POTENTIAL AND VORTEX PARTS OF OPTICAL SPECKLE-FIELDS

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Wavefront dislocations generated by vortex flux of light energy in the vicinity of points with zero field intensity are studied on the basis of a solution to the problem of retrieval of the phase distribution of an optical beam from the measured intensity distribution (the phase problem in optics). Reasons for ambiguity of the phase problem solution are discussed. Concepts of the vortex and potential parts of the phase are introduced. An analytic formula for retrieval of the potential phase from the measured intensity distribution of optical speckle-field has been obtained. It has been suggested to use the potential phase for correction of singular phase distortions with optical adaptive systems.

INTRODUCTION

Wavefront dislocations in optical speckle-fields[1,2] have been the object of investigation for researchers engaged in the problem of the wave propagation through the atmosphere and an indicator of a number of phenomena and processes in nonlinear optics[3,4] and laser physics. Dislocations create main obstacles to the phase aberration compensation with adaptive optical systems.[5] Moreover; a new lead of research connected with the use of the dislocations for diagnostics of natural media[6] is formed. In the present paper, we consider a possibility to reveal the dislocations on the basis of a solution to the problem of retrieval of the wave phase from the measured intensity distribution $I(\rho_0, z)$, $\rho = \{x, y\}$. Our approach involves the reconstruction of the Poynting vector components using the differential eikonal and transfer equations describing the wave propagation in a medium, determination of the transverse phase derivative, and phase retrieval from its partial derivatives (wavefront slopes) on the entrance pupil.

THEORETICAL GROUNDS

The propagation of a monochromatic optical wave $U(\rho, z)$ is described by the parabolic equation

$$2\, ik \frac{\partial U}{\partial z} + \Delta_\perp U + k^2\, \tilde{\varepsilon}(\rho, z)\, U(\rho, z) = 0 \qquad (1)$$

in the half-space $z \geq 0$ filled with a refractive medium with the permittivity $\tilde{\varepsilon}(\rho, z)$. Introducing the phase $S(\rho, z)$ and substituting the field $U(\rho, z) = \{I(\rho, z)\}^{1/2} \exp\{iS(\rho, z)\}$ into Eq. (1), we obtain the system

$$2\, kI^2 \frac{\partial S}{\partial z} + I^2\, \{\nabla_\perp S\}^2 = k^2\, I^2\, \tilde{\varepsilon}(\rho, z) + \frac{1}{2} I\, \Delta_\perp I(\rho, z) - \frac{1}{4} \{\nabla_\perp I(\rho, z)\}^2, \qquad (2)$$

$$\nabla_\perp \{I(\rho, z)\, \nabla_\perp S\} = -\, k \frac{\partial I}{\partial z}, \qquad (3)$$

where $\tilde{\varepsilon}(\rho, z) = (\varepsilon - \langle\varepsilon\rangle)/\langle\varepsilon\rangle$, $k = 2\pi\sqrt{\langle\varepsilon\rangle}/\lambda$, $\langle\varepsilon\rangle$ is the mean value of the permittivity, λ is the wavelength, $\Delta_\perp = \nabla_\perp \cdot \nabla_\perp$, $\nabla_\perp = \mathrm{l} \frac{\partial}{\partial x} + \mathrm{m} \frac{\partial}{\partial y}$.

Equation (3) is the differential form of the energy conservation law for the transverse component of the Poynting vector L $\{\mathrm{L}_\perp, k\, I\}$, $\mathrm{L}_\perp = I(\rho, z)\, \nabla_\perp S(\rho, z)$. The vector field $\mathrm{L}_\perp$ can be represented as a sum of the potential $\mathrm{L}_{\perp \mathrm{p}}$ and vortex $\mathrm{L}_{\perp \mathrm{v}}$ parts[7]

$$I(\rho, z)\, \nabla_\perp S = I(\rho, z)\, \nabla_\perp S_\mathrm{p} + I(\rho, z)\, \nabla_\perp S_\mathrm{v}, \qquad (4)$$

where $S_\mathrm{p}(\rho, z)$ is the potential phase and $S_\mathrm{v}(\rho, z)$ is the vortex phase. It follows from the general principles of the field theory that the potential vector field can be expressed in terms of the potential φ (Ref. 7)

$$\mathrm{L}_{\perp \mathrm{p}} = \mathrm{grad}\, \varphi, \qquad (5)$$

and the divergence of the vortex field is equal to zero, i.e., $\nabla_\perp \cdot \mathrm{L}_{\perp \mathrm{v}} = 0$. Therefore, only the potential part of the Poynting vector can be reconstructed with the use of transfer equation (3). Substituting Eq. (5) into Eq. (3), we obtain the Poisson equation for the potential

$$\Delta_\perp\, \varphi = -\, k \frac{\partial I}{\partial z} (\rho, z). \qquad (6)$$

