma_\text{r}\frac{z}{L}\int_0^{\infty} x(\gamma)\,\mathrm{d}\gamma - \frac{\gamma_\text{r}^2}{2}\left(\frac{z}{L}\right)^2 x(0)\right]. \tag{24}$$

It can be shown[6] that for the small-angle scattering phase function the following relations hold:

$$\int_0^{\infty} x^{(\text{D})}(\gamma)\,\mathrm{d}\gamma = \frac{4}{3\pi^2}k\bar{r}, \quad x^{(\text{D})}(0) = \frac{1}{4\pi}k^2\overline{r^2}, \tag{25}$$

where $\overline{r^2} = \langle r^4\rangle/\langle r^2\rangle$, are valid in the diffraction approximation $x(\gamma) = x^{(\text{D})}(\gamma)$.

Comparing Eqs. 22 and 24, it is easy to see that they are quite identical if the square term in Eq. 24 is not taken into account. Thus, we can come to the conclusion that multiple scattering is not taken into account in the first term $D_0(\xi)$ of the asymptotic series (19), which is defined by the linear contribution of single small-angle scattering. This fact is a corollary of vanishing of the coefficient $b_0 = 0$ in decomposing the Bessel function $J_1(\eta)$ into a series.

In contrast to $D_0(\xi)$, the following term of the series (19) $D_1(\xi) = b_1 A_1 + a_2 B_1(\xi)$, in which $b_1 = 1/2$,

$$A_1 = \int_0^1 \eta F(\eta)\,\mathrm{d}\eta - a_1 + \sum_{k=3}^{\infty}\frac{a_k}{k-1}, \tag{26}$$

$$a_2 = 8\left(\frac{1}{3\pi}\frac{\bar{r}}{R}\tau\right)^2, \tag{27}$$

$$B_1(\xi) = \int_0^1\left[\frac{K(\eta)}{\eta} - \frac{1}{2}\right]\frac{\mathrm{d}\eta}{\eta} + \int_1^{\infty}\frac{K(\eta)}{\eta^2}\mathrm{d}\eta - \frac{1}{2}\ln\xi \tag{28}$$

contains information about scattering of any multiplicity. The integrals in Eq. 28 are defined numerically, and their sum approximately equals 0.308. Finally, for the function $B_1(\xi)$ we obtain a simple formula

$$B_1(\xi) = 0.308 - (1/2)\ln\xi. \tag{29}$$

It is easy to establish the connection between $D_1(\xi)$ and approximation (24), if to replace the function $F(\eta)$ in Eq. 26 by the function $g(\eta)$ and equate all the coefficients a_n but a_1 to zero. It turns out that allowance of the second term of the asymptotic series (19) leads to a result coinciding with the square correction in Eq. 24. This example indicates serviceability of the asymptotic decomposition (19) for small optical thickness within the frameworks of the double scattering approximation. In the following section we present the estimate of efficiency for application of the series (19) with allowance for multiple scattering in the general case on the base of comparison with results of numerical calculations by formula (1).

2. Results of numerical simulation

Application of the considered asymptotic decomposition to describing multiple scattering in a lidar signal requires estimations of accuracy and limits of applicability. For this purpose, by the use of asymptotic formulas that were presented in the previous section, functions $m(\gamma_\text{r})$ were calculated and compared with the results of control calculations performed by the initial formula (1). A plane homogeneous layer formed by particles of radius $R = 10$ μm and having constant optical characteristics was considered as a model media. The distance from the nearest boundary of the layer was 1 km, $z = 2$ km, $\lambda = 0.55$ μm. Numerical study considered influence of the order of the asymptotic approximation and that of optical thickness in the interval $1 \le \tau \le 4$, as well as the scattering phase function of the media. The phase function was considered both in the approximation of Fraunhofer diffraction $x(\gamma) = x^{(D)}(\gamma)$ and under additional allowance of its geometry-optical component.

The results of calculations are presented in Figs. 1–4. Figure 1 presents three first summands of the asymptotic series (19) multiplied by ξ (curves *1*, *2′*, and *3′*) as a function of the angle γ_r. Curves *2* and *3* represent the sum of two and three such summands, respectively. These functions are obtained for the scattering phase function in the *D*-approximation, $x(\gamma) = x^{(D)}(\gamma)$. They permit one to estimate the contribution of terms of different order into partial sums of the series (19).

