On the geometric optics of a light pulse

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Received September 1, 2004

It is shown that the combination of the asymptotical estimate of the integral of spectral expansion and geometric optics immediately reduces the problem on the light pulse to the problem on monochromatic wave propagation.

Introduction

As is well known in the electrodynamics, the spectral expansion reduces the problem on pulse propagation to the problem with a monochromatic (at a frequency ω) field. However, the following integration over ω is very complicated, and, in fact, one is forced to deal with the "time" version. However, asymptotic optics of "spectral" integrals opens some possibilities when the approximation of geometric optics is applicable. The main objective of this paper is to illustrate this thesis.

Although, for simplicity and definiteness, we consider a linear isotropic medium without spatial dispersion, this thesis can be extended to other cases as well.

1. The initial equations

Let

$$\mathbf{E}(\mathbf{r},t)=\int_0^\infty d\omega \mathbf{B}(\mathbf{r},\omega)\,e^{-i\omega t} \tag{1}$$

be an analytical signal of a real electric field $\mathbf{E}(\mathbf{r}, t)$ strength at the point $\mathbf{r}$ and time t; $\mathbf{B}(\omega)$ is the spectral component of $\mathbf{E}(t)$ [Fourier series expansion with $\int_{-\infty}^{+\infty} d\omega(\ldots)$]. For the medium under consideration (Δ is the Laplace operator, c is the speed of light)

$$\Delta\mathbf{E}-\frac{1}{2}\frac{\partial^2\mathbf{E}}{\partial t^2}=\frac{4\pi}{c^2}\frac{\partial^2\mathbf{P}(\mathbf{r},t)}{\partial t^2} \tag{2}$$

and the dipole moment of a unit volume

$$\mathbf{P}(\mathbf{r},t)=\int_0^\infty f(\mathbf{r},\tau)\,\mathbf{E}(t-\tau)d\tau \tag{3}$$

with the relaxation function f [Ref. 1]. After the substitution of Eq. (1) into Eqs. (2) and (3) we have

$$\Delta\mathbf{B}(\mathbf{r},\omega)+\frac{\omega^2}{c^2}\varepsilon(\mathbf{r},\omega)\mathbf{B}(\mathbf{r},\omega)=0 \tag{4}$$

and the dielectric constant

$$\varepsilon(\mathbf{r},\omega)=1+4\pi\int_0^\infty f(\mathbf{r},\tau)\,e^{i\omega\tau}d\tau. \tag{5}$$

Upon solving Eq. (4) in the geometric-optics approximation one obtains

$$\mathbf{B}(\mathbf{r},\omega)=\mathbf{V}(r,\omega)\,e^{i\varphi(\mathbf{r},\omega)}. \tag{6}$$

Here

$$(\mathrm{grad}\varphi)^2=(\omega^2\varepsilon)/c^2 \tag{7}$$

is the equation of eikonal φ and

$$2(\mathrm{grad}\,\varphi\,\mathrm{grad})\mathbf{V}+\mathbf{V}\Delta\varphi=0, \tag{8}$$

when, according to the approximation, $\Delta\mathbf{V}$ is ignored. It is clear from Eq. (8) that the equations for any component V of the vector $\mathbf{V}$ are the same and, multiplying them by V, we obtain the equation referred to as the transfer equation:

$$(\mathrm{grad}\varphi\,\mathrm{grad}V^2)+V^2\Delta\varphi=0. \tag{9}$$

Then,

$$\mathrm{grad}\,\varphi=\frac{\omega}{c}\sqrt{\varepsilon}\,\mathbf{s}\equiv\frac{\omega}{c}m\,\mathbf{s} \tag{10}$$

and the ray unit vector $\mathbf{s}$ is the solution of the problem

$$\mathrm{rot}\,\mathbf{s}=\frac{1}{m}(s\times\mathrm{grad}\,m). \tag{11}$$

Geometric optics of Eq. (2) begins from the solution form which is well analogous to expression (6):

$$\mathbf{E}(\mathbf{r},t)=\mathbf{U}(\mathbf{r},t)\,e^{i\psi(\mathbf{r},t)}, \tag{12}$$

but the following simplifications are much more sophisticated. It is a significant condition that $\mathbf{E}(t-\tau)$ in Eq. (3) can be expanded in a series in terms of τ with the minimum number of terms. For the equation of the type (12) this means that the terms $O(\tau)$ and $O(\tau^2)$ are kept in $\mathbf{U}$ and ψ, respectively, with the following elimination of the terms $O(\tau^3)$ and higher. In fact, this means that

$$\partial^2\mathbf{U}/\partial t^2;\partial^3\psi/\partial t^3;(\partial^2\psi/\partial t^2)(\partial\mathbf{U}/\partial t),(\partial^2\varphi/\partial t^2)^2$$