A solution of this equation with the boundary condition on the beam periphery $\varphi(\infty, \infty, z) = 0$ is the solution of the Dirichlet classical problem.[7] Without specifying the expression for φ we write down directly the relationship for the potential part of the Poynting vector using Eq. (5)

$$L_{\perp p}(\rho, z) = -\frac{k}{2\pi}\iint_{-\infty}^{\infty} d\xi\, d\eta\, \frac{\partial}{\partial z} I(\xi, \eta, z) \times$$

$$\times \frac{l\,(x-\xi) + m\,(y-\eta)}{(x-\xi)^2 + (y-\eta)^2}\,. \quad (7)$$

From Eq. (7) we derive the differential equations for the potential energy vector lines in the 3-D space

$$\frac{dy}{dz} = -\frac{1}{2\pi}\frac{\partial}{\partial z}\left\{\iint d\xi\, d\eta\, \frac{I(\xi, \eta, z)\,(y-\eta)}{(x-\xi)^2 + (y-\eta)^2}\right\}, \quad (8)$$

$$\frac{dx}{dz} = -\frac{1}{2\pi}\frac{\partial}{\partial z}\left\{\iint d\xi\, d\eta\, \frac{I(\xi, \eta, z)\,(x-\xi)}{(x-\xi)^2 + (y-\eta)^2}\right\}, \quad (9)$$

$$\frac{dy}{dx} = \frac{\frac{\partial}{\partial z}\left\{\iint d\xi\, d\eta\, \frac{I(\xi, \eta, z)\,(y-\eta)}{(x-\xi)^2 + (y-\eta)^2}\right\}}{\frac{\partial}{\partial z}\left\{\iint d\xi\, d\eta\, \frac{I(\xi, \eta, z)\,(x-\xi)}{(x-\xi)^2 + (y-\eta)^2}\right\}}\,. \quad (10)$$

This raises the question of unambiguity of the phase retrieval from the relationship

$$L_p(\rho, z) = I(\rho, z)\,\nabla S_p \quad (11)$$

for the potential component of the Poynting vector.

Light diffraction is accompanied with anomalous wavefront behavior in the vicinity of points with zero intensity.[8] If signs of the real and imaginary components of the complex field change to opposite ones after crossing the point or line with zero intensity, the anomaly will be manifested through the phase jump by π. The phase jumps observed when the dark rings of diffraction minima having radius r_d are crossed in the focal plane[8] are examples. The phase of such a speckle-field can be considered as a generalized function whose partial derivatives at the point $\{x, y\}$ have the following form:

$$\frac{\partial S}{\partial x} = \left\{\frac{\partial S}{\partial x}\right\} + \pi\cos(nx)\,\delta\left(\sqrt{x^2 + y^2} - r_d\right),$$

$$\frac{\partial S}{\partial y} = \left\{\frac{\partial S}{\partial y}\right\} + \pi\cos(ny)\delta\left(\sqrt{x^2 + y^2} - r_d\right),$$

where n is the exterior normal to the ring of radius r_d, $\left\{\frac{\partial S}{\partial x}\right\}$ and $\left\{\frac{\partial S}{\partial y}\right\}$ are the piecewise-continuous parts of the derivatives. It was shown in Ref. 8 that in the vicinity of the ring the following relation is true:

$$I(x, y) \sim \left(\sqrt{x^2 + y^2} - r_d\right)^2.$$

From the theory of generalized functions, we have

$$\left(\sqrt{x^2 + y^2} - r_d\right)^2 \delta\left(\sqrt{x^2 + y^2} - r_d\right) = 0,$$

and consequently singularity in the phase derivative in Eq. (11) does not influence the components of the Poynting vector L_p. So, only the piecewise-continuous part of the derivative can be determined from Eq. (11). Therefore, we can obtain the "smoothed" version of the phase $S_p(\rho, z)$. To describe the jumps, additional information is required, for example, the information carried by the analytic extension of $I(\rho, z)$ to the complex plane.[9]

