A GAS OF NONINTERACTING PARTICLES AND FLUCTUATIONS OF WAVE INTENSITY BEHIND THE PHASE SCREEN (THE MODEL OF HEATED BEAMS)

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Following the analogy between the evolution of density in a heated gas of noninteracting particles and the diffraction of monochromatic optical waves upon their passage through a random–phase screen, an expression is developed to describe the correlation function of intensity fluctuations behind such a screen.

Applicability limits are found for that approach. The adequacy of the model is tested using the example of a dynamic phase screen.

The evolution of gas consisting of noninteracting particles, its density being ρ_0 at the initial time moment $t = 0$, is described by the Liouville equation[1] for density f in the phase space (x, υ):

$$\frac{\partial f}{\partial t} + \upsilon \frac{\partial f}{\partial x} = 0, \quad f(x, \upsilon, t = 0) = \rho_0 f_0(x, \upsilon) . \tag{1}$$

The density of the gas at the point x is expressed in terms of the solution of Eq. (1)

$$\rho(x, t) = \int\limits_{-\infty}^{\infty} f(x, \upsilon, t)\, d\upsilon = r_0 \int\limits_{-\infty}^{\infty} f(x, -\upsilon t, \upsilon,)\, d\upsilon .$$

Within the hydrodynamic approximation this solution yields infinite mean square density and variance of density of the particle flux in the vicinities of caustics. In a real gas the field of density is limited due to thermal variability of particle velocities. To account for this variability we prescribe the initial density in the phase space in the form $f_T(\upsilon - \upsilon_0(x))$ where the initial field of velocities $\upsilon_0(x)$ is a random function with known[1] statisitcal properties. In this case the density of such a gas is given by the formula

$$\rho_T(x, t) = \int\limits_{-\infty}^{\infty} f_T(c)\, \rho(x - ct, t)\, dc .$$

One can see from this relation that the thermal variability of velocities results in smoothing the caustic singularities in the densities.

If the initial field of velocities is statistically homogeneous, the correlation function of density fluctuations can be written using the solution of Eq. (1) in the form[1]

$$K_\rho^T(s, t) = \frac{\rho_0^2}{t} \int\limits_{-\infty}^{\infty} f_T\left(c_T - \frac{c}{2}\right) f_T\left(c_T + \frac{c}{2}\right) \times$$

$$\times\, \omega_u\left(\frac{s - s_0}{t} - c;\, s_0\right) ds_0\, dc_T\, dc ,$$

where $\omega_u(u, s_0)$ is the probability distribution of the field of differences between the particle velocities $u(s_0) = \upsilon_0(y + s_0) - \upsilon_0(y)$.

By choosing

$$f_T(c) = \frac{1}{\sqrt{2\pi c_T}} \exp(-c^2/2c_T^2) , \tag{2}$$

for the Gaussian field $\upsilon_0(x)$, with its correlation coefficient $b_0(y)$ and variance σ_0^2, we obtain:

$$K_q^T(y, z) = \frac{\rho_0^2}{\sqrt{\pi}\, z} \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{1 - b_0(y_0) + \varepsilon}} \times$$

$$\times \exp\left[-\frac{(y - y_0)^2}{z^2(1 - b_0(y_0) + \varepsilon)} \right] dy_0 . \tag{3}$$

Here z is the dimensionless longitudinal coordinate, introduced so that $z = 1$ within the range of focusing. In this case $z \ll 1$ and $z \gg 1$ are the ranges of single flux and multi-flux propagation, respectively. The parameter $\varepsilon = c_T^2/s_0^2$ is the gas temperature. The dependence of the coefficient of covariance

$$R_\rho^T(y, z) = \left(K_\rho^T(y, z) - \rho_0^2\right)/\sigma_\rho^2$$

on the dimensionless transverse coordinate y at various z and ε values is shown in Fig. 1. As can be seen from the values of σ_ρ^2 shown in this figure the variance of fluctuations of density increases with decreasing temprature of the gas.

