CALMAN–BUCY FILTRATION IN LIDAR SENSING OF TEMPERATURE BY THE DIAL METHOD

G.M. Igonin

*Institute of Atmospheric Optics,
Siberian Branch of the Russian Academy of Sciences, Tomsk*
Received July 27, 1992

The feasibility of using the optimal Markovian filtration in sensing of the troposphere at altitudes up to 3 km *by the DIAL method is illustrated on the basis of a stochastic model of the altitude behaviour of temperature fluctuations smoothed by lidar pulses. The Calman-Bucy algorithms for optimal estimate of the fluctuating profiles of temperature and its variance are synthesized. Their efficiency is analyzed by a numerical simulation technique as applied to sensing in the line of absorption of the* A*–band of oxygen at the transition* P^p27 *centered at* λ = 768.3802 nm.

Introduction. Routine information about the profiles of meteorological parameters with high spatiotemporal resolution and accuracy is needed for solving many problems in meteorology, ecology, and atmospheric physics. Because the efficiency of temperature lidar sensing by any method is limited by the fluctuations in the temperature and return signals, in particular, by shot fluctuations, it is necessary to increase the energy potential of a lidar and to optimize processing of received signals.

In Refs. 1–8 the feasibility of optimizing with the help of the Markovian filtration was demonstrated as applied to single–frequency and bifrequency sensing of the fluctuating temperature and related parameters with the use of Rayleigh and Raman scattering in the vibrational–rotational spectrum of nitrogen. In Ref. 9 this elaboration was continued as applied to the optimization of signal processing of the differential absorption lidar (DIAL) and the equations of filtration were derived on the basis of the Markovian model of altitude behaviour of fluctuations in a gas concentration smoothed by the lidar pulse.

In this paper the Markovian filtration is used for optimal reconstruction of the fluctuating profiles of temperature in sensing of the temperature in the absorption line of the A–band of oxygen. Concrete estimates are given as applied to sensing at the wavelengths of an alexandrite laser or a titanium–doped sapphire laser in the absorption line of oxygen at the transition $P^p 27$.

Physical premises. We consider the ground–based monostatic lidar generating the pulses with the normalized power function $f(t)$ at the wavelengths λ_1 and λ_0, lying in the center and off the oxygen absorption line, and sensing of the atmosphere in the altitude range $[h_0, h_{\max}]$. The power $P_{si}(h)$ of the signal component at the detector input in the single–scattering approximation from the distance h at λ_i is determined by the lidar equation.

Assuming that smoothing over the running interval $[h - L, h]$ changes substantially only the profiles of the functions $\tilde{Y}_{gi}(0, h)$ describing the transmission at λ_i due to absorption by molecular oxygen and the altitude profiles of the absorption characteristics and related atmospheric thermodynamic parameters, we obtain the following form of the lidar equation[9]:

$$P_{si}(h) = \chi_1 E_0 S_a h^{-2} \beta_i(h) \frac{c}{2} Y_{ai}(0, h) Y_{Ri}(0, h) J_i(h) , \quad (1)$$

$$J_i(h) = \frac{2}{c} \int\limits_0^h \mathrm{d}h' \, f[2(h - h')/c] \, \tilde{Y}^2_{gi}(0, h') , \quad (2)$$

where χ_1 is the efficiency of the entire optical train, E_0 is the pulse energy, S_a is the effective area of the receiving aperture, Y_{ai} and Y_{Ri} are the transmission functions due to aerosol and molecular scattering, c is the speed of light, $L = c\tau_p/2$, τ_p is the effective pulse width, and $\beta_i(h)$ are the profiles of the backscattering and molecular scattering coefficients.

