RADIATION PROPAGATION THROUGH SCATTERING MEDIA (THE EXACT SOLUTION OF A ONE-DIMENSIONAL TRANSFER EQUATION)

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Received June 11, 1998

We consider in this paper, based on the apparatus of radiative transfer equations, the problem on radiation propagation through the random scattering media. The exact solution of a one-dimensional radiative transfer equation has been obtained for such a medium. Analytical expressions for the signal propagated through the medium and the backscattering signal are obtained for special cases of continuous and pulsed signal and a homogeneous medium. Analysis and interpretation of the results in applications to remote sounding of the distributed formations, stratified media, and so on, are presented.

INTRODUCTION

Nowadays, in connection with the needs of remote sounding of spatially distributed formations, an increased interest is observed in theoretical investigations into the propagation of signals through scattering media and in calculation of the backscattering signal. It is of great importance for the development of algorithms for the medium diagnostics to explain the dependence of backscattering signal on the parameters of a medium sounded. The radiative transfer equations provide mathematical tools for solving that type of problems. Many publications can be found in the literature that deal with the transfer equations, among which we will mention the monographs we cite as Refs. 1 to 5. However, only few solutions have so far been derived in an explicit analytical form because of a high complexity of this problem. At the same, for the case of a narrow laser beam propagation through such media one may neglect the scattering along the directions out of the beam and the problem reduces to solving a one-dimensional radiative transfer equation that may be solved in a rigorous way. In this paper the exact solutions of the one-dimensional transfer equation are presented for the cases of homogeneous and piecewise inhomogeneous scattering media. The explicit analytical expressions have been obtained, that describe the laser pulse deformation at propagation through a scattering medium along with the shape of a backscattering signal.

1. SOLUTION OF A ONE-DIMENSIONAL TRANSFER EQUATION

Let us consider a one-dimensional problem on the propagation of radiation through a scattering medium. Let the photons move along the axis z and do not undergo scattering outside the beam. The photons may undergo absorption with the probability $\gamma(z)\mathrm{d}z$, (where $\gamma(z)$ is the absorption coefficient), when propagated along the path $(z, z+\mathrm{d}z)$, reflected with the probability $\sigma(z)\mathrm{d}z$ ($\sigma(z)$ is the reflection coefficient), or survive with the probability $1-\gamma(z)\mathrm{d}z-\sigma(z)\mathrm{d}z = 1-\varepsilon(z)\mathrm{d}z$ ($\varepsilon(z)$ is the extinction coefficient). The situation with the one-dimensional case is characterized by two functions: $p(t, z)$ is the mean number of photons at the point z and at the moment t, that move along the direction of increasing z, and $q(t, z)$ is the mean number of photons, that move backwards. In this case the radiative transfer equation is presented by the following system of differential equations:

$$\frac{1}{V(z)}\frac{\partial p(t, z)}{\partial t}+\frac{\partial p(t, z)}{\partial z}+\varepsilon(z)\, p(t, z)=$$

$$=\sigma(z)\, q(t, z)+\frac{1}{V(z)} f(t, z), \tag{1}$$

$$\frac{1}{V(z)}\frac{\partial q(t, z)}{\partial t}-\frac{\partial q(t, z)}{\partial z}+\varepsilon(z)\, q(t, z)=$$

$$=\sigma(z)\, p(t, z)+\frac{1}{V(z)} g(t, z), \tag{2}$$

where $f(t, z)\mathrm{d}t$ is the mean number of the forward moving photons, created at the point z during the time $\mathrm{d}t$ by an external source (emitter); $g(t, z)\mathrm{d}t$ is similar number of photons moving backwards; and $V(z)$ is the speed of photons.

Let us suppose that the emitter is located at the point $z=0$ and only generates the forward moving photons, that is, $f(t, z)=\delta(z)f(t)$, $g(t, z)=0$, where $f(t)$ is the function that characterizes the shape and energy of a signal, $\delta(z)$ is Dirac delta function, $i=\sqrt{-1}$. By making a Fourier transformation of Eqs. (1) and (2), we obtain

$$\tilde{\varepsilon}(i\omega, z)\, P(i\omega, z)+\frac{\partial P(i\omega, z)}{\partial z}=$$

$$=\sigma(z)\, Q(i\omega, z)+\frac{1}{V} F(i\omega)\, \delta(z), \tag{3}$$

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$$\tilde{\varepsilon}(i\omega, z)\, Q(i\omega, z) - \frac{\partial Q(i\omega, z)}{\partial z} = \sigma(z)\, P(i\omega, z), \qquad (4)$$

where

$$\tilde{\varepsilon}(i\omega, z) = \frac{i\omega}{V(z)} + \varepsilon(z);$$

$$P(i\omega, z) = \int p(t, z)\, e^{-i\omega t}\, dt;$$

$$F(i\omega) = \int f(t)\, e^{-i\omega t}\, dt.$$

As follows from the system of equations (Eqs. (3) and (4)), the function $P(i\omega, z)$ has the only break at the point $z = 0$, while the function $Q(i\omega, z)$ is continuous.

Let us note that within the homogeneous ($\varepsilon = \text{const}$, $\sigma = \text{const}$) intervals with no emitters the functions P and Q satisfy one and the same simple equation:

$$\frac{\partial^2 P}{\partial z^2} - \lambda^2 p = 0, \qquad (5)$$

where $\lambda^2 = \tilde{\varepsilon}^2 - \sigma^2$.