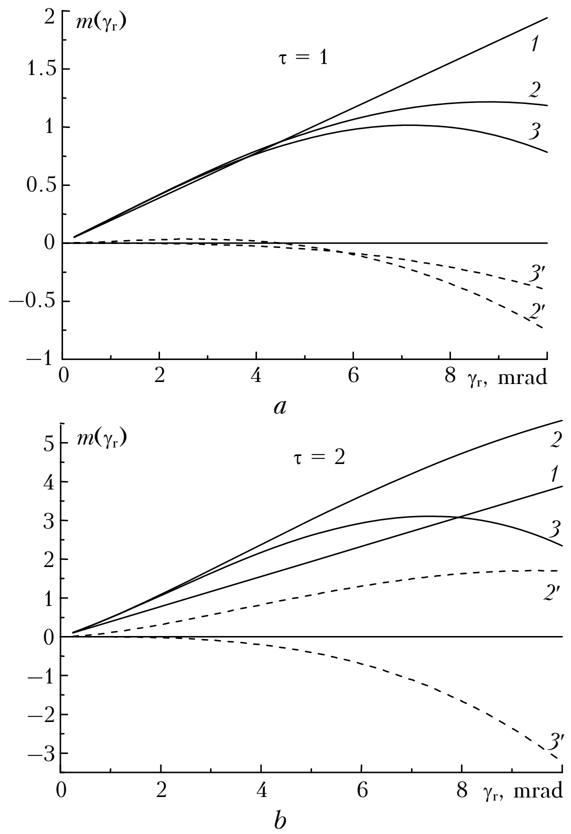

Fig. 1. Angular dependencies of partial sums (curves *1–3*) of the asymptotic series for the function $m(\gamma_r)$ and its linear (*1*), square (*2′*), and cube parts (*3′*) for two values of optical thickness; diffraction approximation.

The linear term (curve *1*) makes main contribution into the sum at a small optical thickness ($\tau = 1$, Fig. 1*a*) and angles $\gamma_r < 5$ mrad. The influence of the square (curves *2*, *2′*) and cube (curves *3*, *3′*) terms decreases with increase of their order and becomes appreciable for $\gamma_r > 5$ mrad. It should be noted that the curve *2′* goes to the range of negative values at a small optical thickness beginning with γ_r, which are a little greater than 4 mrad. This is caused by the influence of logarithmic dependence of the coefficient $B_1(\xi)$ (29), which changes its sign at $\xi = 1.85$. For the simulation conditions, this corresponds to the angle $\gamma_r = 3.9$ mrad. As a result, allowance for the square term leads to lowered deviation of the approximation curve *2* with respect to linear approximation (curve *1*).

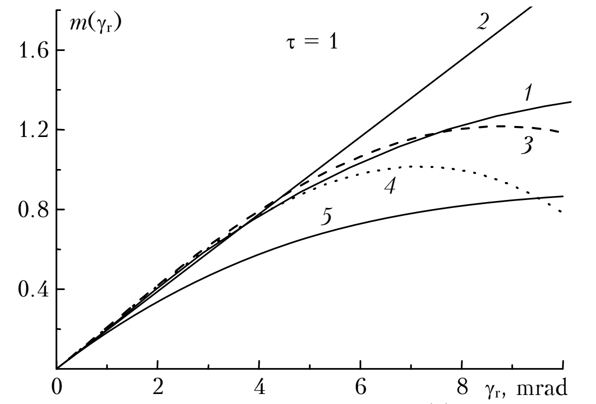

Fig. 2. Comparison of the dependencies $m^{(D)}(\gamma_r)$ calculated by the initial formula (1) (curve *1*) and asymptotic formulas in linear (*2*), square (*3*), and cube (*4*) approximation; the curve *5* is approximation of double scattering.