Using the integral representation for the phase retrieved from the wavefront slopes,[10] we obtain from Eqs. (7) and (11)

$$\{S_p(\rho, z)\} = \frac{k}{4\pi^2}\iint_D \frac{d^2\rho_0}{I(\rho_0, z)} \times$$

$$\times \frac{\partial}{\partial z}\iint_{-\infty}^{\infty} d^2\rho_0'\, \frac{I(\rho_0', z)\,(\rho_0 - \rho_0')\,(\rho - \rho_0)}{(\rho_0 - \rho_0')^2\,(\rho - \rho_0)^2}\,. \quad (12)$$

Integration in Eq.(12) is carried out over the entrance pupil D bounded by the contour c. If on the pupil boundary the phase $S_\Gamma(\rho, z)$ is nonzero, the following line integral should be added into the right side of Eq. (12):

$$\frac{1}{2\pi}\int_\Gamma \frac{S_\Gamma(\xi, \eta, z)}{(\xi - x)^2 + (\eta - y)^2}\,[(\xi - x)\,d\eta - (\eta - y)\,d\xi]. \quad (13)$$

For the vortex component of the Poynting vector, we succeeded to derive the expression

$$L_v(\rho, z) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \text{rot}\, L_v(\xi, \eta, z) \times$$

$$\times\left[l\frac{\partial}{\partial y}\ln\frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} - \right.$$

$$\left. - m\frac{\partial}{\partial x}\ln\frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}\right] d\xi\, d\eta \quad (14)$$

on the basis of Eq. (6) for the vector-potential. As a result, for the phase we obtained the following integro-differential equation:

$$S(\rho, z) = \frac{k}{4\pi^2}\iint_D \frac{d^2\rho_0}{I(\rho_0, z)} \times$$

$$\times \frac{\partial}{\partial z}\iint_{-\infty}^{\infty} d^2\rho_0'\, \frac{I(\rho_0', z)\,(\rho_0 - \rho_0')\,(\rho - \rho_0)}{(\rho_0 - \rho_0')^2\,(\rho - \rho_0)^2} +$$

$$+ \frac{1}{4\pi^2}\iint_D \frac{d^2\rho_0}{I(\rho_0, z)}\frac{\partial}{\partial z}\iint_{-\infty}^{\infty} d^2\rho_0'\, \nabla I(\rho_0', z) \times$$

$$\times \nabla S(\rho_0', z)\,\frac{(\rho - \rho_0)\times(\rho_0 - \rho_0')}{(\rho_0 - \rho_0')^2\,(\rho - \rho_0)^2}\,. \quad (15)$$

Abramochkin and Volostnikov[11] failed to solve Eq. (15) by the iteration technique without its complement by reference values of ∇S containing information on vorticity at least at the finite number of points. In principle, such information can be obtained from Eq. (2). However, this problem has not yet been solved. So we dwell on possible use of Eq. (12) for diagnostics and correction of the phase with adaptive optics systems.

ENERGY VECTOR LINES, PECULIARITIES OF THE VORTEX PHASE, AND RETRIEVAL OF THE POTENTIAL PHASE

To investigate the properties of the optical specle-field, we choose as an object a Laguerre–Gaussian optical beam excited in a resonator with round mirrors. In this case, the initial field distribution on the radiating aperture has the form

$$U(r,\,\phi) = \left(\frac{r}{a}\right)^m \exp\left\{-\frac{r^2}{2a^2} - i\,\frac{k}{2F}\,r^2\right\} \times$$

$$\times\, L_n^m\left(\frac{r^2}{a^2}\right) \exp\{i\,m\,\phi\}, \qquad (16)$$

where $r = \sqrt{x^2 + y^2}$ and $\phi = \text{arc}\tan y/x$ are the polar coordinates, a and F are the beam radius and the wavefront curvature on the radiating aperture, $L_n^m(x)$ is Laguerre polynomial. For the fist axial asymmetric mode ($m = 1$, $n = 0$) at the distance z from the source in the vacuum ($\varepsilon = 0$) the field after substitution of Eq. (16) into the Kirchhoff integral has the form

$$U(x,\,y,\,z) = q\,\Omega\,(1 + \Omega^2)^{-3/2}\,g^{1/2} \times$$

$$\times \exp\left\{3i \arctan\Omega + \frac{i\,\Omega}{2}\,\frac{x^2 + y^2}{(1 + \Omega^2)} -\right.$$

$$\left. - \,i \arctan\frac{\Omega^2\,x^2 - 1}{\Omega\,(1 + \Omega x y)}\right\}, \qquad (17)$$

$$q = \exp\left\{-\frac{\Omega^2\,(x^2 + y^2)}{2(1 + \Omega^2)}\right\},$$

$$g = \Omega^2(1 + \Omega x y)^2 + (\Omega^2 x^2 - 1)^2,$$

where x and y are the Cartesian coordinates normalized to a, $\Omega = ka^2/z$ is the diffraction parameter. It is clear from Eq. (17) that in the beam cross section, the intensity equals zero at the points $x = 1/\Omega$, $y = -1$ and $x = -1/\Omega$, $y = 1$. The intensity distribution is shown in Fig. 1.