The solution of this equation has the following view:

$$P = A\, e^{\lambda z} + B\, e^{-\lambda z} \quad (\operatorname{Re} \lambda > 0), \qquad (6)$$

where A and B are arbitrary constants.

Let us consider the piecewise inhomogeneous medium with the internal interfaces (N is the number of the interfaces): $a_0 = 0 < a_1 < a_2 <\ldots< a_N$. Let the medium be homogeneous within the intervals (a_n, a_{n+1}), (a_{n+1}, ∞) with the parameters there being $\varepsilon(z) = \text{const} = \varepsilon_{n+1}$, $\sigma(z) = \text{const} = \sigma_{n+1}$, $V(z) = \text{const} = V_{n+1}$. According to Eq. (5) we have that within these segments

$$P(i\omega, z) = A_{n+1}\, e^{\lambda_{n+1} z} + B_{n+1}\, e^{-\lambda_{n+1} z};$$

$$Q(i\omega, z) = C_{n+1}\, e^{\lambda_{n+1} z} + D_{n+1}\, e^{-\lambda_{n+1} z}. \qquad (7)$$

With the account of Eq. (4) we obtain

$$A_{n+1}\, e^{\lambda_{n+1} z} + B_{n+1}\, e^{-\lambda_{n+1} z} =$$

$$= \frac{\tilde{\varepsilon}_{n+1} - \lambda_{n+1}}{\sigma_{n+1}} C_{n+1}\, e^{\lambda_{n+1} z} + \frac{\tilde{\varepsilon}_{n+1} + \lambda_{n+1}}{\sigma_{n+1}} D_{n+1}\, e^{-\lambda_{n+1} z},$$

$$A_{n+1} = \frac{\tilde{\varepsilon}_{n+1} - \lambda_{n+1}}{\sigma_{n+1}} C_{n+1} = \frac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1} + \lambda_{n+1}} C_{n+1},$$

$$B_{n+1} = \frac{\tilde{\varepsilon}_{n+1} + \lambda_{n+1}}{\sigma_{n+1}} D_{n+1} = \frac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1} - \lambda_{n+1}} D_{n+1}. \qquad (8)$$

Since the functions $P(i\omega, z)$ and $Q(i\omega, t)$ are continuous at the points a_1, a_2, ..., a_N, we have the following relations:

$$C_{n+1}\, e^{\lambda_{n+1} a_n} + D_{n+1}\, e^{-\lambda_{n+1} a_n} = C_n\, e^{\lambda_n a_n} + D_n\, e^{-\lambda_n a_n}, \qquad (9)$$

$$C_{n+1} \frac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1} + \lambda_{n+1}} e^{\lambda_{n+1} a_n} + D_{n+1} \frac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1} - \lambda_{n+1}} e^{-\lambda_{n+1} a_n} =$$

$$= C_n \frac{\sigma_n}{\tilde{\varepsilon}_n + \lambda_n} e^{\lambda_n a_n} + D_n \frac{\sigma_n}{\tilde{\varepsilon}_n - \lambda_n} e^{-\lambda_n a_n}. \qquad (10)$$

Using matrix designations, we can write these relations in the following form:

$$\begin{pmatrix} e^{\lambda_{n+1} a_n} & e^{-\lambda_{n+1} a_n} \\ \dfrac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1} + \lambda_{n+1}} e^{\lambda_{n+1} a_n} & \dfrac{\sigma_{n+1}}{\tilde{\varepsilon}_{n+1} - \lambda_{n+1}} e^{-\lambda_{n+1} a_n} \end{pmatrix} \begin{pmatrix} C_{n+1} \\ D_{n+1} \end{pmatrix} =$$

$$= \begin{pmatrix} e^{\lambda_n a_n} & e^{-\lambda_n a_n} \\ \dfrac{\sigma_n}{\tilde{\varepsilon}_n + \lambda_n} e^{\lambda_n a_n} & \dfrac{\sigma_n}{\tilde{\varepsilon}_n - \lambda_n} e^{-\lambda_n a_n} \end{pmatrix} \begin{pmatrix} C_n \\ D_n \end{pmatrix}. \qquad (11)$$

Assuming that

$$\begin{pmatrix} \tilde{C}_n \\ \tilde{D}_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \dfrac{\sigma_n}{\tilde{\varepsilon}_n + \lambda_n} & \dfrac{\sigma_n}{\tilde{\varepsilon}_n - \lambda_n} \end{pmatrix} \begin{pmatrix} e^{\lambda_n a_{n-1}} & 0 \\ 0 & e^{-\lambda_n a_{n-1}} \end{pmatrix} \begin{pmatrix} C_n \\ D_n \end{pmatrix}, \qquad (12)$$

