THE MONTE-CARLO METHOD AND THE TRANSFER EQUATION FOR PLANT CANOPIES

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The equation of radiation transfer for a flat layer of vegetationis studied. Two forms of this equation are presented: an integrodifferential equation and an integral equation of the second kind. An algorithm for solving the integral equation by the Monte Carlo method is described. The advantages of this algorithm and the prospects for using it in the solution of direct and inverse problems are pointed out.

1. INTRODUCTION

In the last few years the rapid development of aerospace sounding of vegetation has made it necessary to develop different models of the interaction of sunlight with a plant canopy (PC). There exist different models of the radiation regime of PC: geometric models, models of a turbid medium, mixed models, and statistical models.

The model proposed below is a combination of turbid-medium and statistical models. Great computational difficulties arise when vertical and horizontal nonuniformities, the row arrangement of the plantings, and the dimension of the phytoelements are included in the turbid-medium model. In the statistical model, on the other hand, the geometric factors are taken into account in a natural manner. However, statistical models of the type employed in Ref. 4, are difficult to use to estimate the geometric parameters of a separate plant and the crop as a whole because of the impossibility of calculating the derivatives of the photon trajectories with respect to these parameters.[1]

In this paper we propose a description of the radiation regime of PC with the help of the transfer equation (including in it all important geometric parameters), which we solve with the help of the Monte-Carlo method. The procedure developed makes it possible to extend the technique of solving the direct and inverse problems in atmospheric physics by the Monte-Carlo method[2] to vegetation.

To simplify the presentation, we shall confine our attention to uniform PC. In the first two sections we describe the integrodifferential equation describing radiation transfer in a foliated medium and the transfer to an integral equation. The next two sections are devoted to the solution of this equation by the Monte-Carlo method. Then, we include in our model the sizes of the plants in order to take into account the effect of reflection.

2. THE EQUATION OF TRANSFER IN PLANT CANOPIES

We shall study a plane-parallel, uniform, lamellar medium with thickness H, whose top boundary is illuminated by direct sunlight and whose bottom boundary is a lambertian reflecting surface with albedo r_s. The process of radiation transfer in such a medium is described by the boundary-value problem

$$-\mu \frac{\partial I}{\partial \tau} = G(\Omega) I(\tau, \Omega) = \frac{1}{\pi} \int_{4\pi} \Gamma(\Omega' \to \Omega) I(\tau, \Omega') \, d\Omega';$$

$$I(0, \Omega) = I_0 \delta(\Omega - \Omega_0), \quad \mu < 0;$$

$$I(H, \Omega) = \frac{r_s}{\pi} \int_{2\pi} |\mu| \, I(H, \Omega') \, d\Omega', \tag{1}$$

where $G(\Omega)$ is the average projection of the normals of the lamella on the direction Ω, i.e.,

$$G(\Omega) = \frac{1}{2\pi} \int_{2\pi^+} g_L(\Omega_L) |\Omega_L \Omega| \, d\Omega_L, \tag{2}$$

where $\frac{1}{2\pi} g_L(\Omega_L)$ is the probability distribution of the normals of the lamella (here and below the factors 4π and $2\pi^{\pm}$ appearing in the integrand, denote the entire unit sphere and its upper and lower hemispheres, respectively). The angles φ and θ ($\cos\theta = \mu$) are the azimuthal angle of the unit vector $\Omega = \Omega(\mu, \varphi)$ relative to the outward normal, directed along the negative z-axis. The vector $\Omega_0 = \Omega(\mu_0, \varphi_0)$ is the direction of the sunlight, whose intensity is I_0, and the function Γ/π is the unnormalized scattering phase function of an elementary volume:

$$\Gamma(\vec{\Omega}' \to \Omega) = \frac{1}{2} \int g_L(\vec{\Omega}_L) |\vec{\Omega}' \cdot \vec{\Omega}_L| f(\vec{\Omega}' \to \vec{\Omega}, \vec{\Omega}_L) d\vec{\Omega}_L, \tag{3}$$

where f is the reflection phase function of the surface of a leaf. If the leaf albedo is denoted as ω_L, i.e.,

$$\omega_L = \int_{4\pi} f(\vec{\Omega}' \to \vec{\Omega}, \vec{\Omega}_L)\, d\vec{\Omega}, \tag{4}$$