Calculating the intensity $I(\rho,\,z)$ and $\text{rot}(I\,\nabla S)$ with the use of Eqs. (17), (14), and (12), we obtain the potential and vortex components of the Poynting vector

$$\mathrm{L}_{\perp p} = \frac{\Omega^3}{(1 + \Omega^2)^3}\,q^2\,\{\mathrm{l}\,x + \mathrm{m}\,y\}, \qquad (18)$$

$$\mathrm{L}_{\perp v} = \frac{\Omega^3}{(1 + \Omega^2)^3}\,q^2\,\{\mathrm{l}\,[fx + y(\Omega^2\,x^2 - 1) -$$

$$- 2\Omega x\,(1 + \Omega x y)] + \mathrm{m}\,[fy + x\,(\Omega^2\,x^2 - 1)]\}, \qquad (19)$$

$$f = g/(1 + \Omega^2) - 1.$$

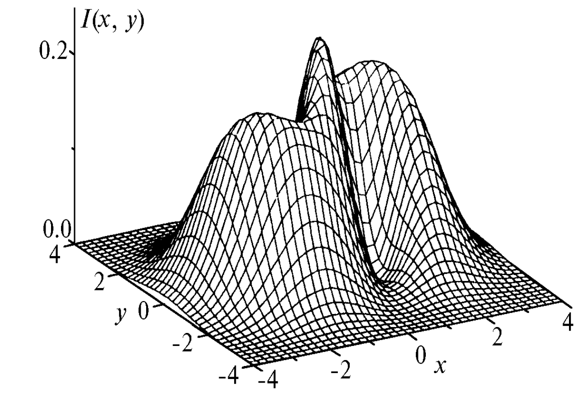


FIG. 1. Beam intensity distribution in the transverse plane.

Relationships (18) and (19) determine the potential and vortex energy flux densities of the vector field. Using the differential equations for the potential energy vector lines

$$\mathrm{d}y/\mathrm{d}z = \Omega\,y/[k\,(1 + \Omega^2)], \qquad (20)$$

$$\mathrm{d}x/\mathrm{d}z = \Omega\,x/[k\,(1 + \Omega^2)], \qquad (21)$$

$$\mathrm{d}y/\mathrm{d}x = y/x, \qquad (22)$$

and for the vortex energy vector lines

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{\Omega^2}{k}\,\frac{f\,y + x(\Omega^2\,x^2 - 1)}{g}, \qquad (23)$$

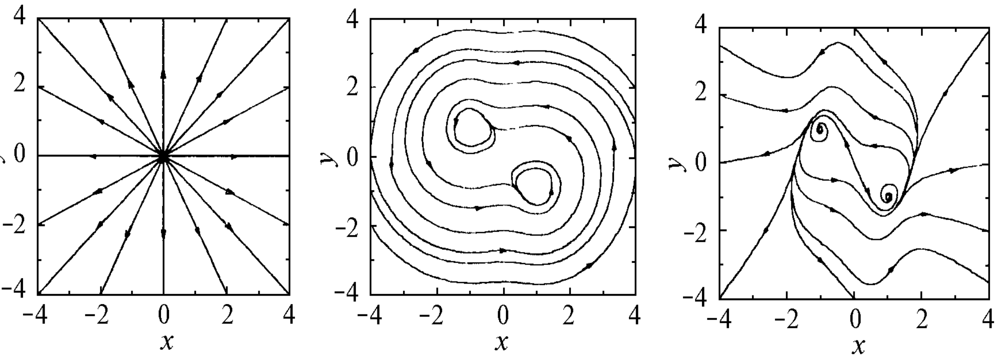
$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{\Omega^2}{k}\,\frac{f\,x + y\,(\Omega^2\,x^2 - 1) - 2\Omega x\,(1 + \Omega x y)}{g}, \qquad (24)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f\,y + x\,(\Omega^2 x^2 - 1)}{f\,x + y\,(\Omega^2 x^2 - 1) - 2\Omega\,x(1 + \Omega\,x y)} \qquad (25)$$

for our example we construct a family of curves, with tangents at each point coinciding with the direction of the vector L. In Fig. 2 such curves are drawn in the plane z = const.

