Synthesis of wavelet basis for analysis of optical signals. Part 1. Orthogonal wavelet basis

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To restore and analyze random optical signals, various orthogonal bases are used. The algorithm enabling one to synthesize orthogonal wavelets obeying the condition of multiple scale analysis is presented along with a large number of new wavelets obtained with this algorithm. Orthogonal symmetric wavelets are synthesized. Examples of image compression and filtering are presented. Local properties of wavelets and the possibility of expanding signal about inhomogeneity scales are demonstrated.

The intensity and phase of optical radiation having passed through the atmosphere with random inhomogeneities are used to reveal the information about the atmosphere or for information transmission through it. Due to interaction with a turbulent medium, the phase and intensity of a wave become topologically complex objects. A smooth shape of the wave front transforms into a discontinuous broken structure with power-law peculiarities and singularities.

For simple and convenient analysis, these complex mathematical objects will conditionally be called random optical signals, which are usually presented by sums of orthogonal components in an infinite number of ways. Since every time the system of orthogonal functions used for expansion is known, the intensity and phase of an optical wave are fully determined by the sets of weighting factors for these functions.

Such sets are spectra of optical signals. The spectrum is the only possible form of analytical presentation of signals within the framework of a linear theory, and the problem reduces to selection of suitable basis set of functions convenient for solution of that or other practical problem. The adequacy of a restored signal to the actual one depends on how successful is selection of the basis set. In solving the problems of field reconstruction, it is necessary to store large data arrays, to remove image noise, to shorten the data processing time, and thus to approach the real time scale of a physical process, and to follow evolution of the signal frequency due to different time and spatial scales of inhomogeneities.

In this paper, we do not consider polynomial functions, whose smoothness makes them to behave in a certain manner, because signals, we deal with in practice, are thought to be continuous, but nondifferentiable. Polynomial functions are not flexible enough to follow up the jumps and power-law peculiarities in realistic signals. Therefore, in this paper we consider wavelet bases possessing the fractal property of self-similarity and allowing singularities and breaks of the studied signal to be followed up. A remarkable property of self-similarity is a cause for other useful properties, such as locality and the possibility of fast transformations, high-degree compression of signals and images, separation of singularities and fractal structure of a random signal, and expansion of the studied signal about scales of inhomogeneities.

Now there exists a wide class of wavelet bases of different nature, but their mathematical description is poorly covered in our domestic literature. The number of publications on synthesis of wavelet transformations is not large. Let us note some of them. Reference 1 considering possible applications of wavelet transformations is interesting and useful. Most informative are Refs. 2 and 3, which present classifications of wavelets and all the needed theorems on synthesis of wavelet transformations. Recently we have got the translations of the outstanding basic books by Daubechies[4] and Chui[5] on wavelet construction. And, finally, the most powerful source of information about wavelets is Internet. On such sites as www.mathsoft.com/wavelets.html, www.wavelet.org, www-stat.stanford.edu/~wavelab, and www.math.spbu.ru/user/dmp, one can find the information about programs, papers, and conferences on the theory and applications of the wavelets. The latter of the sites listed above is the site of St. Petersburg Seminar on Wavelets and Their Applications. For engineers interested in application of specialized software on wavelet transformations, we can recommend the well-known MATLAB-6.1, Mathematica-4, and Mathcad-2001 (Wavelet Extension) software packages.

In this paper, we perform synthesis of new wavelet bases and demonstrate their application to signal processing. These wavelet bases are new, because no one of the wavelets obtained below falls in a known group presented in the literature. Below we will consider the synthesis of symmetric wavelets, which is principally impossible as stated by the authors of many

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papers devoted to this subject. From a wide class of wavelets, we will construct only orthogonal wavelets and only those of them that have the multiple scale features. It is just this feature that allows realization of efficient fast expansion and restoration algorithms.

For convenience of analysis, the studied signal $f(x)$ is expanded into a series over orthonormal wavelet functions

$$\Psi_{jk}(x) = 2^{-j/2}\,\Psi(2^{-j}\,x - k);$$

$$f(x) = \sum_{jk}^{N} c_{jk}\,\Psi_{jk}(x). \quad (1)$$

The expansion coefficients are determined by the scalar product

$$c_{jk} = \langle f, \Psi_{jk}\rangle = \int_{-\infty}^{\infty} f(x)\,\Psi_{jk}(x)\,\mathrm{d}x. \quad (2)$$

Let us construct the basis function $\Psi(x)$ used in Eq. (1). In so doing let us choose the scaling function $\varphi(x)$ having the fractal multiple-scale feature:

$$\varphi(x) = \sum_{k=0}^{N} p_k\,\varphi(2x - k). \quad (3)$$

This equation is called the scaling equation and is the basic one in the wavelet theory. It serves a tool for construction of new wavelets and gives a significant gain in the computation speed. Note that the function $\varphi(x)$ has no analytical form. It is formed as a result of compression and shift of self-similar functions. The form of the function $\varphi(x)$ is determined by the expansion coefficients and their number, that is, the upper summation index. The upper index N is also the support of the function $\varphi(x)$, $N = \mathrm{supp}\,\varphi(x)$. To find this function means determination of the coefficients of this expansion.