The molecular mass absorption coefficient has the form

$$K_g(\lambda_i, h) = S_g(\lambda_i, h) \, f_g(\nu - \nu_0) , \quad (3)$$

where $S_g(\lambda_i, h)$ is the intensity of the absorption line depending on $T(h)$, while $f_g(\nu - \nu_0)$ describes the shape of the absorption line. In the troposphere below 3 km the collisional broadening prevails, and taking this into account for $\nu = \nu_0$, at the absorption line center we have $f(0) = 1/[\pi\gamma_L(\lambda_i, h)]$, where $\gamma_L(\lambda_i, h)$ is the Lorentz halfwidth of the absorption line depending on the temperature $T(h)$ and pressure $P(h)$. In Ref. 10 for the Voigt shape of the O_2 absorption line the analytical approximation

$$f(0) = \frac{1}{3\gamma_L(\lambda, h)} \left[1 - \frac{e}{10 \, b_{DL}} \right] \quad (4)$$

was proposed accurate to 0.1% at altitudes up to 2 km and to 0.7% in the troposphere, where $b_{DL} = (\gamma_L/\gamma_D)\sqrt{\ln 2}$ and $\gamma_D(\lambda, h)$ is the line halfwidth due to Doppler broadening.

Following the approach adopted in Refs. 2 and 8, we represent the random temperature values in the form $T(h) = \overline{T}(h) + \Delta\tilde{T}(h)$, where $\overline{T}(h)$ is the mean profile known *a priori* with sufficient statistics and the bar denotes averaging over the ensemble of temperature fluctuations. The use of Eq. (4) permits us to write down the relation for the profiles $K_{O_2}(\lambda_1, h)$ of the O_2 absorption for arbitrary $T(h)$ and $P(h)$ and $\overline{K}_{O_2}(\lambda_1, h)$ for $\overline{T}(h)$ and $\overline{P}(h)$. The bifrequency method of temperature sensing is based on

0235-6880/92/10 690-05 $02.00 © 1992 Institute of Atmospheric Optics

measuring the volume scattering coefficient $\gamma_1(h; T(h), P(h))$ in the maximum of the O_2 absorption line.[10,11] Therefore, taking into account the fact that

$$\gamma_1(h; T, P) = q_0(1 - q(h))\, b(h) K_{O_2}(\lambda_1, h)$$

we have

$$\gamma_1(h; T, P) = \gamma_1(h, \overline{T}, \overline{P}) \left(\frac{\overline{T}}{T}\right)^{3/2} \times$$

$$\times \exp\left\{\frac{h_P c}{k} E''_1 \left(\frac{1}{\overline{T}} - \frac{1}{T}\right)\right\}, \quad (5)$$

where $q_0 = 0.2095$ is the volume content of oxygen in the dry atmosphere, $q(h)$ is the volume content of water vapour at the altitude h, $b(h)$ is the density profile of air, and h_P and k are the Planck and Boltzmann constants, respectively.

As far as in the atmosphere $\sigma_T(h) \ll (h)$, where $\sigma_T(h)$ is the rms error of $T(h)$, Eq. (5) can be linearized with respect to natural fluctuations $\Delta\tilde{T}(h)$, i.e.,

$$\gamma_1(h; T, P) = \gamma_1(h; \overline{T}, \overline{P}) \exp\left\{B(h) \frac{\Delta\tilde{T}(h)}{\overline{T}(h)}\right\}, \quad (6)$$

where

$$B(h) = 1.439 \frac{E''_1}{\overline{T}(h)} - \frac{3}{2}. \quad (7)$$

In Ref. 11 it was shown that the transition $P^P 27$ centered at $\lambda_1 = 768.3802$ nm with the energy of the lower level $E''_1 = 1085.206\ \text{cm}^{-1}$ is most suitable for determining the temperature at the altitudes up to 3 km. It is evident from Eq. (7) that in the troposphere for real $\overline{T}(h)$ at this transition $B(h) < 10$. For this reason Eq. (6) admits further linearization with the fluctuations $\Delta\gamma_1 = \gamma_1 - \overline{\gamma}_1$ in the O_2 absorption coefficient and the temperature fluctuations $\Delta\tilde{T}(h)$ being related linearly since

$$\gamma_1(h; T, P) \simeq \gamma_1(h; \overline{T}, \overline{P}) \left[1 + B(h) \frac{\Delta\tilde{T}(h)}{\overline{T}(h)}\right]. \quad (8)$$