Eq. (11) can be reduced to the following view:

$$\begin{pmatrix} \tilde{C}_{n+1} \\ \tilde{D}_{n+1} \end{pmatrix} = \frac{1}{2} \left\{ e^{\lambda_n \Delta a_n} \begin{pmatrix} \dfrac{\tilde{\varepsilon}_n}{\lambda_n} + 1 & -\dfrac{\sigma_n}{\lambda_n} \\ \dfrac{\sigma_n}{\lambda_n} & -\dfrac{\tilde{\varepsilon}_n}{\lambda_n} + 1 \end{pmatrix} + \right.$$

$$\left. + \; e^{-\lambda_n \Delta a_n} \begin{pmatrix} -\dfrac{\tilde{\varepsilon}_n}{\lambda_n} + 1 & \dfrac{\sigma_n}{\lambda_n} \\ -\dfrac{\sigma_n}{\lambda_n} & \dfrac{\tilde{\varepsilon}_n}{\lambda_n} + 1 \end{pmatrix} \right\} \begin{pmatrix} \tilde{C}_n \\ \tilde{D}_n \end{pmatrix} = g_n{}^* \begin{pmatrix} \tilde{C}_n \\ \tilde{D}_n \end{pmatrix},$$

where $\Delta a_n = a_n - a_{n-1}$.;

$$g_n = \left\{ e^{\lambda_n \Delta a_n} \begin{pmatrix} x_n \\ y_n \end{pmatrix} (x_n, -y_n) + e^{-\lambda_n \Delta a_n} \begin{pmatrix} y_n \\ x_n \end{pmatrix} (-y_n, x_n) \right\} =$$

$$= \sum_{\theta = \pm 1} e^{\theta \lambda_n \Delta a_n} \begin{pmatrix} \dfrac{u_n + \theta \nu_n}{2} \\ \dfrac{u_n - \theta \nu_n}{2} \end{pmatrix} \left(\frac{\nu_n + \theta u_n}{2}, \frac{\nu_n - \theta u_n}{2} \right). \qquad (13)$$

Here, the following designations are used:

$$x_n = \frac{1}{2} \left(\sqrt{\frac{\tilde{\varepsilon}_n + \sigma_n}{\lambda_n}} + \sqrt{\frac{\tilde{\varepsilon}_n - \sigma_n}{\lambda_n}} \right) = \frac{u_n + \nu_n}{2};$$

$$y_n = \frac{1}{2} \left(\sqrt{\frac{\tilde{\varepsilon}_n + \sigma_n}{\lambda_n}} - \sqrt{\frac{\tilde{\varepsilon}_n - \sigma_n}{\lambda_n}} \right) = \frac{u_n - \nu_n}{2}, \qquad (14)$$

where

$$u_n = \sqrt{\frac{\tilde{\varepsilon}_n + \sigma_n}{\lambda_n}}; \quad \nu_n = \sqrt{\frac{\tilde{\varepsilon}_n - \sigma_n}{\lambda_n}} = \frac{1}{u_n}.$$

As a result one may write that

$$\begin{pmatrix} \tilde{C}_{N+1} \\ \tilde{D}_{N+1} \end{pmatrix} = \prod_{n=1}^{N} g_n \begin{pmatrix} \tilde{C}_1 \\ \tilde{D}_1 \end{pmatrix}.$$

Using Eq. (13), we obtain

$$\prod_{n=1}^{N} g_n = \sum_{\theta_1,\, \theta_2,\, \dots,\, \theta_N = \pm 1} e^{\sum_{n=1}^{N} \theta_n \lambda_n \Delta a_n} \times$$

$$\times \prod_{n=2}^{N} \frac{1}{2} (u_{n-1}\, \nu_n + \theta_n\, \theta_{n-1}\, \nu_{n-1}\, u_n) \times$$

$$\times \begin{pmatrix} \dfrac{u_N + \theta_N \nu_N}{2} \\ \dfrac{u_N - \theta_N \nu_N}{2} \end{pmatrix} \left(\frac{\nu_1 + \theta_1 u_1}{2}, \frac{\nu_1 - \theta_1 u_1}{2} \right).$$

In the final result we obtain

$$C_{N+1} = \frac{\sigma_{N+1}}{2\lambda_{N+1}} e^{-\lambda_{N+1} a_N} \sum_{\theta_1,\, \theta_2,\, \dots,\, \theta_N = \pm 1} e^{\sum_{n=1}^{N} \theta_n \lambda_n \Delta a_n} \times$$

$$\times \prod_{n=2}^{N} \frac{1}{2} (u_{n-1}\, \nu_n + \theta_n\, \theta_{n-1}\, \nu_{n-1}\, u_n) \times$$

$$\times \frac{1}{2} \frac{\nu_{N+1}\, u_N + \theta_N\, u_{N+1}\, \nu_N}{u_{N+1} - \nu_{N+1}} \times$$

$$\times \left(\left[\frac{1 + \theta_1}{u_1 + \nu_1} \frac{\sigma_1}{\tilde{\varepsilon} + \lambda_1} \right] C_1 + \left[\frac{1 - \theta_1}{u_1 - \nu_1} \frac{\sigma_1}{\tilde{\varepsilon} - \lambda_1} \right] D_1 \right). \quad (15)$$

To determine D_1, we make use of the following considerations. It is obvious that

$$P(i\omega, z) = \begin{cases} \dfrac{\tilde{\varepsilon}_1 - \lambda_1}{\sigma_1} C_0\, e^{\lambda_1 z}, & -\infty < z \le 0, \\ \dfrac{\tilde{\varepsilon}_1 - \lambda_1}{\sigma_1} C_1 e^{\lambda_1 z} + \dfrac{\tilde{\varepsilon}_1 + \lambda_1}{\sigma_1} D_1 e^{-\lambda_1 z}, & 0 \le z \le a_1. \end{cases}$$