then the function

$$P(\vec{\Omega}' \to \vec{\Omega}) = \frac{\Gamma(\vec{\Omega}' \to \vec{\Omega})}{\pi G(\vec{\Omega}')} \tag{5}$$

is the scattering phase function normalized to the leaf albedo

$$\int_{4\pi} P(\vec{\Omega}' \to \vec{\Omega})\, d\Omega = \omega_L. \tag{6}$$

3. THE INTEGRAL TRANSFER EQUATION

We denote

$$J(\tau, \vec{\Omega}) = I(\tau, \vec{\Omega})\, G(\vec{\Omega}). \tag{7}$$

The integrodifferential equation (1) for the function $I(\tau, \vec{\Omega})$, can be reduced in the standard manner to an integral equation for the function, $J(\tau, \vec{\Omega})$. Omitting the intermediate calculations, we write out this integral equation:

$$J(\tau, \vec{\Omega}) = \int_{4\pi} \int_0^H k[(\tau', \vec{\Omega}') \to (\tau, \vec{\Omega})] \times J(\tau', \vec{\Omega}')\, d\tau' d\vec{\Omega}' + F(\tau, \vec{\Omega}), \tag{8}$$

where

$$k(x' \to x) = \begin{cases} k_1(x' \to x), & \mu < 0, \\ k_2(x' \to x), & \mu > 0, \end{cases} \tag{9}$$

and $x = (\tau, \vec{\Omega})$ is a point in the phase space X. Here

$$k_1(x' \to x) = \begin{cases} P(\vec{\Omega}' \to \vec{\Omega}) \dfrac{G(\vec{\Omega})}{|\mu|} \exp\left[-\dfrac{G(\vec{\Omega})}{|\mu|}(\tau - \tau')\right], & 0 \le \tau' \le \tau, \\ 0, & \tau < \tau' \le H; \end{cases} \tag{10}$$

$$k_2(x' \to x) = \begin{cases} 0, & 0 \le \tau' \le \tau, \\ P(\vec{\Omega}' \to \vec{\Omega}) \dfrac{G(\vec{\Omega})}{\mu} \exp\left[-\dfrac{G(\vec{\Omega})}{\mu}(\tau' - \tau)\right], & \tau < \tau' \le H; \end{cases} \tag{11}$$

and the function $F(\tau, \vec{\Omega})$ is

$$F(\tau, \vec{\Omega}) = J(0, \vec{\Omega}) \exp\left[-\frac{G(\vec{\Omega})}{|\mu|}\tau\right], \quad \mu < 0. \tag{12}$$

We note that the integral equation (8) can also be derived directly from physical considerations.

4. SOLUTION OF THE INTEGRAL EQUATION BY THE MONTE CARLO-METHOD

We write the integral equation (8) in the operator form

$$J = KJ + F, \tag{13}$$

where the kernel of the integral operator K, is defined in Eqs. (9)–(11). We shall give the probability interpretation of this equation. The kernel $k(x' \to x)$ is the probability distribution of collisions at the point x in the phase space under the condition that the preceding collision occurred at the point x'. Here, $x = (\tau, \vec{\Omega})$, where τ is the depth of the collision point and $\vec{\Omega}$ is the direction of motion of the photon immediately prior to the collision; $J(\tau, \vec{\Omega})$ is the phase density of the collisions; $F(\tau, \vec{\Omega})/|\mu|$ is the density of the first collisions of the photons emanating from the source with the radiation density $I(0, \vec{\Omega})$ in the direction $\vec{\Omega}$.

It is well known that if $\|K\| < 1$, then Eq. (13) has a unique solution, which can be written in the form of a Neumann series

$$J = F + KF + K^2F + \ldots . \tag{14}$$

It is often sufficient to find not the entire solution, but rather only some functional of it. For example, to calculate the spectral brightness coefficient (SBC) we need to know $J(0, \vec{\Omega}^*)$, $\vec{\Omega}^* = \vec{\Omega}(\mu^*, \varphi^*)$, $\mu > 0$, i.e., the reflection from the top surface in the direction $\vec{\Omega}^*$. Then

$$J(0, \vec{\Omega}^*) = (J, \delta_{x^*}) = \int_X J(x)\delta_{x^*}(x)\, dx,$$

where δ_{x^*} is the Dirac delta-function and $x^* = (0, \vec{\Omega}^*)$. Substituting Eq. (14), the last equation can be put into the form

$$J(0, \vec{\Omega}^*) = (F, \delta_{x^*}) + (KF, \delta_{x^*}) + (K^2F, \delta_{x^*}) + \ldots$$
$$= (F, K^*\delta_{x^*}) + (KF, K^*\delta_{x^*}) + (KF^2, K^*\delta_{x^*}) + \ldots .$$