Let us write the needed system of equations for the coefficients. For this purpose, subject the function $\varphi(x)$ to the normalization condition

$$\int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x = 1. \quad (4)$$

For the coefficients this equation transforms into the following form:

$$\sum_{k=1}^{N} p_k = 1. \quad (5)$$

Assume that $\varphi(x)$ satisfies the conditions of orthogonality

$$\langle \varphi(x), \varphi(x-k)\rangle = \delta_{0k}. \quad (6)$$

Substituting Eq. (3) into Eq. (6) we obtain for the coefficients

$$\sum_{k=0}^{N} p_k\,p_{k+2m} = \delta_{0m}, \quad m = 0, \ldots, N/2. \quad (7)$$

The wavelet function $\Psi(x)$ in this case is determined by the equation[4,5]:

$$\Psi(x) = \sum_{k=0}^{N} (-1)^k\,p_{N-k}\,\varphi(2x - k), \quad (8)$$

which is a corollary of the equation

$$\int_{-\infty}^{\infty} \Psi(x)\,\mathrm{d}x = 0.$$

The system of equations (5) and (7) is needed to solve the variational problem on determination of the coefficients to construct wavelets. For example, at $N = 4$ we have the equations

$$\sum_{k=0}^{4} p_k^2 = 1, \quad p_0\,p_2 + p_1\,p_3 = 0. \quad (9)$$

To restrict ambiguity, in Ref. 3 it is proposed to invoke the equation of moments for the function $\varphi(x)$: $\int_{-\infty}^{\infty} \varphi(x)\,x^m\,\mathrm{d}x = M$. Then we obtain some more equations

$$\sum_{k=0}^{N} (-1)^k\,k^m\,p_k = M, \quad m = 0, \ldots, N/2. \quad (10)$$

However, under such additional conditions, the convergence of numerical, variational algorithms depends, to a high degree, on the proper selection of the initial values of the coefficients. This requires certain skills from an investigator. If the initial, starting values are not properly chosen, the iteration algorithm may fail to converge. Wavelets not obeying the restriction (10), that is, wavelets with lower smoothness, are excluded from the solution. Therefore, we need rigorous conditions, which, on the one hand, should allow some freedom in choosing the initial values of the variational algorithm, but, on the other hand, they would allow us to find all possible solutions.

I succeeded in obtaining a criterion, which is, in essence, a more rigorous requirement for orthogonality and can be expressed by the equations

$$\sum_{k=0}^{N} p_k\,p_{k+1} = b1, \quad (11)$$

$$\sum_{k=0}^{N} p_k\,p_{k+3} = b3; \quad (12)$$

$$b3 = 0.5 - b1, \quad (13)$$

where $b1 \le 1/\sqrt{3}$ and $b0 = 1$, $b2 = b4 = 0$ according to Eq. (7). Note that the index of the coefficients b corresponds to the fixed index in Eqs. (7), (10), and (11). Continuously decreasing the value of the coefficient $b1$ from the maximum of ≈ 0.56 and using Eqs. (11)–(13) along with Eq. (7), we can obtain all possible values of the coefficients for construction of orthogonal wavelets. With this algorithm, we can, for example, follow the evolution of transition of the known second Daubechies wavelet into the first Haar–Daubechies wavelet. In the gap between the first and second wavelets, this algorithm shows a cascade of wavelets with different degree of smoothness (Fig. 1). Using this algorithm, I have obtained a wide variety of wavelets $\varphi(x)$ with different degree of smoothness and different N supports, and, starting from $N = \text{supp}\varphi(x) = 5$, one of a set of solutions was for a symmetric wavelet. Some of solutions were obtained analytically.