Let us expand

$$\tilde{Y}^2_{g1}(0, h) = \exp\left\{-2 \int_0^h \mathrm{d}h'\, \gamma_1(h'; T, P)\right\}$$

in a Taylor series expansion in terms of the profile $\Delta\tilde{T}(h)$ about the altitude realization smoothed by the sensing pulse

$$\Delta T(h) = \frac{2}{c} \int_0^h \mathrm{d}h'\, f[2(h - h')/c]\, \Delta\tilde{T}(h'). \quad (9)$$

Because the profiles $\Delta\tilde{T}(h)$ and $\Delta T(h)$ are close in values, for the wavelength λ_1 the functional given by Eq. (2) can be written down in the form $J_1(h) \simeq \overline{Y}^2_{g1}(0, h) \exp[-2\,\Delta\tau_1(0, h)]$, where $\overline{Y}_{g1}(0, h)$ is the transmission due to absorption by oxygen for $T(h)$ and $P(h)$, while

$$\Delta\tau_1(0, h) = \int_0^h \mathrm{d}h'\, g_1(h'; \overline{T}, \overline{P})\, B(h')\, \Delta T(h')/\overline{T}(h') \quad (10)$$

are the fluctuations of the optical thickness. As far as the optimal mean optical thickness[11] $\overline{\tau}_1(0, h) \simeq 1.1$, then with the use of the Bunyakowskii–Schwarz inequality it can be shown that $\sqrt{\overline{\Delta\tau^2_1(0, h)}} \ll 1$.

By linearizing the relation for $J_1(h)$ we obtain

$$J_1(h) \simeq \overline{Y}^2_{g1}(0, h)\left[1 - 2\,\Delta\tau_1(0, h)\right]. \quad (11)$$

Thus, when the profiles of the Rayleigh and aerosol backscattering coefficients and the transmission functions are deterministic but unknown altitude function, the statistical structure of the return power profile $P_{s1}(h)$ is determined by the linearly related temperature fluctuations $\Delta T(h)$ smoothed efficiently in accordance with Eq. (9) over the spatial range L.

Signal and noise model. Let $L \gg h^n_{cT}$, when h^n_{cT} is the vertical correlation distance of the nonsmoothed fluctuations $\Delta\tilde{T}((h)$. Then for the normalized fluctuations $\eta_1(\tau) = \Delta T(c\tau/2)/\sigma_T(h)$ the approximation in the form of the Gaussian Markovian process, which was proposed in Refs. 1–3, is acceptable. We introduce the state variable $\eta_2(\tau) = \Delta\tau_1(0, h)/\mu(h)$, where $\mu(h) = \sigma_T(h)/\overline{T}(h)$ is the variation coefficient of fluctuations in $T(h)$, and differentiate Eq. (10). Following Refs. 8 and 9 we conclude that the complete statistical description of the temperature fluctuations and the related absorption characteristics is provided by the two–dimensional vector–process $\eta = \{\eta_1, \eta_2\}^T$, the stochastic differential equation (SDE) for which has the form

$$\frac{\mathrm{d}\eta}{\mathrm{d}\tau} = A(h)\, \eta(\tau) + w(\tau), \quad (12)$$

where $w_1(\tau) = \{w_1(\tau), 0\}^T$, $w_1(\tau)$ is the Gaussian white noise:

$$\langle w_1(\tau)\rangle = 0, \quad \langle w_1(\tau)\, w_1(\tau')\rangle = 2\alpha\, \delta(\tau - \tau'), \quad \alpha = 1/\tau_p,$$

$$A(h) = \begin{pmatrix} -\alpha & 0 \\ c\overline{\gamma}_1(h)\, B(h)/2 & 0 \end{pmatrix}.$$