At the point $z = 0$, $P(i\omega, z)$ undergoes a break that amounts to $F(i\omega)/V_1$. From that we obtain

$$(q_1 - q_0) \frac{\tilde{\varepsilon}_1 - \lambda_1}{\sigma_1} + D_1 \frac{\tilde{\varepsilon}_1 + \lambda_1}{\sigma_1} = \frac{1}{V_1} F(i\omega).$$

Then

$$Q(i\omega, z) = \begin{cases} C_0\, e^{\lambda_1 z}, & -\infty < z \le 0, \\ C_1\, e^{\lambda_1 z} + D_1\, e^{-\lambda_1 z}, & 0 \le z \le a_1. \end{cases}$$

The function $Q(i\omega, z)$ is continuous at the point $z = 0$. Therefore $C_1 - C_0 + D_1 = 0$. Hence

$$D_1 = \sigma_1 / (2\, \lambda_1\, V_1)\, F(i\omega). \quad (16)$$

It is clear, from the physical point of view, that on the interval (a_N, ∞) the function $Q(i\omega, z) = D_{N+1} e^{-\lambda_{N+1}}$, i.e., $C_{N+1} = 0$. As a consequence, we obtain from Eq. (15) the following equation for determining C_1:

$$0 = \sum_{\theta_1,\, \theta_2,\, \dots,\, \theta_N = \pm 1} e^{\sum_{n=1}^{N} \theta_n \lambda_n \Delta a_n} \prod_{n=2}^{N+1} (u_{n-1} \nu_n + \theta_n \theta_{n-1} \nu_{n-1} u_n) \times$$

$$\times \left(\frac{1 + \theta_1}{u_1 + \nu_1} C_1 + \frac{1 - \theta_1}{u_1 - V_1} \frac{\sigma_1}{2\lambda_1\, V_1} F(i\omega) \right).$$

Hence

$$q_1 = -\frac{u_1 + \nu_1}{u_1 - \nu_1} \frac{\sigma_1}{2\lambda_1\, V_1} F(i\omega)\, e^{-2\lambda_1 \Delta a_1} \left\{ 1 - 2\, V_1 u_2 \frac{\displaystyle\sum_{\theta_2,\, \dots,\, \theta_N = \pm 1} e^{\sum_{n=2}^{N} \theta_n \lambda_n \Delta a_n} \prod_{n=3}^{N+1} (u_{n-1}\, \nu_n + \theta_n\, \theta_{n-1}\, \nu_{n-1}\, u_n)}{\displaystyle\sum_{\theta_2,\, \dots,\, \theta_N = \pm 1} e^{\sum_{n=2}^{N} \theta_n \lambda_n \Delta a_n} \prod_{n=2}^{N+1} (u_{n-1}\, \nu_n + \theta_n\, \theta_{n-1}\, \nu_{n-1}\, u_n)} \right\} =$$

$$= -\frac{\tilde{\varepsilon}_1 + \lambda_1}{\sigma_1} \frac{\sigma_1}{2\lambda_1\, V_1} F(i\omega)\, e^{-2\lambda_1 \Delta a_1} \left\{ 1 - 2 \frac{\tilde{\varepsilon}_2 + \sigma_2}{\lambda_2} \frac{\displaystyle\sum_{\theta_2,\, \dots,\, \theta_N = \pm 1} e^{\sum_{n=2}^{N} \theta_n \lambda_n \Delta a_n} \theta_2 \prod_{n=3}^{N+1} \left(\frac{\tilde{\varepsilon}_{n-1} + \sigma_{n-1}}{\lambda_{n-1}} + \theta_n\, \theta_{n-1} \frac{\tilde{\varepsilon}_n + \sigma_n}{\lambda_n} \right)}{\displaystyle\sum_{\theta_2,\, \dots,\, \theta_N = \pm 1} e^{\sum_{n=2}^{N} \theta_n \lambda_n \Delta a_n} \prod_{n=2}^{N+1} \left(\frac{\tilde{\varepsilon}_{n-1} + \sigma_{n-1}}{\lambda_{n-1}} + \theta_n\, \theta_{n-1} \frac{\tilde{\varepsilon}_n + \sigma_n}{\lambda_n} \right)} \right\}. \quad (17)$$

The backscattered signal Q at the point $z = 0$ is as follows:

$$Q(i\omega, 0) = q_0 = q_1 + D_1.$$

Thus, C_1 and D_1 are found.

It is of a primary interest to determine the backscattered signal at the point of the emitter location, i.e., at $z = 0$. Obviously, the backscattered signal at the point $z = 0$ is as follows:

$$q_0 = q_1 + D_1 = \sigma_1/(2\lambda_1 V_1)\, F(i\omega) \left\{ 1 + \frac{\tilde{\varepsilon}_1 + \lambda_1}{\sigma_1} \exp\{-2\,\lambda_1\, \Delta a_1\} - 2 \left(\frac{\tilde{\varepsilon}_1 + \lambda_1}{\sigma_1}\right) \frac{\tilde{\varepsilon}_1 + \sigma_1}{\lambda_1} \exp\{-2\,\lambda_1\, \Delta a_1\} \times \right.$$