We took into account the fact that $(F, \delta_{x^*}) = 0$, because the sunlight and the reflected radiation are

oppositely directed. The operator K^* is the operator conjugate to K, namely,

$$(K^*\Psi)(x) = \int_X k(x \to x')\,\Psi(x')\,dx'.$$

From here we have

$$(K^*\delta_{x^*})(\tau, \vec{\Omega}) = P(\vec{\Omega} \to \vec{\Omega}^*)\,\frac{G(\vec{\Omega}^*)}{\mu^*}\exp\left[-\tau\,\frac{G(\vec{\Omega}^*)}{\mu^*}\right].$$

Denoting the right side of the last equation by $\Psi(\tau, \vec{\Omega})\,G(\vec{\Omega}^*)$ we obtain

$$I(0, \vec{\Omega}^*) = \frac{J(0, \vec{\Omega}^*)}{G(\vec{\Omega}^*)} = (F, \Psi) + (KF, \Psi) + (K^2F, \Psi) + \ldots, \tag{15}$$

where the contribution function

$$\Psi(\tau, \vec{\Omega}) = P(\vec{\Omega} \to \vec{\Omega}^*)\exp\left[-\frac{G(\vec{\Omega}^*)}{\mu^*}\tau\right]\Big/\mu^*. \tag{16}$$

Here $\vec{\Omega}^*$ is the direction of observation, and the observation point is located on the top boundary of the medium; $P(\vec{\Omega} \to \vec{\Omega}^*)$ is calculated from the formulas (2), (3), and (4).

In the case of reflection from an underlying lambertian surface

$$\Psi(H, \vec{\Omega}) = r_s \exp[-G(\vec{\Omega}^*)\,H/\mu^*]/\pi$$

The i-th term in Eq. (15) is the contribution of photons from the i-th collisions to the SBC. The expansion (15) corresponds to the Monte-Carlo algorithm for calculating the spectral brightness coefficient, namely, the photon trajectories $x_0^n \to x_1^n \to \ldots \to x_m^n$, where x_1^n are the collision points of the n-th photon in the phase space and m is the number of the last collision before the photon escapes or is absorbed, are modeled according to the two distribution of the direction of scattering $P(\vec{\Omega} \to \vec{\Omega})$, appearing in the definition of the kernel $k(x' \to x)$. After each successive collision at a point the contribution $W_i^n\Psi(x_i^n)$ is added to the statistical estimate of I and

$$I(0, \vec{\Omega}^*) \approx \frac{1}{N}\sum_{n=1}^{N}\sum_{i=0}^{m} W_i^n\,\Psi(x_i^n),$$

where N is the number of trajectories and W_i^n is the "weight" of the n-th photon after the i-th collision.

5. MODELING OF THE MARKOV CHAIN

To solve the transfer equation it remains only to model the photon trajectory. The modeling process is based on the kernel k, which consists of two probability distribution of the photon free path length in the direction $\vec{\Omega}$ and the probability distribution of scattering in the direction $\vec{\Omega}$.

1) *Modeling of the free path length*. The optical free path length is

$$\bar{\tau} = -\ln\alpha,$$

where α is a random quantity distributed uniformly in the interval (0,1). From here the optical free path depth is

$$\tau = -\frac{\mu}{G(\vec{\Omega})}\ln\alpha.$$

2) *Modeling of the direction of scattering*. Let $\vec{\Omega}'$ be the direction of propagation of the photon prior to a collision. We shall study the question of modeling the direction $\vec{\Omega}$ after the collision. The probability distribution of a transition from the state $\vec{\Omega}'$ into the state $\vec{\Omega}$ has the form

$$P(\vec{\Omega}' \to \vec{\Omega}) = \frac{1}{2\pi}\int_{2\pi^+} g(\vec{\Omega}_L)\,\frac{|\vec{\Omega}'\vec{\Omega}|}{G(\vec{\Omega}')}\,f(\vec{\Omega}' \to \vec{\Omega}, \vec{\Omega}_L)\,d\vec{\Omega}_L.$$

We note that the modeling density must satisfy two requirements: 1) it must be convenient for modeling and 2) it must be universal.