For the given realizations $\eta(\tau)$ the current of the photodetector in the ith channel is

$$y_i(\tau) = s_i(\tau; \eta, \boldsymbol{u}_i) + n_i(\tau), \quad (13)$$

where s_i is the signal current averaged over the ensemble of shot fluctuations,

$$s_i(\tau; \eta, \boldsymbol{u}_i) = \overline{s}_i(\tau; \boldsymbol{u}_i)\,(1 + \boldsymbol{C}^T\eta), \quad \boldsymbol{C} = \{0, -2\mu(h)\}^T,$$

$$\overline{s}_i(\tau; \boldsymbol{u}_i) = \xi_i\, \overline{P}_{si}(h), \quad \xi_i = \frac{\chi_d\, q_e}{h_P\, c}\lambda_i, \quad \boldsymbol{u}_i = \{\beta_i, Y_{Ri}\},$$

$Y_{\Sigma i} = Y_{ai} Y_{Ri} \overline{Y}_{gi}$, q_e is the charge of electron, $n_i(\tau)$ is the Gaussian process with zero mean including the shot fluctuations of the signal, background, and dark current in the ith channel. Under the condition $\Pi_i \tau_p \gg 1$, where Π_i is the bandwidth of the postdetector filter in the ith channel, the process $n_i(\tau)$ may be considered as white with the spectral power density

$$N_{0i}/2 = q_e [\, \overline{s}_i + \xi_i P_{\mathrm{bg}\, i} + s_{\mathrm{dc}\, i}] ,$$

where $P_{\mathrm{bg}\, i}$ and $s_{\mathrm{dc}\, i}$ are the background power at the input and the dark current of the ith detector and χ_{d} is the quantum efficiency of photodetector.

Filtration equations. We search for the technique of processing of the photocurrents $\boldsymbol{y}(\tau) = \{y_i(\tau)\}$, $i = 0,1$ providing the optimal (in the sense of maximum *a posteriori* probability density) estimate of the realization $\eta(\tau)$. By virtue of the assumptions adopted above it is necessary to estimate simultaneously the unknown profiles $\boldsymbol{u}_i$. Following Refs. 5, 7, and 8 we avoid the *a priori* ambiguity in $\boldsymbol{u}_i$ by adopting the variant of maximum likelihood that in our case of the additive Gaussian noise reduces to solving the equation

$$y_0(\tau) = \overline{s}_0(\tau;\, \tilde{\boldsymbol{u}}_0) , \tag{14}$$

where $\tilde{\boldsymbol{u}}_0$ is the estimate of $\boldsymbol{u}_i$ at λ_0. It should be noted that Eq. (14) is derived under condition that the O_2 absorption at λ_0 can be neglected. As λ_0 and λ_1 are closely spaced, the wavelength dependence of the aerosol and molecular scattering coefficients can be disregarded and thereby the estimates $\tilde{\boldsymbol{u}}_0$ obtained at λ_0 can be used in processing of the signals at λ_1.

By applying the Gaussian approximation of the *a posteriori* probability density η, we arrive at the Calman–Bucy system of the equations for quasioptimal filtration[5,12]

$$\dot{\mathbf{h}}^* = A(h)\, \boldsymbol{\eta}^* + \frac{2}{N_1} K\boldsymbol{C} \left[\, y_1(\tau) - \overset{\simeq}{s}_1(\tau)(1 - 2\mu(h)\, \eta_2^+ \right] , \tag{15}$$

$$\dot{K} = AK + KA^{\mathrm{T}} + b - \frac{2\, \overset{\simeq}{s}{}_1^2}{N_1} K\boldsymbol{C}\boldsymbol{C}^{\mathrm{T}} K , \tag{16}$$