0235-6880/04/11 770-04 $02.00

etc., are ignored, and the calculation of $\mathbf{P}(\mathbf{r}, t)$ involves derivatives of Eq. (5) with respect to ω with the following substitution:

$$\omega \to \Omega = -\partial\psi/\partial t, \tag{13}$$

and $\Delta\mathbf{U}$ is removed as before. The result reads as:

$$(\mathrm{grad}\,\psi)^2 = \frac{1}{c^2}\left(\frac{\partial\psi}{\partial\tau}\right)^2 \varepsilon(\mathbf{r} - \partial\psi/\partial t), \tag{14}$$

$$2(\mathrm{grad}\,\psi\ \ \mathrm{grad}\,U^2) + U^2\Delta\psi + \frac{1}{2c^2}\frac{\partial}{\partial t}\left(U^2\frac{\partial(\omega^2\varepsilon)}{\partial\omega}\right) = 0. \tag{15}$$

The analog of Eq. (9) for U — the vector $\mathbf{U}$ component — is presented. The equation (14) becomes an equivalent of eikonal (7) after the substitution of Eq. (12) into Eq. (14), and after differentiation of Eq. (12) with respect to ω the same substitution should be done in Eq. (15).

All mathematical details of Eqs. (6)–(11), (12)–(15), and their corresponding physical aspects can be found in Refs. 2–5.

2. Relation of ψ, U with φ, V

The asymptotic estimation of the integral (1) after the substitution of Eq. (6) gives (accurate to a constant factor determined by a particular approach: saddle-point technique, method of stationary phase, etc.[6]) the equation

$$\left.\frac{\mathbf{V}(\mathbf{r},\omega)}{\sqrt{\partial^2\varphi(\mathbf{r},\omega)/(\partial\omega^2)}}\right|_{\tilde\omega} e^{i\varphi(\mathbf{r},\tilde\omega)-i\tilde\omega t}, \tag{16}$$

where $\tilde\omega(r,\omega)$ is the root of the equation

$$\frac{\partial\varphi(\mathbf{r},\omega)}{\partial\omega} - t = 0 \tag{17}$$

with respect to ω. The possibility of this action is guaranteed by the fact that for a light pulse its duration exceeds the period of oscillations $(1/\omega)$.

The comparison of Eqs. (16) and (12) indicates that

$$\psi(\mathbf{r},t) = \varphi(\mathbf{r},\tilde\omega(\mathbf{r},t)) - t\,\tilde\omega(\mathbf{r},t); \tag{18}$$

$$\mathbf{U}(\mathbf{r},t) = \frac{\mathbf{V}(\mathbf{r},\tilde\omega(\mathbf{r},t))}{\left.\sqrt{\partial^2\varphi(\mathbf{r},\omega)/(\partial\omega^2)}\right|_{\tilde\omega}}, \tag{19}$$

and we should check whether Eqs. (18) and (19) satisfy Eqs. (14) and (15).

In Eq. (18)

$$\mathrm{grad}\,\psi = \mathrm{grad}\,\varphi(\mathbf{r},\tilde\omega) - t\,\mathrm{grad}\,\tilde\omega(\mathbf{r},t),$$

$$\mathrm{grad}\varphi(\mathbf{r},\tilde\omega) = (\mathrm{grad}\,\varphi(\mathbf{r},\omega))_{\tilde\omega} + \left.\frac{\partial\varphi(\mathbf{r},\omega)}{\partial\omega}\right|_{\tilde\omega}\mathrm{grad}\,\tilde\omega - t\,\mathrm{grad}\,\tilde\omega = $$
$$= (\mathrm{grad}\,\varphi(\mathbf{r},\omega))_{\tilde\omega}$$

in view of Eq. (17). Similarly,

$$\frac{\partial\psi}{\partial t} = \left.\frac{\partial\varphi}{\partial\omega}\right|_{\tilde\omega}\frac{\partial\omega}{\partial t} - \tilde\omega - t\frac{\partial\tilde\omega}{\partial t} = -\tilde\omega.$$