We shall explain. Universality means that the density employed in the computational algorithm should be independent of the real distribution. Otherwise it would be necessary to construct a specific modeling formula for each computational variant, and the computational algorithm would not be universal. In the Monte-Carlo method the modeling is performed based on some uniformed and convenient density, and to compensate for the bias of the statistical estimate the "weight" of a particle in each collision is multiplied by a corresponding factor. The "weight" appears as a factor in the contribution function.

Thus we shall represent the function $P(\vec{\Omega}' \to \vec{\Omega})$ as a superposition of two distributions:

$$P(\vec{\Omega}' \to \vec{\Omega}) = \int_{2\pi^+}\frac{1}{2\pi}g_L(\vec{\Omega}_L)\frac{|\vec{\Omega}_L\vec{\Omega}'|}{G(\vec{\Omega}')}\,f(\vec{\Omega}' \to \vec{\Omega}, \vec{\Omega}_L)\,d\vec{\Omega}_L.$$

The direction $\vec{\Omega}_L$ can now be modeled as follows. We model the normal of a leaf $\vec{\Omega}_L$ using the distribution

$$P(\vec{\Omega}_L) = \frac{1}{2\pi}\,g_L(\vec{\Omega}_L)\,\frac{|\vec{\Omega}'\ \vec{\Omega}_L|}{G(\vec{\Omega}')},$$

and then, knowing $\vec{\Omega}_L$, we model the direction $\vec{\Omega}$ using the distribution $f(\vec{\Omega}' \to \vec{\Omega}, \vec{\Omega}_L)$. However the

density $P(\vec{\Omega}_L)$ does not satisfy the two requirements mentioned above: simple modeling formulas do not exist and the distribution function of the leaf normals $g_L(\vec{\Omega}_L)$ is, in the general case, a multiparametric function.

Assuming that the leaf normals are distributed uniformly over the azimuth φ and using a three-parameter family for their distribution over μ,[3] we represent the density $g_L(\vec{\Omega}_L)/2\pi$ in the form

$$\frac{1}{2\pi} g_L(\vec{\Omega}_L) = \frac{2}{\pi} \frac{1}{\sqrt{1-\mu^2}} \frac{1}{2\pi} g^*(\mu),$$

where

$$g^*(\mu) = a + b\,(2\mu^2 - 1) + c\,(8\mu^4 - 8\mu^2 + 1),$$

a, b, and c are const.

We shall regard the function $\frac{1}{2\pi}\frac{2}{\pi\sqrt{1-\mu^2}}$ as a new probably density for the direction $\vec{\Omega}_L \sim (\mu_L, \varphi_L)$, where $1/2\pi$ is the probability density along φ_L and $\frac{2}{\pi\sqrt{1-\mu^2}}$ is the probability density along μ_L. Thus the density along $\vec{\Omega}_L$ is convenient owing to the simplicity of the modeling formulas: $\varphi_L = 2\pi\alpha$, $\mu_1 = \sin(\pi\beta/2)$, where α and β are random (independent) quantities, uniformly distributed in the interval (0, 1).

We now separate from function $P(\vec{\Omega}_L)$ the density mentioned above:

$$P(\vec{\Omega}_L) = \left[\frac{1}{2\pi}\frac{}{\pi\sqrt{1-\mu^2}}\right] \times \left[g^*(\mu)\frac{|\vec{\Omega}'\vec{\Omega}|}{G(\vec{\Omega}')}\right].$$

The first factor is used as a new density for $\vec{\Omega}_L$ and the second factor is multiplied by the "weight" of a particle in each collision. It should be noted that the factorization can also be performed in a different manner, but the method we have chosen is convenient because the unbounded cofactor $1/\sqrt{1-\mu^2}$ is included in the density, and correspondingly, it is excluded from the "weight", so that the estimate remains bounded. Unboundedness of the estimate would result in large computational errors.