which must be completed by Eq. (14) to obtain the estimate $\overset{\downarrow}{s}_0$ and thereby $\overset{\simeq}{s}_1 = \overset{\simeq}{s}_0 \overline{Y}{}_{g1}^2$, where $K = \langle(\eta - \eta^*)(\eta - \eta^*)^{\mathrm{T}}\rangle$ is the *a posteriori* correlation matrix η, $b = \{b_{ij}\}$ is the matrix of diffusion coefficients, $b_{11} = 2\alpha$, and $b_{ij} = 0$ for $(i, j) \neq (1,1)$. The initial conditions are prescribed at $\tau_0 = 2h_0/c$: $\boldsymbol{\eta}^*(\tau_0) = 0$, $K_{11}(\tau_0) = 1$, and $K_{ij}(\tau_0) = 0$ for $(i, j) \neq (1, 1)$.

The optimal processing consists in simultaneous solution of the system of equations (14)–(16) as the sampled data $y_i(\tau)$ become available with the use of the *a priori* profiles $\overline{T}(h)$, $\sigma_T(h)$, $\gamma_1(h;\, \overline{T},\, \overline{P})$, and so on, initial conditions, and appropriate finite–difference method. The solution of this system yields the optimal estimate η_1^* and hence the estimate of the profile $T(h)$

$$T^*(h) = \overline{T}(h)[\, 1 + \mu(h)\, \eta_1^*(\tau)] . \tag{17}$$

Analysis of the filtration efficiency. If in the calculations the estimate $\overset{\downarrow}{s}_1$ is replaced by its mean value or by the profile constructed on the basis of the atmospheric optical model, then Eq. (16) is independent of the chosen realizations $\boldsymbol{y}(\tau)$ and can be analyzed *a priori*. As a figure of performance of the filtration we consider the altitude dependence of the variance of temperature estimate. According to Eq. (17)

$$D[T^*(h)] = \mu^2(h)\, \overline{T}{}^2(h)\, D[\eta_1^*(\tau)] , \tag{18}$$

where $D[\eta_1^*(\tau)] = K_{11}(\tau)$ is the corresponding diagonal element of the matrix K satisfying Eq. (16). In its turn from Eq. (18) the relation for $K_{11}(h = c\tau/2)$ can be derived in the following form:

$$K_{11}(h) = D[T^*(h)]/D[T(h)] ,$$

since $D[T(h)] = \mu^2(h) \overline{T}{}^2(h)$. Thus, $K_{11}(h)$ represents the ratio of the *a posteriori* variance of the estimate $T^*(h)$ to the *a priori* variance of fluctuating temperature profile $T(h)$.

The dynamics of the filtration efficiency can be analyzed if for $K_{12}(h)$ we write down the approximation

$$K_{12}(h) \simeq \overline{\gamma}_1(h)\, LB(h)\, K_{11}(h) ,$$

which is valid for $L \ll h - h_0$. In this case the equation for $K_{11}(h)$ can be integrated independently of the other equations of system (16). As a result, analogously to Ref. 9 we have

$$\frac{\mathrm{d}K_{11}(h)}{\mathrm{d}h} = -\frac{2}{L} \left[K_{11}(h) - 1 + Q(h;\, \lambda,\, E_1'') K_{11}^2(h) \right] , \tag{19}$$

where

$$Q(h;\, \lambda,\, E_1'') = \frac{4 \overline{s}{}_1^2(h)\, \mu^2(h)}{N_1 \alpha} [\overline{\gamma}_1(h)\, L]^2 B^2(h) . \tag{20}$$

The quantity $Q(h;\, \lambda,\, E_1'')$, whose value determines the filtration efficiency, is referred to as the generalized signal–to–noise ratio, much as it was done in a number of our previous papers.[1–9]

However, in contrast to the case of sensing of T based on elastic and Raman scattering in which the generalized signal–to–noise ratio depends on the signal–to–noise ratio at λ_1 due to elastic scattering [the term $\overline{s}{}_1^2/(N_1\alpha)$] and on the variation coefficient $\mu(h)$, in this case it depends additionally on the optical thickness (the term $\overline{\gamma}_1(h)\, L$) of differential absorption in the layer $[h - L,\, h]$ and on the energy E_1'' of the lower level of the chosen transition [the term $B(h)$].