After the substitution

$$\mathrm{grad}\psi = (\mathrm{grad}\varphi(\mathbf{r},\omega))_{\tilde\omega},\quad \frac{\partial\psi}{\partial t} = -\tilde\omega, \tag{20}$$

the left-hand side of Eq. (14) becomes

$$\left.(\mathrm{grad}\,\varphi(\mathbf{r},\omega))^2 - (1/c^2)\omega^2\varepsilon(\mathbf{r},\omega)\right|_{\tilde\omega} = 0$$

because of Eq. (7). From the first of the equalities (20) and from Eq. (17), it also follows that

$$\Delta\psi(\mathbf{r},t) = \Delta\varphi(\mathbf{r},\omega)|_{\tilde\omega}. \tag{21}$$

Then introduce the designations:

$$V^2 = A(\mathbf{r},\omega);\ g = \frac{\partial^2 U(\mathbf{r},\omega)}{\partial\omega^2};\ U^2 = \left.\frac{A}{g}\right|_{\tilde\omega},$$

which will be substituted into Eq. (15), keeping in mind Eq. (19). It is clear that

$$\mathrm{grad}\,U^2 = \left(\mathrm{grad}\frac{A(\mathbf{r},\omega)}{g(\mathbf{r},\omega)}\right)_{\tilde\omega} + \left(\frac{\partial}{\partial\omega}\frac{A(\mathbf{r},\omega)}{g(\mathbf{r},\omega)}\right)_{\tilde\omega}\mathrm{grad}\,\tilde\omega =$$
$$= \underline{\left(\frac{1}{g}\mathrm{grad}\,A(\mathbf{r},\omega)\right)_{\tilde\omega}} - \left(\frac{A}{g^2}\mathrm{grad}\,g(\mathbf{r},\omega)\right)_{\tilde\omega} + \left(\frac{\partial}{\partial\omega}\frac{A}{g}\right)_{\tilde\omega}\mathrm{grad}\,\tilde\omega.$$

The sum of Eq. (21) and the underlined term is zero in view of Eq. (9). The left-hand side of Eq. (15), due to grad U^2, now includes the following term

$$\left(\left(\frac{\partial}{\partial\omega}\frac{A}{g}\right)\mathrm{grad}\,\tilde\omega\ \mathrm{grad}\,\varphi(\mathbf{r},\omega)\right)_{\tilde\omega} -$$
$$-\left(\frac{A}{g}\mathrm{grad}\,g(\mathbf{r},\omega)\,\mathrm{grad}\,\varphi(\mathbf{r},\omega)\right)_{\tilde\omega}. \tag{22}$$

The third term from the left-hand side of Eq. (15) can be directly reduced to the form

$$\left(\frac{\partial}{\partial\omega}\left(\frac{A}{g}\right)\frac{\partial(\omega^2\varepsilon(\omega))}{\partial\omega}\right)_{\tilde\omega}\frac{\partial\tilde\omega}{\partial t}\frac{1}{2c^2} +$$
$$+\ \frac{1}{2c^2}\left(\frac{A}{g}\frac{\partial^2(\omega^2\varepsilon(\omega))}{\partial\omega^2}\right)_{\tilde\omega}\frac{\partial\tilde\omega}{\partial t}. \tag{23}$$

Upon differentiating Eq. (14) with respect to t and invoking Eq. (20), it becomes clear that the first terms in Eqs. (22) and (23) mutually annihilate. Analogous identical transformations with the application of the cited equations and the additional equation, following from Eq. (17),

$$\frac{\partial\tilde\omega}{\partial t} = \frac{1}{g(\mathbf{r},\tilde\omega)} \tag{24}$$

allow us to see that the second terms of Eqs. (22) and (23) mutually annihilate as well. Therefore, it can be stated that Eq. (19) satisfies Eq. (15).