After determination of the coefficients, it is necessary to present graphically the scaling $\varphi(x)$ and wavelet $\Psi(x)$ functions. As was said above, these functions have no analytical form. Therefore, let us describe an algorithm for graphical representation of the functions $\varphi(x)$ and $\Psi(x)$. For this purpose, we will use Eq. (3)

$$\varphi(x) = \sum_{k=0}^{N} p_k\, \varphi(2x - k). \tag{14}$$

Let us take discrete values $s = 1, 2, \ldots, N - 1$ as x, then Eq. (3) can be rewritten in the form

$$\varphi(s) = \sum_{k=0}^{N} p_{2s-k}\, \varphi(k). \tag{15}$$

It can be easily seen that it is an equation for eigenvalues, which can be presented as

$$\mathbf{A}\varphi = \varphi. \tag{16}$$

In particular, at $N = 4$ we obtain the equation, in which the matrix $\mathbf{A}$ has 3×3 dimensionality:

$$\mathbf{A} = \begin{pmatrix} p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 \\ 0 & p_4 & p_3 \end{pmatrix}. \tag{17}$$

Equation (16) for eigenvalues and the normalization condition $\sum_{k=0}^{N} \varphi(k) = 1$ allow us to determine $\varphi(x)$ at the points $x = 1, 2, 3, \ldots, N - 1$. Recall that $\varphi(x) = 0$ at the points $x = 0$ and $x = N$. Then we can find $\varphi(x)$ at the points $x/2$ ($x = 1, 2, \ldots, N - 1$) using the recurrence equation (14). Thus, iterations are continued until the required accuracy is reached. The value of the wavelet function $\Psi(x)$ is determined using Eq. (8) in parallel with the scaling function $\varphi(x)$ at the same points.

Let us present the coefficients obtained for the functions $\varphi(x)$ and $\Psi(x)$ through solution of the variational problem with the use of Eqs. (7)–(11) at $N = 4$:

$$a = 3.3431457;\ \ p_0 = \frac{1-\sqrt{a}}{12},\ \ p_1 = \frac{3-\sqrt{a}}{6},$$

$$p_2 = \frac{5+\sqrt{a}}{6},\ \ p_3 = \frac{3+\sqrt{a}}{6},\ \ p_4 = \frac{1-\sqrt{a}}{12},$$

and the graphical dependences of the functions $\varphi(x)$ and $\Psi(x)$ obtained with the use of the algorithm described above (Fig. 2*a*).

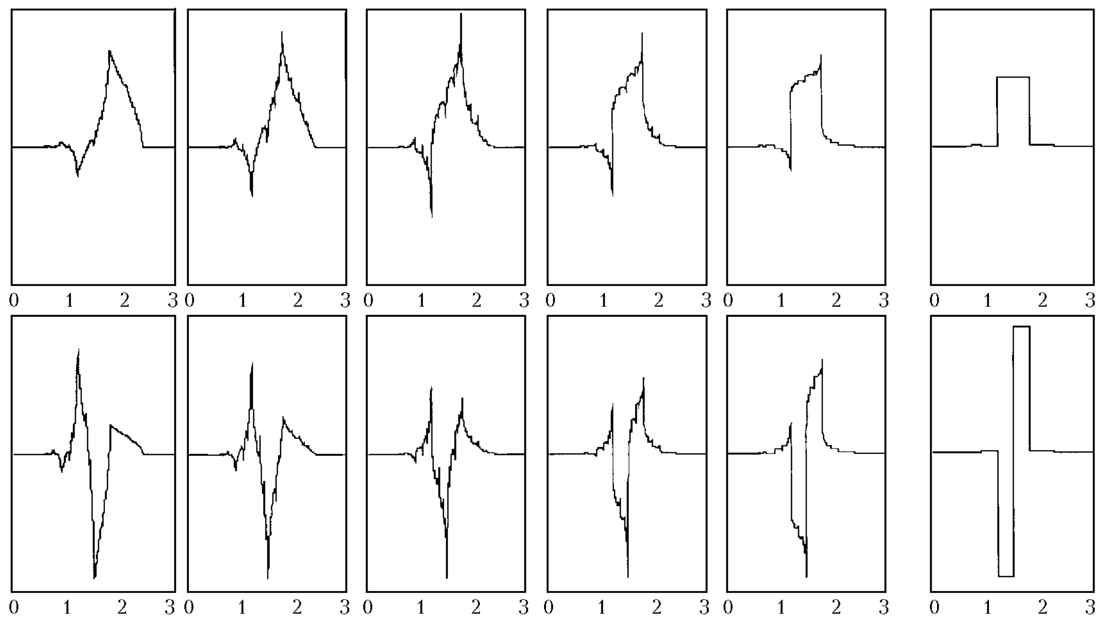


Fig. 1. Evolution of transformation of the Daubechies wavelet 2 into the Haar wavelet; scaling functions (top row) and wavelets (bottom row).