The filtration makes sense only at the altitudes where $K_{11}(h) \ll 1$, what is possible only for $Q \gg 1$ (see Refs. 1–9). The profiles $Q(h;\, \lambda,\, E_1'')$ at the transition $P^p\, 27$ of the O_2

absorption line centered at $\lambda_1 = 768.3802$ nm have been calculated for analysis of the altitude behaviour of $Q(h;\ \lambda,\ E_1'')$ in the regime of threshold signal sensitivity preset by the shot noise. An alexandrite or titanium–doped sapphire laser was used as the source of radiation. Opto–meteorological models were borrowed from McClatchey.[13] The lidar specifications[14] used in calculation were the following:

Wavelength, nm	768.3802
Pulse energy, mJ	100
Pulse repetition frequency, Hz	10
Pulse width, μs	0.66; 1.33
Diameter of the receiving aperture, m	0.5
Width of the lasing line, cm^{-1}	0.02
Quantum efficiency of the photodetector	0.24

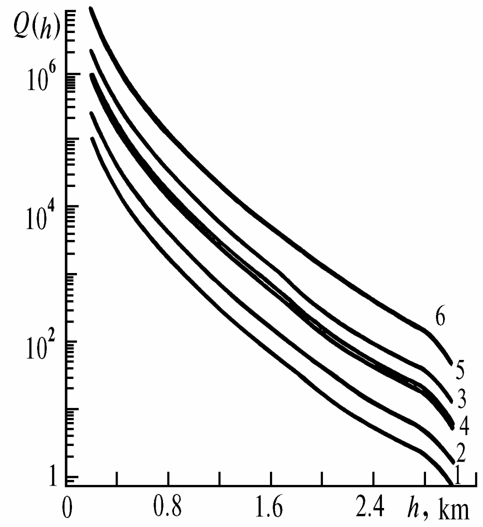

FIG. 1. The profiles of the generalized signal–to–noise ratio over the period $\Delta t_s = 1$ min *for* $L = 100$ m *and* $\mu = 0.33\cdot10^{-2}$ *(1),* $0.5\cdot10^{-2}$ *(2), and* 10^{-2} *(3); for* $L = 200$ m *and* $\mu = 0.33\cdot10^{-2}$ *(4),* $0.5\cdot10^{-2}$ *(5), and* 10^{-2} *(6).*

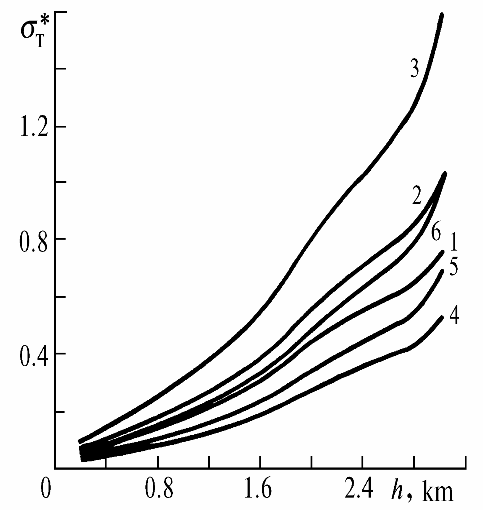

FIG. 2. The profiles of the errors in optimal reconstructing T(h) for the same parameters (see Fig. 1).

In Fig. 1 the profiles $Q(h)$ are shown for different pulse energies E_0, the spatial resolution L of the lidar, and the numbers $M = f_p\,\Delta t_s$ of sensing pulses, where f_p is the pulse repetition frequency and Δt_s is the sensing period. It can be seen from the figure that for the real lidar parameters the required values are obtained even at the altitudes up to 3 km.