Essentially, the thesis, formulated in the Introduction, that in the geometric-optics approximation the asymptotic estimation of the "spectral" integral allows us to directly represent the solution of the problem of pulse propagation in terms of the "monochromatic" approximation is already justified.

3. Additional arguments

In the general case, the complex characteristic (5) is $\varepsilon = \varepsilon' + i\varepsilon''$. This naturally implies the complex $m = n + i\kappa$ with the refractive index n and the absorption coefficient κ, the complex eikonal $\varphi = \varphi' + i\varphi''$ and unit vector $\mathbf{s} = \mathbf{s}' + i\mathbf{s}''$ in Eq. (9). After separation of the real and imaginary parts in Eq. (7)

$$(\mathrm{grad}\,\varphi')^2 - (\mathrm{grad}\,\varphi'')^2 = \frac{\omega^2}{c^2}\varepsilon' = \frac{\omega^2}{c^2}(n^2 - \kappa^2), \quad (25)$$

$$\mathrm{grad}\,\varphi'\,\mathrm{grad}\varphi'' = \frac{\omega^2}{c^2}\frac{\varepsilon''}{2} = \frac{\omega^2}{c^2}n\kappa. \quad (26)$$

Interpretation of the geometric-optics characteristics is usually connected with studying the structure of the field of type (6) or (12) in the $\boldsymbol{\rho}$-vicinity of an arbitrary point $\mathbf{r}$. The possibility of $\mathbf{U}(\mathbf{r} + \boldsymbol{\rho}) \to \mathbf{U}(\mathbf{r})$ was already commented, and

$$\begin{aligned}\psi(\mathbf{r} + \boldsymbol{\rho}, t) &= \psi(\mathbf{r}, t) + \left(\mathrm{grad}\psi(\mathbf{r}, t)\right)\boldsymbol{\rho} = \\ &= \varphi(\mathbf{r}, \tilde{\omega}) - t\tilde{\omega}(\mathbf{r}, t) + \left(\mathrm{grad}\varphi(\mathbf{r}, \omega)\right)_{\tilde{\omega}}\boldsymbol{\rho} = \\ &= \varphi(\mathbf{r}, \tilde{\omega}) - t\tilde{\omega}(\mathbf{r}, t) + \left(\mathrm{grad}\varphi'(\mathbf{r}, \omega)\right)_{\tilde{\omega}}\boldsymbol{\rho} + \\ &\quad + i\left(\mathrm{grad}\varphi''(\mathbf{r}, \omega)\right)_{\tilde{\omega}}\boldsymbol{\rho}\end{aligned}$$

after invoking Eqs. (18) and (20) and a comment on the complex character of the eikonal. Now $\mathbf{U}\exp i\varphi(\mathbf{r},\tilde{\omega})$ can be declared the wave amplitude at the point $\mathbf{r}$ and $\mathrm{grad}\,\varphi'(\mathbf{r},\tilde{\omega})$ can be declared the wave vector κ. For the field with this κ and when $\boldsymbol{\rho} \sim \boldsymbol{\kappa}$, the term $\left(\mathrm{grad}\,\varphi''(\mathbf{r},\tilde{\omega})\right)\boldsymbol{\rho}$ describes, as follows from Eq. (21), the absorption if this wave in the $\boldsymbol{\rho}$-vicinity. Certainly, the amplitude should be supplemented with the corresponding factor.

It becomes clear that the oscillating factor in the combination (1) is $\varphi' - \omega t$; just this factor creates prerequisites for asymptotical estimation of the integral (1). Therefore Eq. (17), describing a stationary point, should be rewritten as

$$\frac{\partial\varphi'(\mathbf{r},\omega)}{\partial\omega} = t. \quad (27)$$

From Eq. (27) we can see that its root $\tilde{\omega}(\mathbf{r},t)$ is real. Therefore, the asymptotical estimation of Eq. (1) is performed by the method of stationary phase, and Eq. (16) acquires the factor $\exp i(\pi/4)$. Then, Eq. (27) automatically introduces the restriction on the time interval t (pulse (!)) – once (at some t) the root of Eq. (27) appears to be complex, the method of stationary phase is replaced by the saddle-point method, which requires the exponentially small (asymptotically vanishing) field value.