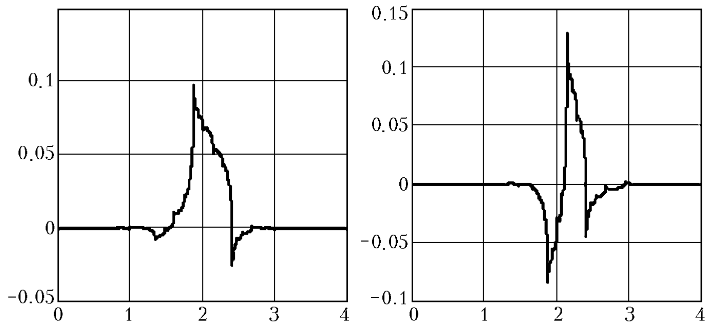
$p_0 = -0.04867$

$p_1 = 0.138$

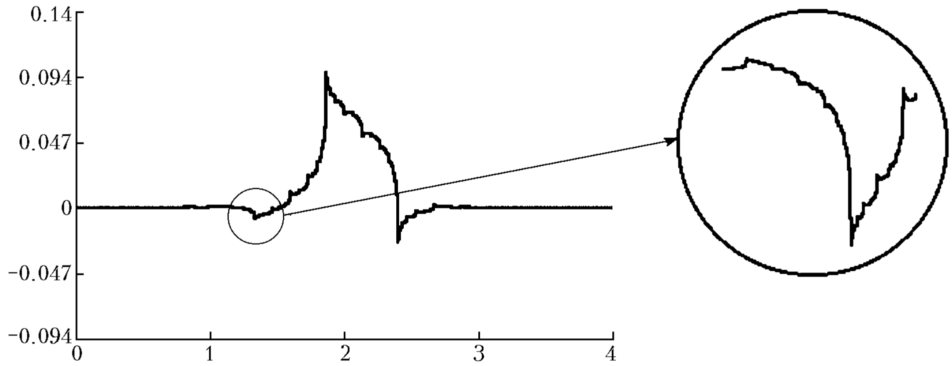
$p_2 = 0.80646$

$p_3 = 0.57079$

$p_4 = -0.049$

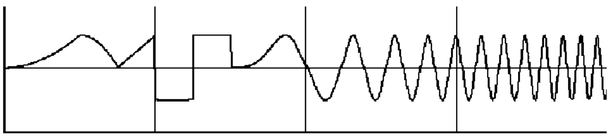


a

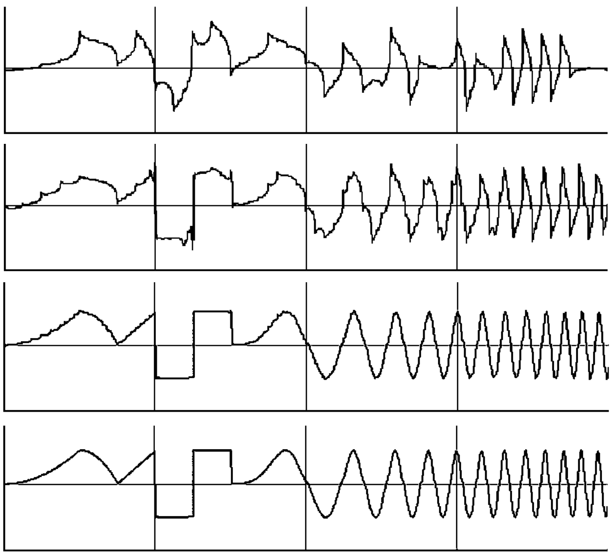


b

Fig. 2. Coefficients, φ (left) and Ψ (right) functions at $N = 4$ (*a*); demonstration of fractal structure of scaling function (*b*).



a



b

Fig. 3. Model signal (*a*) and wavelet expansion of signal at $K = 2^4, 2^6, 2^8, 2^{10}$ from top to bottom, respectively (*b*).

As can be seen from Fig. 2*b*, the scaling function $\varphi(x)$ has a fractal character. A distinguished small fragment enlarged several times clearly demonstrates the fractal character of the function $\varphi(x)$. Figure 3*a* shows the model function with derivative jumps, and Fig. 3*b* shows the successive stages of wavelet restoration of the model function using an increasing number of wavelets. At restoration, the coefficients were sorted and less significant ones were rejected. This process is called signal compression. The largest number of the coefficients K in the expansion was $2^{10} = 1024$.

To synthesize a smoother wavelet, increase the coefficient $b1$ in Eq. (11). After variation of the coefficients, we obtain the following values and graphical dependences for the scaling and wavelet functions, respectively (Fig. 4*a*).

Let us present synthesis of a symmetric wavelet. The symmetric wavelet, because of the symmetry of the coefficients, allows the time needed for signal expansion and restoration to be significantly shortened. Besides, it is preferable because it usually has more zero moments, what leads to better signal compression. In the regions, where the signal is smooth, its expansion coefficients for small scales are zero.