$$\left. \times \frac{\sum\limits_{\theta_2,\, \ldots,\, \theta_N = \pm 1} \exp\left\{\sum\limits_{n=2}^{N} \theta_n \lambda_n \Delta a_n\right\} \prod\limits_{n=3}^{N+1} \left(\frac{\tilde{\varepsilon}_{n-1} + \sigma_{n-1}}{\lambda_{n-1}} + \theta_n\, \theta_{n-1} \frac{\tilde{\varepsilon}_n + \sigma_n}{\lambda_n}\right)}{\sum\limits_{\theta_2,\, \ldots,\, \theta_N = \pm 1} \exp\left\{\sum\limits_{n=2}^{N} \theta_n \lambda_n \Delta a_n\right\} \prod\limits_{n=2}^{N+1} \left(\frac{\tilde{\varepsilon}_{n-1} + \sigma_{n-1}}{\lambda_{n-1}} + \theta_n\, \theta_{n-1} \frac{\tilde{\varepsilon}_n + \sigma_n}{\lambda_n}\right)} \right\}. \tag{18}$$

It is assumed therewith that $\theta_1 = \theta_{N+1} = 1$.

2. ANALYSIS OF THE SOLUTION

2.1. For the case of a steady-state signal, the equation for C_0 is as follows:

$$q_0(t, z = 0) = q_0(t) = q_0 = \frac{\sigma_1}{2\lambda_1 V_1} f \left\{ 1 + \frac{\varepsilon_1 + \lambda_1}{\sigma_1} \exp\{-2\,\lambda_1\, \Delta a_1\} - 2 \frac{\varepsilon_1 + \lambda_1}{\sigma_1} \frac{\varepsilon_1 + \sigma_1}{\lambda_1} \exp\{-2\,\lambda_1\, \Delta a_1\} \times \right.$$

$$\left. \times \frac{\sum\limits_{\theta_2,\, \ldots,\, \theta_N = \pm 1} \exp\left\{\sum\limits_{n=2}^{N} \theta_n \lambda_n \Delta a_n\right\} \prod\limits_{n=3}^{N+1} \left(\frac{\varepsilon_{n-1} + \sigma_{n-1}}{\lambda_{n-1}} + \theta_n\, \theta_{n-1} \frac{\varepsilon_n + \sigma_n}{\lambda_n}\right)}{\sum\limits_{\theta_2,\, \ldots,\, \theta_N = \pm 1} \exp\left\{\sum\limits_{n=2}^{N} \theta_n \lambda_n \Delta a_n\right\} \prod\limits_{n=2}^{N+1} \left(\frac{\varepsilon_{n-1} + \sigma_{n-1}}{\lambda_{n-1}} + \theta_n\, \theta_{n-1} \frac{\varepsilon_n + \sigma_n}{\lambda_n}\right)} \right\}. \tag{19}$$

Let us introduce the following designation:

$$\frac{q_0}{f/(2V_1)} = \hat{C}_0\,. \tag{20}$$

Consider now the case of the 3-layer medium. According to Eq. (19) at $N = 3$, we obtain

$$\hat{C}_0 = \frac{\sigma_1}{\lambda_1} \left\{ 1 + \frac{\varepsilon_1 + \lambda_1}{\sigma_1} \exp\{-2\,\lambda_1\, \Delta a_1\} - 2 \frac{\varepsilon_1 + \lambda_1}{\sigma_1} \frac{\varepsilon_1 + \sigma_1}{\lambda_1} \exp\{-2\,\lambda_1\, \Delta a_1\} \times \right.$$

$$\times \frac{\exp\{\lambda_2(a_2 - a_1) + \lambda_3(a_3 - a_2)\} \left(\frac{\varepsilon_2 + \sigma_2}{\lambda_2} + \frac{\varepsilon_3 + \sigma_3}{\lambda_3}\right) \times}{\exp\{\lambda_2(a_2 - a_1) + \lambda_3(a_3 - a_2)\} \left(\frac{\varepsilon_1 + \sigma_1}{\lambda_1} + \frac{\varepsilon_2 + \sigma_2}{\lambda_2}\right) \left(\frac{\varepsilon_2 + \sigma_2}{\lambda_2} + \frac{\varepsilon_3 + \sigma_3}{\lambda_3}\right) \times}$$

$$\frac{\times \left(\frac{\varepsilon_3 + \sigma_3}{\lambda_3} + \frac{\varepsilon_4 + \sigma_4}{\lambda_4}\right) + \exp\{\lambda_2(a_2 - a_1) - \lambda_3(a_3 - a_2)\} \times}{\times \left(\frac{\varepsilon_3 + \sigma_3}{\lambda_3} + \frac{\varepsilon_4 + \sigma_4}{\lambda_4}\right) + \exp\{\lambda_2(a_2 - a_1) - \lambda_3(a_3 - a_2)\} \times}$$

$$\frac{\times \left(\frac{\varepsilon_2 + \sigma_2}{\lambda_2} - \frac{\varepsilon_3 + \sigma_3}{\lambda_3}\right) \left(\frac{\varepsilon_3 + \sigma_3}{\lambda_3} - \frac{\varepsilon_4 + \sigma_4}{\lambda_4}\right) +}{\times \left(\frac{\varepsilon_1 + \sigma_1}{\lambda_1} + \frac{\varepsilon_2 + \sigma_2}{\lambda_2}\right) \left(\frac{\varepsilon_2 + \sigma_2}{\lambda_2} - \frac{\varepsilon_3 + \sigma_3}{\lambda_3}\right) \left(\frac{\varepsilon_3 + \sigma_3}{\lambda_3} - \frac{\varepsilon_4 + \sigma_4}{\lambda_4}\right) +}$$