By assuming $dK_{11}/dh = 0$ in Eq. (19), we write down the solution $K_{11}(h)$ of the quadratic equation for $\bar{K}_{11}(h)$ in the form[8,9]

$$\bar{K}_{11}(h) = \frac{1}{2Q(h)}\left\{\sqrt{1 + 4\,Q(h)} - 1\right\}. \tag{21}$$

The random error $\sigma_T^* = \sqrt{D(T^*)}$ in optimal estimating the temperature is

$$\sigma_T^* = \sigma_T(h)\sqrt{K_{11}(h)}\,, \tag{22}$$

and for this reason using the obtained values of $Q(h)$ the profiles $K_{11}(h)$ [according to Eq. (21)] and the errors σ_T^* in reconstructing $T^*(h)$ were determined. They are shown in Fig. 2. It can be seen from the figure that the optimization of processing with the help of the algorithm for the Calman–Bucy filtration will ensure the effective reconstruction of fluctuating temperature profiles disregarding the systematic errors of different kinds, which have been analyzed in Ref. 11.

Conclusion. The algorithm for the Calman–Bucy filtration is synthesized which enables us to optimize processing of signals of the differential absorption lidar operating in the current regime in sensing of the troposphere at the altitudes up to 3 km. It has been shown that the efficiency of filtration of spatial temperature realizations depends on the generalized signal–to–noise ratio (introduced in the paper) which accumulates all the factors determining the efficiency of implementation of the method.

The author would like to acknowledge Prof. G.N. Glasov for useful remarks and advice and N.A. Shefer for her assistance in some calculations.

REFERENCES

1. G.N. Glazov and G.M. Igonin, in: *Abstracts of Reports at the Twelfth International Laser Radar Conference*, Aix–En–Provance (1984), pp. 125–126.
2. G.N. Glazov, Gen.N. Glazov, and G.M. Igonin, Avtometriya, No. 4, 46–51 (1985).
3. V.E. Zuev, G.N. Glazov, and G.M. Igonin, in: *Abstracts of Reports at the Thirteenth International Laser Radar Conference*, Toronto (1986), pp. 117–118.
4. G.M. Igonin, in: *Opto–Meteorological Investigations of the Earth's Atmosphere* (Novosibirsk, Nauka, 1987), pp. 150–162.
5. G.N. Glazov and G.M. Igonin, ibid., pp. 163–172.
6. G.M. Igonin, ibid., pp. 172–178.
7. G.N. Glazov and G.M. Igonin, Issled. Zemli iz Kosmosa, No. 2, 51–56 (1987).
8. G.M. Igonin, "*Optimal filtration of signals in laser sensing of the atmosphere,*" Candidate's Dissertation in Physical and Mathematical Sciences, Institute of Atmospheric Optics, Siberian Branch of the Russian Academy of Sciences, Tomsk (1989), 227 pp.
9. G.N. Glazov and G.M. Igonin, Atm. Oceanic Opt. **5**, No. 6, 417–422 (1992).

10. J.K. Kalshoven, C.L. Korb, and G.K. Schwemmer, Appl. Opt. **20**, No. 11, 1967–1971 (1981).
11. B. Grossman, *Study of Water and Oxygen Molecules by the Method of Spectroscopy of Laser Radiation Absorption in Continuous and Pulsed Regimes. Application for Sensing of the Atmosphere* [Russian translation] (Vses. Tsentr Perevodov, Moscow, 1988), 194 pp.
12. V.I. Tikhonov, *Optimal Signal Reception* (Radio i svyaz', Moscow, 1983), 320 pp.
13. R.A. McClatchey, R.W. Fenn, J.E.A. Selby, et al., *Optical Properties of the Atmosphere*, Report AFCRL–71–0279, Belford (1971), 86 pp.
14. G.K. Schwemmer, M. Dombrovsky, and C.L. Korb, IGARSS'80, No. 1, 26–29 (1980).