References 4 and 5 present synthesis of symmetric wavelets within the framework of multiple-scale analysis, but, unfortunately, they are either nonorthogonal or weakly symmetric (symlets, Coiflets). The categorical statement of impossibility to construct symmetric orthogonal wavelets[4] likely deterred investigators from this problem. In our case, wavelets acquire symmetry due to a small loss in smoothness and accuracy. (Speaking about inaccuracy, we mean the error not exceeding the desired level of accuracy in signal restoration.) To synthesize a symmetric wavelet, we have to solve the variational problem for Eqs. (9), (11), and (12) under the additional condition of symmetric coefficients. After some simple transformations, we obtain the following system of equations:

$$2p_0^2 + 2p_1^2 + p_2^2 = 1, \tag{18}$$

$$2p_0\, p_2 = -\, p_1^2, \tag{19}$$

$$2p_0 - 2p_1 + p_2 = 0. \tag{20}$$

The solution of this system is

$$p_0 = -\frac{\sqrt{15134 - 10304\sqrt{2}}}{322}, \quad p_4 = p_0,$$

$$p_2 = \frac{4\sqrt{2093 - 322\sqrt{2}} - \sqrt{15134 - 10304\sqrt{2}}}{161},$$

$$p_1 = \sqrt{-2p_2\, p_0}, \quad p_3 = p_1.$$

The obtained coefficients, scaling and wavelet functions are shown in Fig. 4*b*.

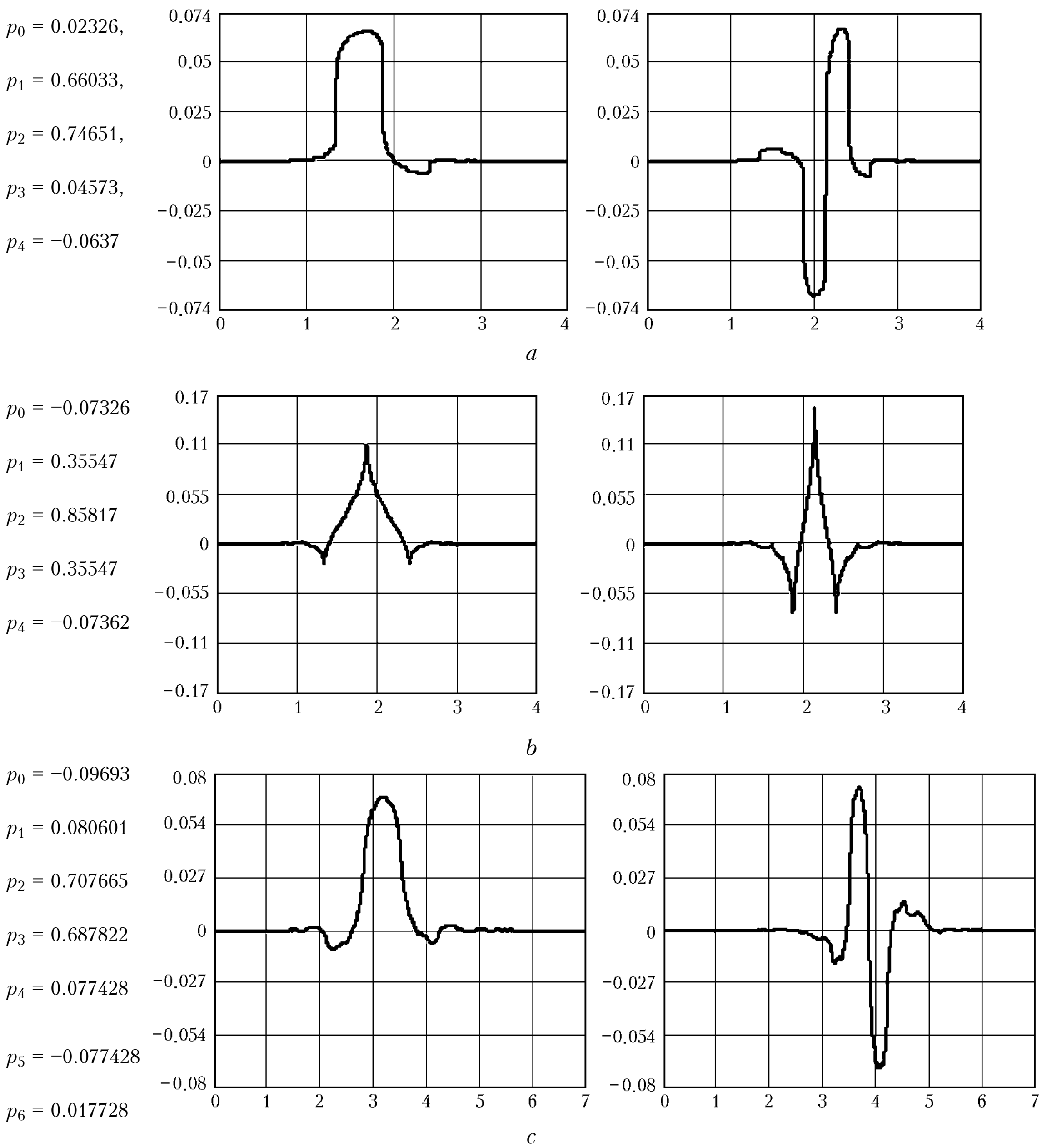


Fig. 4. Scaling function φ (left) and wavelet function Ψ (right).