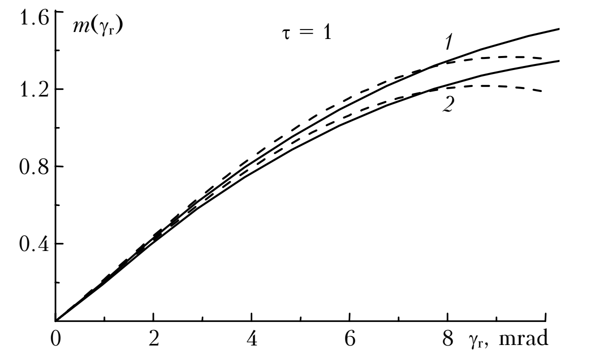

Fig. 3. Angular dependencies $m(\gamma_r)$ with allowance for the geometrical optics component of the scattering phase function (*1*) and without it (*2*); solid lines show the calculation by exact formulas in the small-angle approximation; dashed lines denote the calculation by asymptotic formulas in the square approximation.

With allowance for the cubic summand that makes a negative contribution at any γ_r, the deviation is even larger (curves *3*, *3′*), because $b_2 = 0$ and $B_2(\xi) = \text{const} = -1/3$ for the coefficient of the asymptotic series $D_2(\xi) = b_2 A_2 + a_3 B_2(\xi)$. So, the final expression for the cube term is

$$D_2 \xi^3 = -(1/18)[m_1(\gamma_r)]^3, \qquad (30)$$

where $m_1(\gamma_r)$ describes the linear part of the asymptotic approximation according to the formula (22). The coefficient a_n is defined by small-angle scattering of multiplicity n. Therefore, it can be considered that the cube term of the asymptotic series is defined by the small-angle triple scattering. Its contribution increases the proportionally to the third powers of mean particles' radius and optical thickness. This can be observed in comparison of the curves *3* and *3′* in Fig. 1. Within the frameworks of the approximation considered, the cube term, as is seen from Fig. 1*b*, is a single factor that makes a negative contribution into the partial sum of the asymptotic series with increase of the optical thickness.

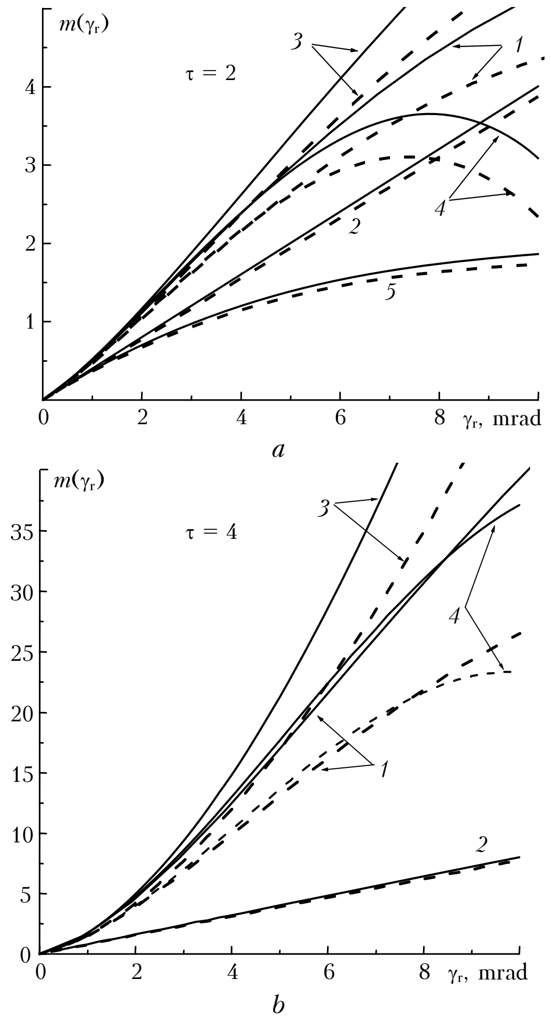

Fig. 4. Linear (*2*), square (*3*), and cube (*4*) approximations of the function $m(\gamma_r)$ by an asymptotic series; *1* denotes calculation by exact formulas in the small-angle approximation; *5* is double scattering approximation; calculation without (dashed lines) and with allowance for the geometry optics component of the phase function (solid lines).

Figure 2 visualizes the relation between the exact behavior of $m(\gamma_r) = m^{(D)}(\gamma_r)$ (curve *1*) and its approximations as partial sums of the asymptotic series (curves *2*–*4*). For comparison, we present the function $\hat{m}^{(D)}(\gamma_r)$ (23) in the approximation of double scattering (curve *5*). As is seen from Fig. 2, the square approximation (curve *3*) provides for high accuracy up to $\gamma_r = 9$ mrad ($\delta = 5.5\%$, δ is the relative error). If we, in addition, take into account the cube term (curve *4*), we obtain a little increase in accuracy, but only in a more narrow range of viewing angles, and the error δ exceeds 6% already at $\gamma_r > 6$.