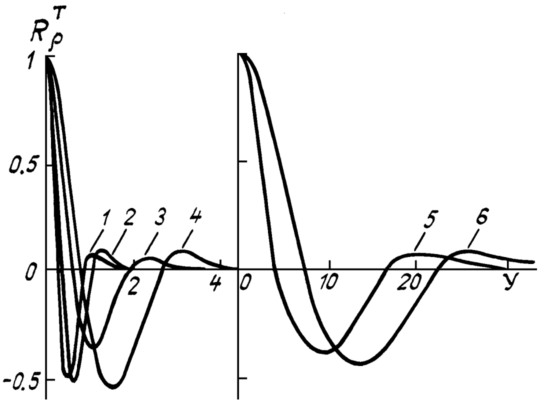

FIG. 1. Covariance coefficient $R_\rho^T(y, z)$. Parameters of the curves: 1) $z = 0.1$, $\varepsilon = 0.1$, $\sigma_\rho^2 = 0.07388$, 2) $z = 0.1$, $\varepsilon = 1$, $\sigma_\rho^2 = 0.06119$, 3) $z = 1$, $\varepsilon = 0.1$, $\sigma_\rho^2 = 0.28388$, 4) $z = 1$, $\varepsilon = 1$, $\sigma_\rho^2 = 0.0444$, 5) $z = 10$, $\varepsilon = 0.1$, $\sigma_\rho^2 = 0.02398$, and 6) $z = 10$, $\varepsilon = 1$, $\sigma_\rho^2 = 0.00279$

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An invariant follows from Eq. (3) (see Ref. 1)

$$\int_{-\infty}^{\infty}[K_{\rho}^{\mathrm{T}}(y, z) - \rho_0^2]\,\mathrm{d}y = 0\ ,$$

which denotes that condensing of the medium ($\rho > \rho_0$) is accompanied by its rarefaction in the adjacent ranges ($\rho < \rho_0$).

The suggested approach appears to be particularly convenient for solving the problem of wave propagation behind the phase screen. It is well known[2] that the traditional way of solving this problem meets with difficulties since the resulting integrals are hard to compute. At the same time the analogy[1] between the evolution of density of a heated gas of noninteracting particles, and the diffraction of monochromatic optical waves, passed through a random phase screen, shows a way of relatively easy arriving at statistical characteristics of the wave benind such a screen. The screen introduces phase distortions $k\psi(x)$ into the propagating wave. To find the analogy we are searching for, one has to proceed from the equation of quasioptics for the complex amplitude $E(x, t)$ to the equation for the function

$$f(x, \upsilon, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E\left(x + \frac{z}{2k}\ ,\ t\right) E^*\left(x - \frac{z}{2k}\ ,\ t\right) \times$$

$$\times \exp\,(-i\upsilon z)\,\mathrm{d}z\ , \qquad (4)$$

which may be interpreted as the beam density of a wave, arriving at the point (x, t) at the angle υ with respect to the t axis. This function statisfies Eq. (1) with the same boundary condition if the initial density ρ_0 is substituted by the initial intensity I_0.

By solving Eq. (1) for a phase screen, and neglecting "thermal variability" one obtains the expression analogous the geometric optics (GO) approximation. The optical analogue of thermal scatter of particle velocities is the phenomenon of diffraction smoothing of caustic singularities in the wave field. To account for the diffraction the initial angle of the beam arrival $\upsilon_0(x) = \psi'(x)$ is assumed to be a random function with known statistical properties, the initial beam density being equal to $f_{\mathrm{T}}(\upsilon - \upsilon_0(x))$.

To construct the optical analogue of the phase density let us isolate two parts in the phase factor (4), one of them determining the GO–propagation, and the other one being responsible for the diffraction blooming. Semi–qualitative account for the effect of diffraction, wich limits fluctuations of the intensity, allows one to obtain the boundary condition

$$f_{\mathrm{T}}(x, \upsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(z, k)\exp\,[\,-i(\upsilon - \upsilon_0(x))\,z]\,\mathrm{d}z\ ,$$

where the random diffraction factor is substituted by its average value

$$g(z, k) = \left\langle \exp\left\{ ik \int_{x-z/2k}^{x+z/2k} [\upsilon_0(y) - \upsilon_0(x)]\,\mathrm{d}y \right\}\right\rangle .$$

If we have a statistically homogeneous Gaussian field $\upsilon_0(x)$ the beam density $f_{\mathrm{T}}(c)$ has the form (2), so that Eq. (3) holds for the corelation function of intensity fluctuations, provided that ρ_0 is substituted by I_0. The parameter ε now determines the ratio of the width of the diffraction blooming to the rams initial angle of the beam arrival. The lower is ε, more accurate is the GO–approximation.