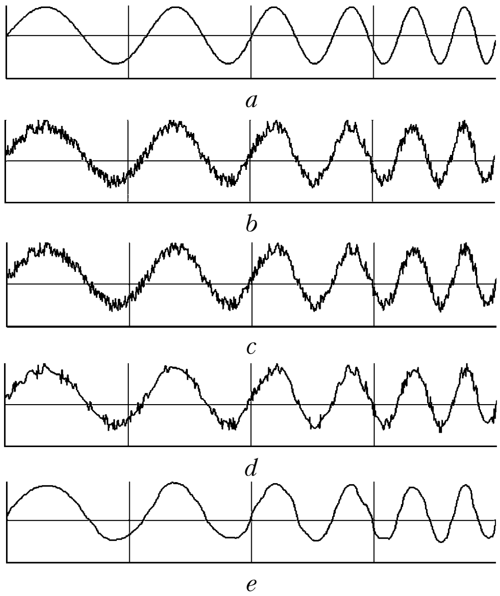


Fig. 5. Filtering of model signal; K is the number of expansion coefficients: model signal (*a*), model signal with 5% noise (*b*), signals restored at $K = 2^9$ (*c*), 2^7 (*d*), and 2^5 (*e*).

For a wavelet to have higher smoothness, increase the support of the scaling function $N = \mathrm{supp}\,\varphi(x)$. At $N = 6$ we obtain the coefficients and plots for $\varphi(x)$ and $\Psi(x)$ that are shown in Fig. 4*c*. The wavelet obtained is smoother and close to the symmetric one. Let us use the synthesized wavelets for signal compression. Impose 5% noise on the model signal (Fig. 5*a*) and then try to remove it. The results of signal filtering are shown in Figs. 5*c*–*e*. The maximum number of the expansion coefficients K was 2^9.

To improve the symmetry of the scaling function, we should increase the support N and take into account the additional condition of symmetric coefficients. For example, if at $N = 7$ we additionally assume $p_0 = p_7$, $p_1 = p_6$, $p_2 = p_5$, $p_3 = p_4$, then the system of equations reduces to four equations. Having solved this system, we obtain the group of wavelets shown in Fig. 6. It should be noted that $p_1 = -p_2$ is valid for the first and second coefficients. So, the system of equations could be reduced by one equation more. Thus, it is sufficient to vary only three coefficients.