Modeling of the direction $\vec{\Omega}$ with known $\vec{\Omega}_L$ is performed using the density of the interaction of a photon with the surface of a leaf. In the case of a bilambertian scattering law

$$f(\vec{\Omega}' \to \vec{\Omega}, \vec{\Omega}_L) = \begin{cases} r_L |\vec{\Omega}\vec{\Omega}_L|/\pi, & (\vec{\Omega}\vec{\Omega}_L)(\vec{\Omega}'\vec{\Omega}_L) < 0, \\ t_L |\vec{\Omega}\vec{\Omega}_L|/\pi, & (\vec{\Omega}\vec{\Omega}_L)(\vec{\Omega}'\vec{\Omega}_L) > 0, \end{cases}$$

where r_L and t_L are differential reflection and transmission coefficients. Then $\vec{\Omega}$ is modeled by the formulas

$$\varphi = 2\pi\alpha, \quad \mu = \delta\sqrt{\beta},$$

where μ and φ are the coordinates of the vector $\vec{\Omega}$ in a spherical coordinate system with axis Ω_L, and

$$\delta = \begin{cases} +1, & [(\vec{\Omega}\vec{\Omega}_L) > 0] \;\&\; [\gamma > r_L/(r_L + t_L)] \text{ or} \\ & [(\vec{\Omega}'\vec{\Omega}_L) < 0] \;\&\; [\gamma < r_L/(r_L + t_L)], \\ -1, & \text{otherwise}, \end{cases}$$

where α, β, and γ are random quantities uniformly distributed in the interval (0, 1).

3) *Calculation of the contribution function.* For the contribution function we must use not the exact quantity $\Psi(\tau, \Omega)$, which is difficult to calculate after each collision, but rather its estimate randomized over Ω_L $f(\vec{\Omega} \to \vec{\Omega}^*, \vec{\Omega}_L)\exp(-\tau^*)/\mu^*$. Here $\vec{\Omega}$ is the leaf normal modeled after a collision and τ^* is the optical thickness of the layer from the collision point to the detector in the direction of observation $\vec{\Omega}^*$.

6. INCLUSION OF REFLECTION GLARE

The model constructed above approximates well PC in which the phytoelements are negligibly small compared with the height of the plants and only the probability distribution of their normals is given. In real PC, however, the phytoelements have finite sizes. In results in the formation of the so-called reflection glare in the SBC: the brightness of the PC increases significantly in the direction of the sun and this increase depends on the geometric structure of both a single plant and the crop as a whole.[4] The extinction coefficient of the radiation flux in the direction $\vec{\Omega}$ after a collision depends not only on the distribution of the normals $g_L(\vec{\Omega}_L)$ (see Eq. (2)), but also on the direction of motion prior to the collision. If the cross-correlation coefficient of the indicator functions of the presence of lamella in the differential layer in the direction $\vec{\Omega}$ and $\vec{\Omega}'$ is known,[5] the new extinction coefficient σ_χ, which depends on the dimensions of the phytoelements can be calculated. In the case of uniform PC we have

$$\sigma_\chi[(\tau, \vec{\Omega}) \to (\tau', \vec{\Omega}')] =$$

$$= \begin{cases} G(\vec{\Omega}), & \mu\mu' > 0, \\ G(\vec{\Omega})\left[1 - \sqrt{A'/A}\,\exp\left(-\frac{\Delta(\vec{\Omega}\vec{\Omega}')|\tau - \tau'|}{\chi}\right)\right], & \mu\mu' < 0, \end{cases}$$

where $A = G(\vec{\Omega})/|\mu|$, $A' = G(\vec{\Omega}')/|\mu'|$, and $\Delta(\vec{\Omega}, \vec{\Omega}') = (\mu^{-2} + \mu'^{-1} + 2(\vec{\Omega}'\cdot\vec{\Omega})/|\mu\mu_0|)^{1/2}$, where χ is

the average length of the chord of a phytoelement. Understandably, in the case $\vec{\Omega} = -\vec{\Omega}'$, i.e., when the reflection occurs strictly backwards, there is no extinction and $\sigma_\chi = 0$.

Thus in modeling the free path length of a photon we use instead of $G(\vec{\Omega})$ the new extinction coefficient σ_χ. Here χ are the average linear parameters of a leaf.[5]

4. CONCLUSION

The method proposed above for solving the integral equation of radiation transfer (for vegetation) by the Monte-Carlo method makes it possible to estimate the spectral brightness coefficient of the "soil - vegetation" system for different types of agricultural crops.

The results are in good agreement with the solution of the integrodifferential equation by the method of discrete ordinance.

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