Let us now find the applicability limits of the model of a heated beam from the condition that thermal blooming does not leads to overlapping of the beam tubes:

$$c_{\mathrm{T}} \ll \sigma_0/N\ . \qquad (5)$$

The value in the right side of inequality (5) is the scale of a single beam tube, when the total number of such tubes is N. Assuming the size of the spatial inhomogeneity to be l_0, we obtain $N_0 \sim \sigma_0 t/l_0$ (see Ref. 1), while it follows from the definition of the diffractional scale that $c_{\mathrm{T}} \sim (\sigma_0/(kl_0)^2)^{1/3}$. Substitution of this value into Eq. (5) yields:

$$z \ll \varepsilon^{-1/2}\ . \qquad (6)$$

Adequacy of the proposed model was tested using the dynamic sine phase screen. This screen transforms the wave phase as follows:

$$k\psi(x) = k\psi_0\sin(\mathrm{k}x + \varphi_0)\ ,$$

where the constant initial phase shift φ_0 is homogeneously distributed over the range $[\,-\pi, \pi]$. Expanding the phase into a series and taking only its first term, which describes the diffraction blooming, and averaging over φ_0 we find the intensity within the proposed model of heated beam as

$$I_{\mathrm{T}}(y, z) = \frac{1}{\sqrt{\pi\varepsilon}\ z}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{\varepsilon}\left[\frac{y_0}{z} - \cos(y - y_0)\right]^2\right\}\mathrm{d}y_0\ ,$$

and also we obtain the exact GO–solution:

$$I_{\mathrm{GO}}(y, z) = \frac{1}{z}\sum_{i}\frac{1}{|\sin(y - y_{0\,i}) - 1/z|}\ ,$$

where the summation is conducted over all roots of the equation

$$\cos(y - y_0) - y_0/z = 0\ .$$

On the other hand, our model of the screen makes it possible to find the exact solution of the equation of quasioptics (QO)

$$I_{\mathrm{QO}}(y, z) = |\,-iJ_0(\alpha) - i\sum_{n=1}^{\infty}\exp(-\,in^2z/2\alpha)\ \times$$

$$\times J_n(\alpha)[\exp(iny) + (-1)^n\exp(-iny)]\,|^2,\quad \alpha = (3x^*)^{-1/2}\varepsilon^{-3/4}\ ,$$

where x^* is the first zero of the Bessel function J_0.

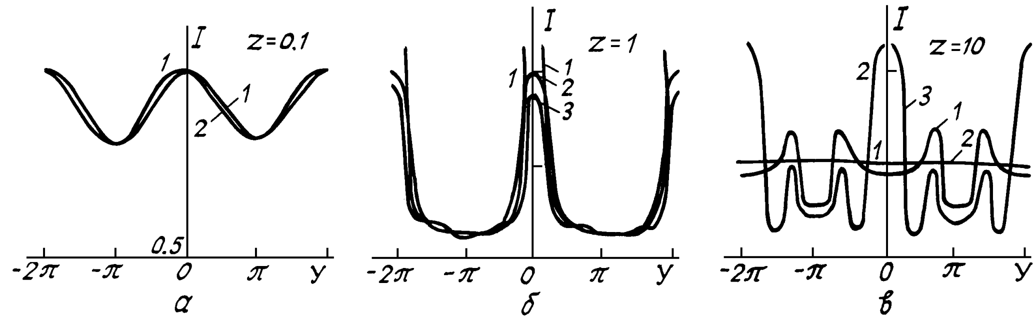

FIG. 2. Average wave intensity behind the dynamic sine phase screen: a) $z = 0.1$, I_{GO} *(1) and* I_T *(2) at* $\varepsilon = 0.1$; *b)* $z = 1$, I_{GO} *(1) and* I_T *(2) at* $\varepsilon = 0.1$; *and c)* $z = 10$, I_{GO} *(1),* I_T *(2), and* I_{QO} *(3) at* $\varepsilon = 0.1$.

It can be seen from Fig. 2. in which the function $I(y)$ is shown as a function of the parameter z, that in the case of single beam propagation the solution obtained within the model of heated beams completely coincides with the GO–solution, provided that ε is not too large. In the range of focusing the diffraction smoothing of caustic singularities is adequately accounted for, what can easily be seen from the comparison of I_{T} with I_{QO}, while outside the caustics the solution agrees with I_{GO}. However within the range of multi-beam propagation ($z \gg 1$) our model fails, as it follows from condition (6), since it yields a constant solution (with exception for the case of $\varepsilon \to 0$, in which it transforms into the GO–solution), while the exact solution in this range is a statistically homogeneous random Gaussian field.[2]

REFERENCES

1. C.N. Gurbatov, A.N. Malakhov, and A.I. Saichev, *Nonlinear Random Waves in Dispersion–Free Media* (Moscow, Nauka, 1990).
2. Yu.A. Kravtsov, S.M. Rytov, and V.I. Tatarskii, Usp. Fiz. Nauk **115**, No. 2, 239–262 (1975).