Besides, it is interesting to note the following fact. As stated above, the linear part of the asymptotic approximation (22) coincides with that of approximation of double scattering (24). They are shown in Fig. 2 by the line *1*, which appreciably diverges even for small γ_r from $\hat{m}^{(D)}(\gamma_r)$ (curve *5*) calculated by the formula (23). However, it remains sufficiently close to the exact dependence $m^{(D)}(\gamma_r)$ (curve *1*) till $\gamma_r = 4$ mrad.

The above-mentioned properties of the asymptotic decomposition of $m(\gamma_r)$ remain valid if we add the geometrical optics component to the scattering phase function. For instance, the dependencies $m(\gamma_r)$ calculated by exact formulas (solid lines) and by asymptotic formulas of the second order (dashed lines) for scattering phase functions with contribution of the geometrical optics component (curves *1*) and without it (curves *2*) are compared in Fig. 3. As is seen from the presented results, the range of angles γ_r, for which the asymptotic decomposition provides acceptable accuracy, essentially does not depend on the choice of a model for the small-angle scattering phase function.

Figure 4 presents the behavior of partial sums of the asymptotic series with increase of optical thickness τ. Comparing the results (Fig. 4*a*) for $\tau = 2$ with the case considered above ($\tau = 1$; see Figs. 2 and 3), one can see the following features. First, the domain of viewing angles, in which it is possible to be satisfied by a linear approximation (presented in Fig. 4*a* by lines *2* for two types of scattering phase functions) is considerably narrowed (less than 1 mrad) with increase of τ. Neglect by scattering of high multiplicity, which begins to play a significant part for $\tau = 2$ and sufficiently large γ_r, in the linear approximation causes a lower position of lines *2* relative to exact dependencies $m(\gamma_r)$ (curves *1*).

The square approximation also does not provide for sufficient accuracy and leads to excessive values of $m(\gamma_r)$ (curves *3*). Finally, if the negative cubic term is taken into account, it compensates the influence of the first two summands and, as a result, the error of such approximation decreases, for instance, to 10–11% at $\gamma_r = 7$ mrad (with variations in the choice of the scattering phase function).

These trends manifest themselves even to a greater extent with increase of optical thickness and can be illustrated by calculations of the function $m(\gamma_r)$ presented in Fig. 4*b* at $\tau = 4$. With allowance for the cubic term, the approximation error decreases in this example from 58–72 to 2–6% for $\gamma_r = 9$ mrad depending on the type of the scattering phase function.

Conclusions

We considered a new mathematical model intended to determine the contribution of multiple scattering (correcting factor) in lidar signals in sensing dense media.

The well-known integral expression for the correcting factor obtained within the frameworks of small-angle approximation with allowance for single scattering by large angles was taken as the initial. By the use of the method of consequent decomposition for the correcting factor we have constructed an asymptotic series by powers of a small parameter, the viewing angle of a lidar's receiver. The construction is based on taking into account of properties of the optical transfer function in the range of high spatial frequencies. The coefficients of the asymptotic series are shown to depend on integral parameters of the media's microstructure and can be easily calculated by simple analytical formulas. In particular, for all odd terms of the decomposition the only of such parameters is mean particles' radius defined by ratio of the third moment of the distribution function of particles' size to the second moment.

The accuracy and limits of applicability of the developed asymptotic model are estimated by numerical simulation. In particular, not more than three terms of the asymptotic series turned to be sufficient to take into account the contribution of multiple scattering in lidar signals in sounding droplet clouds. The error of the asymptotic approximation does not exceed 5–10% for viewing angles less than 9 mrad. These estimates are valid for the small-angle scattering phase function both in the diffraction approximation and with allowance for the geometrical optics component. Optical thickness also has a weak effect on the accuracy of the asymptotic approximation in the range of the considered values from 1 to 4.

The considered asymptotic approximation for the correcting factor can be applied to developing algorithms for recovering cloud droplet sizes from lidar measurements.

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