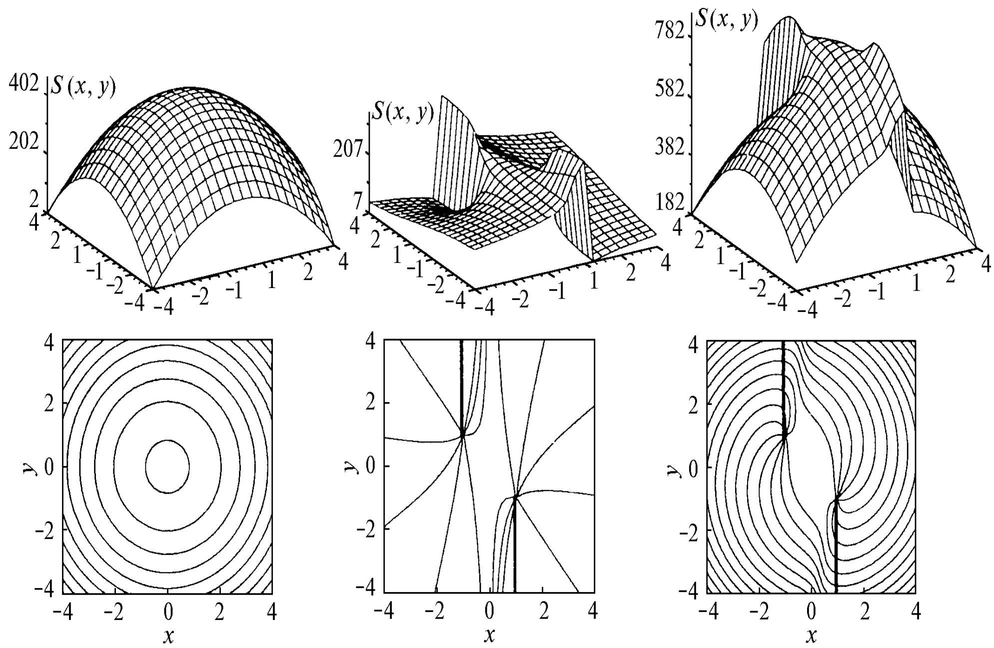
It is clear from Fig. 2*a* that only one singularity is seen for the potential vector field (the singularities can be analyzed on the basis of the theory constructed to study the phase trajectories of autonomous dynamic system[12]). This singular point is the unstable node at the center of coordinates where the numerator and denominator in the right side of differential equation (22) simultaneously vanish. Figure 2*b* shows the energy vector lines in the transverse section of the vortex vector field $\mathrm{L}_{\perp v}$. Here, three singular points can be seen. The first singularity is at the beam axis and is called the saddle ($x = 0$, $y = 0$), the second and third singularities are the complex foci at the points with zero intensity. Figure 2*c* shows the energy vector lines of the total vector field $L_{\perp}$.

Figure 3 shows the potential phase $\{S_p(\rho, z)\}$ (case *a*) retrieved by formula (12), vortex phase S_v (case *b*), and total phase S (case *c*). It is clear from Fig. 3 that the potential wavefront is the paraboloid of revolution, and the singularities of the vortex vector field are the centers of vortex dislocations with opposite twists separated by the saddle, where $\nabla S(\rho, z) = 0$. The total wavefront is the superposition of the potential and vortex wavefronts.



a *b* *c*

FIG. 2. Energy vector lines for the potential (a), vortex (b), and total (c) vector fields.



a *b* *c*

FIG. 3. Potential (a), vortex (b), and total (c) phases with corresponding vector lines.

DISCUSSION

In the optical adaptive systems with phase conjugation, the problem arises of the correction of turbulent wave distortions. The correction of phase dislocations at points with zero intensity is a very difficult problem. Due to the complexity of the wavefront retrieval in the region of dislocations with the use of control devices–correctors a problem arises to obtain the smoothed approximation of the wavefront close to the real one, for example, by the root-mean-square criterion. In particular, a similar problem was solved in Refs. 13 and 14. The potential phase can be considered as such a regularized approximation of the wavefront. The regularization of this type converts the phase being a generalized multidimensional function into an ordinary aberration phase without singularities.

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