$$\frac{+ \exp\{-\lambda_2(a_2 - a_1) + \lambda_3(a_3 - a_2)\} \left(\frac{\varepsilon_2 + \sigma_2}{\lambda_2} - \frac{\varepsilon_3 + \sigma_3}{\lambda_3}\right) \times}{+ \exp\{-\lambda_2(a_2 - a_1) + \lambda_3(a_3 - a_2)\} \left(\frac{\varepsilon_1 + \sigma_1}{\lambda_1} - \frac{\varepsilon_2 + \sigma_2}{\lambda_2}\right) \left(\frac{\varepsilon_2 + \sigma_2}{\lambda_2} - \frac{\varepsilon_3 + \sigma_3}{\lambda_3}\right) \times}$$

$$\frac{\times\left(\frac{\varepsilon_3+\sigma_3}{\lambda_3}+\frac{\varepsilon_4+\sigma_4}{\lambda_4}\right)+\exp\{-\lambda_2(a_2-a_1)-\lambda_3(a_3-a_2)\}\times}{\times\left(\frac{\varepsilon_3+\sigma_3}{\lambda_3}+\frac{\varepsilon_4+\sigma_4}{\lambda_4}\right)+\exp\{-\lambda_2(a_2-a_1)+\lambda_3(a_3-a_2)\}\times}$$

$$\frac{\times\left(\frac{\varepsilon_2+\sigma_2}{\lambda_2}+\frac{\varepsilon_3+\sigma_3}{\lambda_3}\right)\left(\frac{\varepsilon_3+\sigma_3}{\lambda_3}-\frac{\varepsilon_4+\sigma_4}{\lambda_4}\right)}{\times\left(\frac{\varepsilon_1+\sigma_1}{\lambda_1}-\frac{\varepsilon_2+\sigma_2}{\lambda_2}\right)\left(\frac{\varepsilon_2+\sigma_2}{\lambda_2}+\frac{\varepsilon_3+\sigma_3}{\lambda_3}\right)\left(\frac{\varepsilon_3+\sigma_3}{\lambda_3}-\frac{\varepsilon_4+\sigma_4}{\lambda_4}\right)}. \qquad (21)$$

At $\frac{\varepsilon_3+\sigma_3}{\lambda_3} \ll \frac{\varepsilon_4+\sigma_4}{\lambda_4}$ the terms containing the factor $(\varepsilon_4+\sigma_4)/\lambda_4$ can be canceled, and Eq. (21) takes a simpler form

$$\hat{C}_0=\frac{\sigma_1}{\lambda_1}\left\{1+\frac{\varepsilon_1+\lambda_1}{\sigma_1}\exp\{-2\,\lambda_1\,\Delta a_1\}-2\,\frac{\varepsilon_1+\lambda_1}{\sigma_1}\,\frac{\varepsilon_1+\sigma_1}{\lambda_1}\exp\{-2\,\lambda_1\,\Delta a_1\}\times\right.$$

$$\times\frac{\exp\{\lambda_2(a_2-a_1)+\lambda_3(a_3-a_2)\}\left(\frac{\varepsilon_2+\sigma_2}{\lambda_2}+\frac{\varepsilon_3+\sigma_3}{\lambda_3}\right)-}{\exp\{\lambda_2(a_2-a_1)+\lambda_3(a_3-a_2)\}\left(\frac{\varepsilon_1+\sigma_1}{\lambda_1}+\frac{\varepsilon_2+\sigma_2}{\lambda_2}\right)\left(\frac{\varepsilon_2+\sigma_2}{\lambda_2}+\frac{\varepsilon_3+\sigma_3}{\lambda_3}\right)-}$$

$$\frac{-\exp\{\lambda_2(a_2-a_1)-\lambda_3(a_3-a_2)\}\left(\frac{\varepsilon_2+\sigma_2}{\lambda_2}-\frac{\varepsilon_3+\sigma_3}{\lambda_3}\right)+}{-\exp\{\lambda_2(a_2-a_1)-\lambda_3(a_3-a_2)\}\left(\frac{\varepsilon_1+\sigma_1}{\lambda_1}+\frac{\varepsilon_2+\sigma_2}{\lambda_2}\right)\left(\frac{\varepsilon_2+\sigma_2}{\lambda_2}-\frac{\varepsilon_3+\sigma_3}{\lambda_3}\right)+}$$

$$\frac{+\exp\{-\lambda_2(a_2-a_1)+\lambda_3(a_3-a_2)\}\left(\frac{\varepsilon_2+\sigma_2}{\lambda_2}-\frac{\varepsilon_3+\sigma_3}{\lambda_3}\right)-}{+\exp\{-\lambda_2(a_2-a_1)+\lambda_3(a_3-a_2)\}\left(\frac{\varepsilon_1+\sigma_1}{\lambda_1}-\frac{\varepsilon_2+\sigma_2}{\lambda_2}\right)\left(\frac{\varepsilon_2+\sigma_2}{\lambda_2}-\frac{\varepsilon_3+\sigma_3}{\lambda_3}\right)-}$$

$$\frac{-\exp\{-\lambda_2(a_2-a_1)-\lambda_3(a_3-a_2)\}\left(\frac{\varepsilon_2+\sigma_2}{\lambda_2}+\frac{\varepsilon_3+\sigma_3}{\lambda_3}\right)}{-\exp\{-\lambda_2(a_2-a_1)-\lambda_3(a_3-a_2)\}\left(\frac{\varepsilon_1+\sigma_1}{\lambda_1}-\frac{\varepsilon_2+\sigma_2}{\lambda_2}\right)\left(\frac{\varepsilon_2+\sigma_2}{\lambda_2}+\frac{\varepsilon_3+\sigma_3}{\lambda_3}\right)}. \qquad (22)$$