$p_0 = p_7 = 0.003396$

$p_1 = p_6 = -0.049$

$p_2 = p_5 = 0.048765$

$p_3 = p_4 = 0.703711$

$p_0 = p_7 = 0.00543$

$p_1 = p_6 = 0.061967$

$p_2 = p_5 = -0.061491$

$p_3 = p_4 = 0.701676$

$p_0 = p_7 = 0.000176$

$p_1 = p_6 = -0.011213$

$p_2 = p_5 = 0.01121$

$p_3 = p_4 = 0.706929$

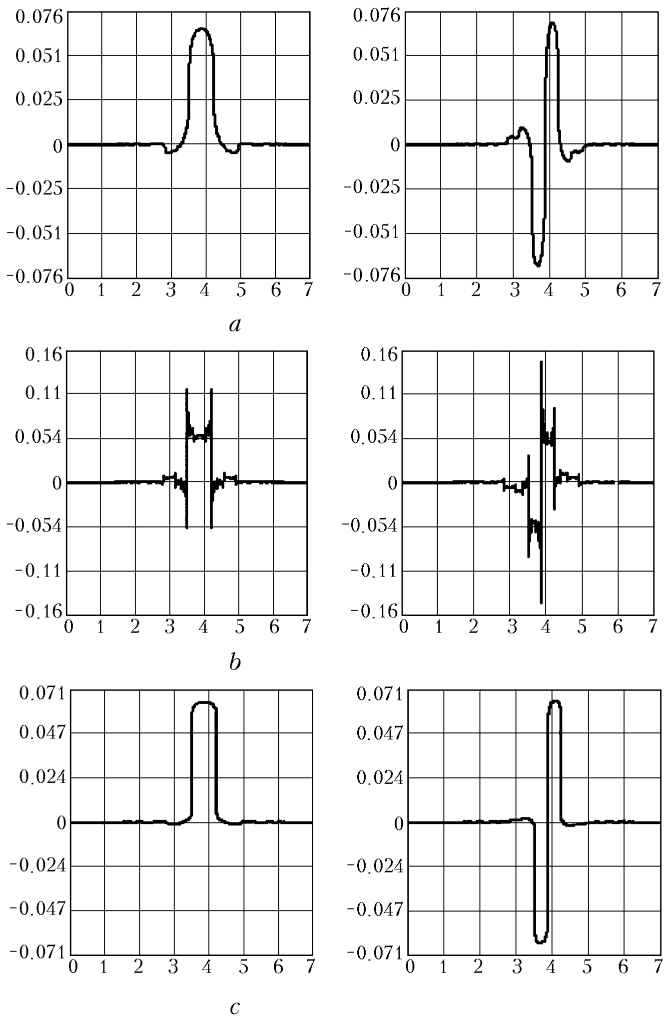


Fig. 6. Scaling function φ (left) and wavelet function Ψ (right).

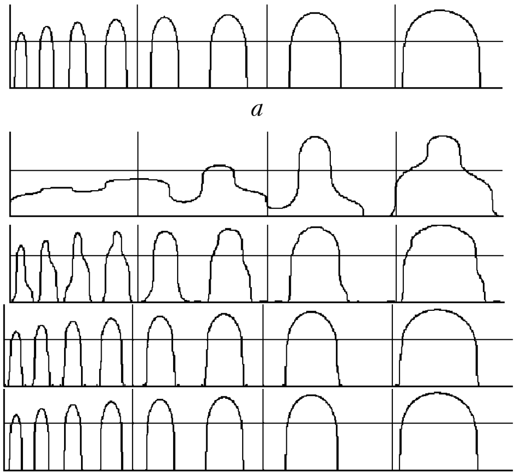


Fig. 7. Model signal (*a*) and wavelet expansion of signal at $K = 2^4, 2^6, 2^8, 2^{10}$ from top to bottom, respectively (*b*).

Let us present the wavelet expansion of a broken model signal shown in Fig. 7*a*. This expansion demonstrates good local properties of wavelets. As a basis one, we took the symmetric wavelet shown in Fig. 6*a*.

Present the coefficients and plots for some more wavelets obtained by me by the algorithm described above at $N = 8$ (Fig. 8*a*), $N = 9$ (Fig. 8*b*), and $N = 10$ (Fig. 8*c*). Demonstrate the possibility of wavelets to separate a signal according to the scales of inhomogeneities. As an example, we will use the wavelet shown in Fig. 8*c*.

The model signal is taken as a sum of two sinusoids (Fig. 9*a*). The frequency of one sinusoid is 10 times higher than that of the other. The amplitude of the high-frequency sinusoid is half as high as that of the low-frequency one.

Figure 9*b* shows the low-frequency trend obtained by retention of only first 2^5 terms of the expansion from 2^{10} along with the high-frequency trend obtained