Another evidence of the reasonableness of the discussed approach is connected with the calculation of the total pulse energy: Eq. (16) should be subject to the operation $\int \mathrm{d}t|\ldots|^2$. The following change of the variable $\tilde{\omega}(\mathbf{r},t) = \omega$ in this integral and Eq. (24) transform the considered expression into $\int \mathrm{d}\omega|\mathbf{B}|^2$ – integration with respect to frequency, as it must be done in the general case.

4. Absorbing medium

Significant $\kappa \neq 0$ (light absorption cannot be ignored), and the need to seek φ (because of the significant role of Eq. (27)) stipulate the need to solve the system of Eqs. (25) and (26). In this case, the conditions of applicability of the geometric optics considerably simplify the corresponding characteristics.

Then, already from Eq. (27), we can find $\tilde{\omega}(\mathbf{r},t)$, represent, using Eqs. (10) and (11), $\mathbf{s}'$ and $\mathbf{s}''$ through $\mathrm{grad}\varphi'$ and $\mathrm{grad}\varphi''$, calculate the unit vector

$$\boldsymbol{\kappa} = \left.\frac{n\,\mathbf{s}' - \kappa\,\mathbf{s}''}{\sqrt{n^2 s'^2 + \kappa^2 s''^2}}\right|_{\tilde{\omega}} \quad (28)$$

of the wave vector, and then solve the equation $\mathrm{d}\mathbf{y}/\mathrm{d}l = \boldsymbol{\kappa}_0$ for the ray trajectory $\mathbf{y}$ as a function of its length l. (Certainly, it becomes possible to use the standard approach[3,5] for construction of the geometric-optics rays).

The relationships discussed lead to an evident result: at low absorption ($\kappa \to 0$, and then $n \approx$ const as a function of ω), $t \to t - (1/c)\int \varphi'\,\mathrm{d}l$ in the expression for $\boldsymbol{\varepsilon}(\Sigma, t)$ (certainly, with a correction for the beam divergence and low absorption of the beam energy).

The analysis shows that in the case of strong absorption $\cos\beta \approx 1$ for the angle β between $\mathrm{grad}\varphi'$ and $\mathrm{grad}\varphi''$. Then from Eqs. (25) and (26) it follows that $(\mathrm{grad}\varphi')^2 \sim n^2$. Just n describes the spectral dependence of φ' in this case, and, therefore, the value of $\tilde{\omega}$ is determined by the areas of anomalous refraction (n changes within an absorption line[2,7]). This, already physical, circumstance correlates well with the general analysis of the problem.

The latter means also that Eqs. (27) can have several roots – corresponding to the number of anomalous dispersion areas covered by the pulse spectrum in the asymptotic estimation of the "spectral" integral. This leads to the sum of the corresponding plane waves, which, in turn, means that geometric-optics rays intersect at the point $\mathbf{r}$.

However, there is no contradiction with the condition of non-intersection, which follows from

Eq. (11), because this condition is formulated for a certain frequency $\omega = \text{const}$, whereas the non-intersection of rays keeps true for Eq. (28) as well. However, now $\omega \to \tilde{\omega}(\mathbf{r}, t)$, which simply means actual co-existence of rays with different frequencies.

References

1. L.D. Landau and E.M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon Press, Oxford, 1984).
2. M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, New York, 1959).
3. Yu.A. Kravtsov and Yu.I. Orlov, *Geometric Optics of Inhomogeneous Media* (Nauka, Moscow, 1980), 304 pp.
4. V.M. Babich and V.S. Buldyrev, *Asymptotic Methods in Problems of Diffraction of Short Waves* (Nauka, Moscow, 1972), 456 pp.
5. Yu.N. Podil'chuk and Yu.K. Rubtsov, *Ray Methods in Theory of Wave Propagation and Scattering* (Naukova Dumka, Kiev, 1988), 216 pp.
6. N.G. de Bruijn, *Asymptotic Methods in Analysis* (Dover, New York, 1981).
7. I.L. Fabelinskii, *Molecular Scattering of Light* (Nauka, Moscow, 1965), 541 pp.