The results calculated by Eq. (22) using different parameters ε_i and σ_i, are shown in Fig. 1. These results allow one, in particular, to draw a conclusion, that an object with the characteristics ε_4 and σ_4 may be better seen from the side of the absorbing ("blackB) medium with the characteristics ε_2 and σ_2 than in the opposite direction, given certain combination of parameters of the non-mixed layers of different media and that the medium with ε_3 and σ_3 characteristics is a reflective ("whiteB) medium.

The above obtained dependences make up the theoretical basis for the developments on creating stratified media that have preset characteristics that may be controlled.

Let us also present the view of the direct and reflected continuous signal in the case of a homogeneous medium:

$$P(t,\,z)=\frac{1}{2V}\,f\left(\frac{\varepsilon}{\lambda}+\mathrm{sign}(z)\right)\exp(-\lambda\,|z|\,);$$

$$Q(t,\,z)=\frac{1}{2V}\,f\,\frac{\sigma}{\lambda}\exp(-\lambda\,|z|\,),$$

$$Q(t,\,z=0)=\frac{f}{2V}\,\frac{\sigma}{\lambda}. \qquad (23)$$

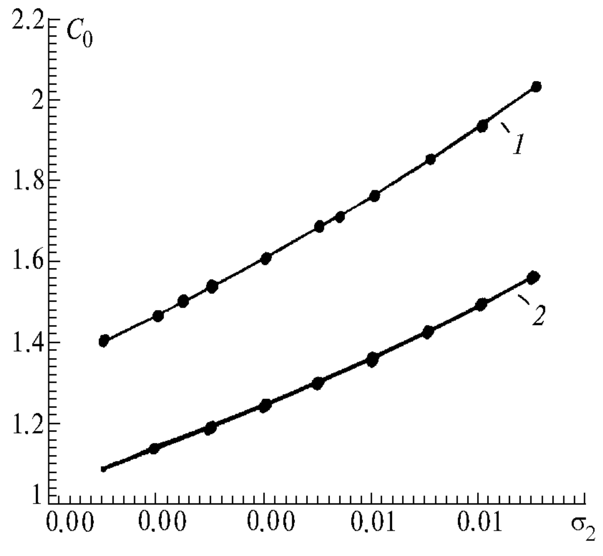

FIG. 1. The dependence of $\hat{C}$ on σ_2: $\sigma_3 = 0.8\cdot10^{-2}$ m^{-1} (1); $\sigma_3 = 0.2\cdot10^{-2}$ m^{-1} (2); for $a_1 = 1000$ m, $a_2 = 1020$ m, $a_3 = 1040$ m, $\varepsilon_3 = 10^{-4}$ m^{-1}; $\sigma_1 = 0.5\cdot10^{-1}$ m^{-1}; and $\varepsilon_2 = \varepsilon_3 = 10^{-2}$ m^{-1}.

2.2. In the case of a non-stationary signal a transmitter emits a δ-pulse, that means that $f(t) = f\delta(t)$ and $f(i\omega) = \text{const} = f$.

In the case of a homogeneous medium one can calculate the inverse Fourier transform

$$P(t, z) = \frac{f}{V} e^{-\varepsilon|z|} \delta\left(t - \frac{z}{V}\right) \eta(z) + \frac{1}{2} f\sigma \sqrt{\frac{Vt + z}{Vt - z}} \times$$

$$\times I_1\left(\sigma \sqrt{(Vt)^2 - z^2}\right) e^{-\varepsilon Vt} \eta(Vt - |z|); \quad (24)$$

$$Q(t, z) = \frac{1}{2} f\sigma e^{-\varepsilon Vt} I_0\left(\sigma \sqrt{(Vt)^2 - z^2}\right) \eta(Vt - |z|), \quad (25)$$

where $I_1(...)$ and $I_0(...)$ are the modified Bessel functions of the first kind and the first and zero order, respectively.

These are the exact solutions of a one-dimensional radiative transfer equation of the problem on pulse propagation through a homogeneous scattering medium.

The backscattering signal received at the point $z = 0$ is as follows:

$$Q(t, 0) = \frac{1}{2} f\sigma e^{-\varepsilon Vt} I_0(\sigma Vt) \eta(t). \quad (26)$$

Here $\eta(t)$ is the unit-step function.

Let us note that the direct (Eq. (4)) and backward (Eq. (25)) scattered signals fill the interval $-Vt < z < Vt$, that is being extended with increasing t. Hence, the backward scattered signal is symmetrical relative to $z = 0$. The calculated curve of the relative value of the direct signal P/P_{max} distribution over z at an arbitrary moment is shown in Fig. 2.