$p_0 = p_8 = 0.025923$

$p_1 = p_7 = -0.074482$

$p_2 = p_6 = 0.068904$

$p_3 = p_5 = -0.282907$

$p_4 = -0.90443$

$p_0 = p_9 = -0.004968$

$p_1 = p_8 = -0.004968$

$p_2 = p_7 = 0.08323$

$p_3 = p_6 = -0.08323$

$p_4 = p_5 = -0.697206$

$p_0 = p_{10} = 0.000828$

$p_1 = p_9 = 0.018714$

$p_2 = p_8 = -0.045872$

$p_3 = p_7 = -0.073234$

$p_4 = p_6 = 0.398704$

$p_5 = 0.816354$

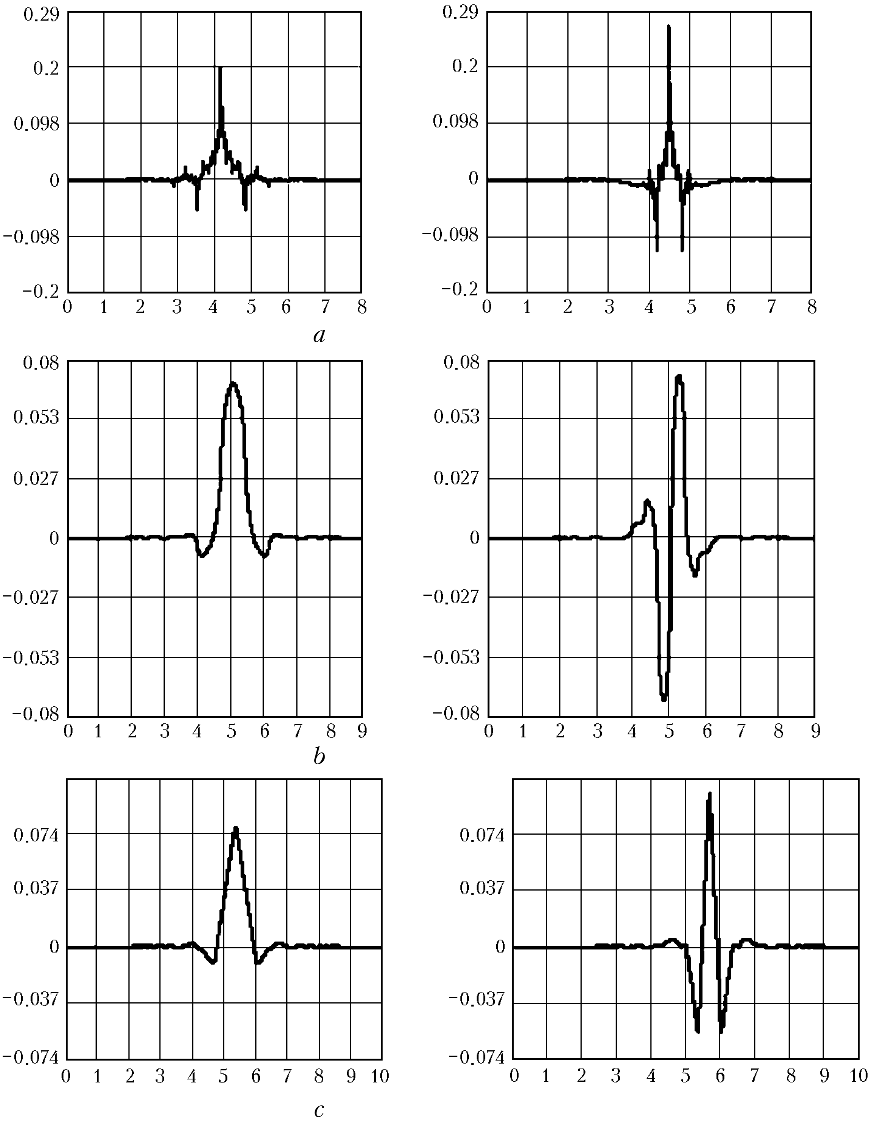


Fig. 8. Scaling function φ (left) and wavelet function Ψ (right).

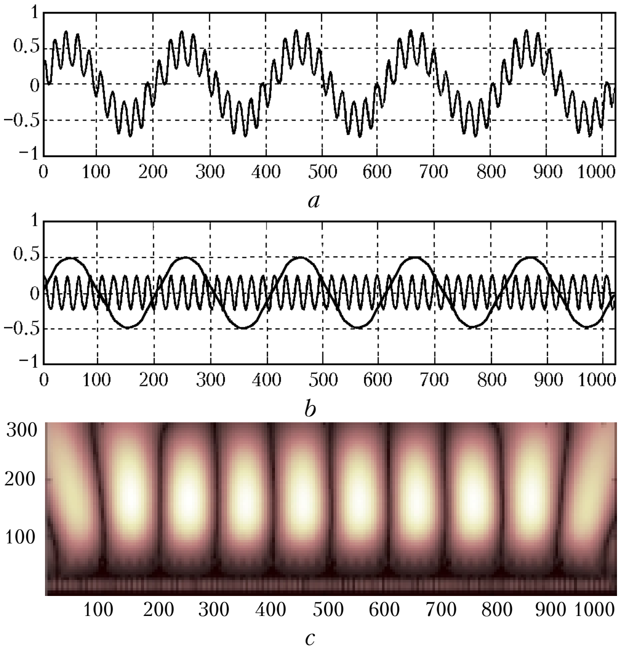


Fig. 9. Model signal as a sum of two sinusoids (*a*), separation of model signal according to scales of inhomogeneities with the use of wavelet expansion (*b*), absolute values of the coefficients of wavelet expansion of the model signal (*c*).

by subtraction of the first 2^5 terms of the expansion from 2^{10} terms. Figure 9*c* depicts the pattern of absolute values of the expansion coefficients that is a scan of inhomogeneity scales of the model signal.

In this paper, a variational algorithm for determining the coefficients of the scaling function in the scaling equation is presented. The algorithm has rigorous restrictions providing for its fast convergence and the possibility to determine all possible solutions at the given support of the scaling function.

The variational algorithm allows synthesis of orthogonal wavelets obeying the condition of multiple scale analysis. A large number of new wavelets synthesized by this algorithm are presented. Orthogonal symmetric wavelets obeying the multiple-scale feature are obtained for the first time. Some examples of image compression and filtering are presented. Local properties of wavelets are demonstrated along with the possibility of expanding signals according to the inhomogeneity scales.

In the second part of this paper, I plan to describe in detail the algorithm for synthesis of nonorthogonal symmetric wavelets. Algorithms for presentation of differential and integral operators through wavelet bases will be presented. These algorithms can be useful, in particular, for modeling wave front slopes of a wave having a broken structure (direct problem) and for reconstruction of a signal from its local slopes with fading areas (inverse problem).

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