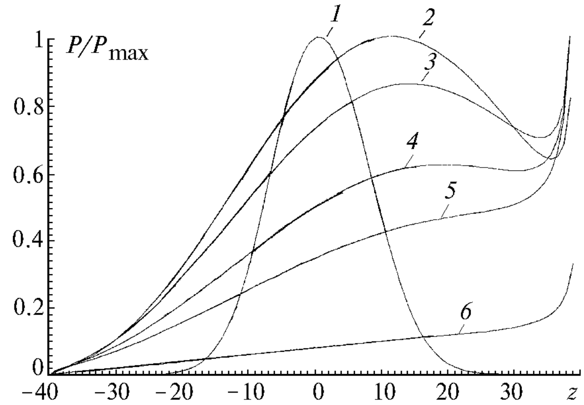

FIG. 2. The dependence of P/P_{max} on the distance z, (m), to the emitter at $Vt = 40$, $a = 30$ (1), 4 (2), 3.5 (3), 3 (4), 2.5 (5), and 2 (6), $a = \sigma Vt$.

It is worth noting the nonmonotonic behavior of the direct signal distribution that appears in the interval $-Vt < z < Vt$ at large values of the reflection coefficient σ with the signal extrema being at the points inside the interval $-Vt < z < Vt$.

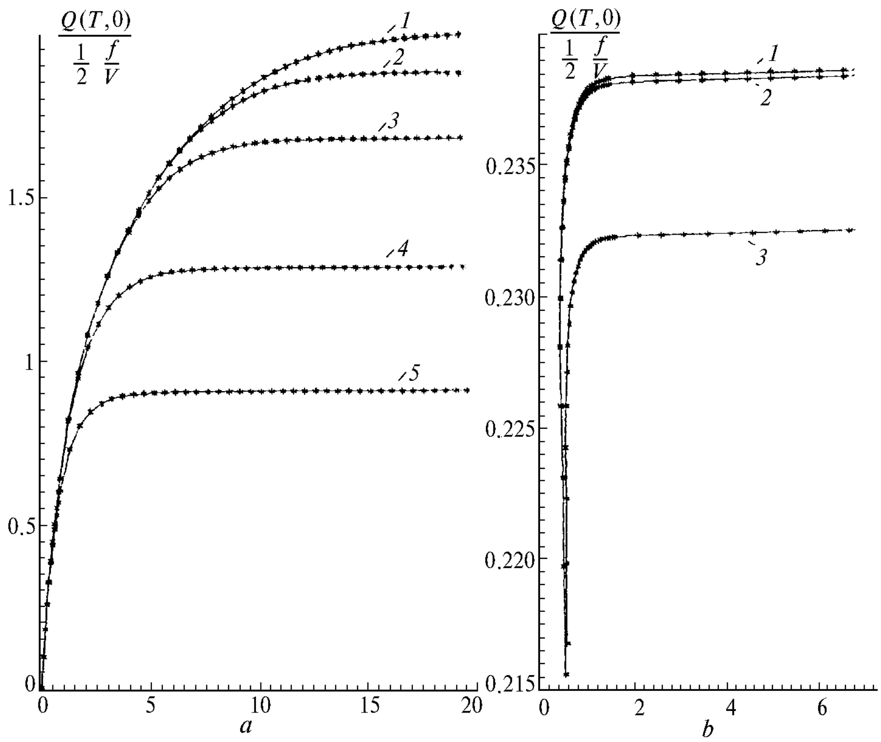

FIG. 3. The calculated results: (a) dependence of $Q(T, 0)/(f/2V)$ on a at $b = 1.1$ for $k = 8$ (1), 6 (2), 4 (3), 2 (4), and 0 (5); (b) dependence of $Q(T, 0)/(f/2V)$ on a at $b = 4.4$ for $k = 4, 6, 8$ (1), 2 (2), and 0 (3).

Although the above results have been obtained for total scattering, that is, with the allowance for all orders of multiple scattering, it is interesting to address the question on which order of scattering ought to be accounted for, or what role plays the account of one or other order of scattering, because that essentially determines the complexity of multidimensional models, and the bulk of the calculations needed. Equation (26), or, better to say, its modification, allows one to answer this question in a more useful and simpler way than that in Ref. 5.

Let T be the time during which the backscattered signal at the point $z = 0$ is recorded, then we obtain

$$Q(T, 0) = \frac{1}{2} f \frac{1}{V} \int\limits_0^a \mathrm{e}^{-bx} I_0(x)\, \mathrm{d}x, \qquad (27)$$

where $a = \sigma VT$; $b = \varepsilon/\sigma$. Having in mind that,

$$I_0(x) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{x}{2}\right)^{2k},$$

and at $k = 0$, the function $Q(T, 0)\big|_{k=0}$ describes the single scattering; at $k = 2$, $Q(T, 0)\big|_{k=2}$ describes the double scattering (that actually is the triple scattering from the common point of view). Figure 3 presents the calculated dependence $Q(T, 0)/[f/(2V)]$ for a set of the parameters a, b, and k. One can see from this figure that in the majority of cases presented by the parameter sets the account of third order of scattering is quite sufficient, and, moreover, in many cases the account for the double scattering provides quite good results. The above obtained results can be used when developing methods for sounding different media, as well as in the development of stratified non-mixing media that provide for optimal camouflage. The results may also be useful when analyzing the detectability of objects, observed through a scattering medium, synthesizing information media for the controlled screens, and so on.

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