PARAMETRIZATION OF THE BACKSCATTERING PHASE MATRIX OF NONRECIPROCAL MEDIA

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The problems are considered of optimal parametrization of asymmetrical complex operators from the space of congruent transformations. An optimal group of the backscattering phase matrix parameters is proposed which is sufficient and nonredundant characteristic of the electromagnetic wave backscattering properties of an arbitrary medium and has strict physical interpretation. It is shown that nonreciprocal properties of the medium manifest through its backscattering phase matrix invariant characteristics that cause its asymmetry.

The experimental data available now make it possible to say that in the case in which magnetic or electric field excited by external sources is present in the region of effective backscattering, the backscattering phase matrix (BPM) is asymmetrical.[1] This fact raises a problem of the study of asymmetrical complex operators in the space of congruent transformations. This is the type of transformations that describes the BPM representations in various polarization bases. For the symmetrical matrices (describing reciprocal media), this transformation makes it possible to define their canonical (diagonal) form and to introduce sufficient and nonredundant group of the parameters being strictly interpreted that characterize the "internal" scattering properties of the media described by these operators. Let us briefly note the physical meaning of these parameters: ε_0 is the ellipticity angle of the BPM eigenbasis, θ_0 is the orientation angle of the BPM eigenbasis with respect to a laboratory coordinate system, and $\dot{\lambda}_1$ and $\dot{\lambda}_2$ are the BPM eigenvalues.

Ellipticity angle ε_0 and orientation angle θ_0 specify the ellipticity and orientation of the major axis of the polarization ellipse of electromagnetic wave incident on the medium under investigation (described by BPM), for which the power of a reflected signal at the exit from a reciprocal single–channel analyzer–shaper goes to its extreme values. Then the reflection coefficients of two orthogonal waves with the ellipticity angles ε_0 and $-\varepsilon_0$, and orientation angles θ_0 and $\theta_0 \pm \pi/2$, respectively, are proportional to the BPM eigenvalues λ_1 and λ_2 of the investigated medium. These eigenvalues specify the extreme values of the reflectivity in the single–channel single–point method.

The single–channel single–point method is described by the expression:

$$\dot{U}_p(t) = \dot{U}_0(t)\ \tilde{\mathbf{h}}\ S\ \mathbf{h}\ , \tag{1}$$

where the tilde denotes transposition; $\dot{U}_p(t)$ is the observed reflected scalar signal, S is the medium BPM, $\dot{U}_0(t)$ is the scalar signal exciting a field within the aperture of the analyzer–shaper of a measurement system, and

$$\mathbf{h} = L \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2}$$

is the vector describing the radiation field of the single–channel reciprocal analyzer–shaper excited by the scalar signal $\dot{U}_0(t)$. The operator L in Eq. (2) describes the polarization properties of the field analyzer–shaper and belongs to the Jones vector rotation group in the space of its stereographic projection to the Poincare sphere. It can be represented in the multiplicative form as

$$L = R_\theta\ F_\varepsilon = \begin{pmatrix} \cos\theta\ ; & -\sin\theta \\ \sin\theta\ ; & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\varepsilon\ ; & j\sin\varepsilon \\ j\sin\varepsilon\ ; & \cos\varepsilon \end{pmatrix} \tag{3}$$

and can be parametrized by two independent parameters ε and θ specifying the ellipticity and orientation of the vector $\mathbf{h}$ in Eq. (2). For some $\varepsilon = \pm\varepsilon_0$ and $\theta = \theta_0 \pm \pi/2$, the observed signal $\dot{U}_p(t)$ in Eq. (1) goes to its extreme value (in power) proportional to the BPM eigenvalues $\dot{\lambda}_1$ and $\dot{\lambda}_2$. Using Eqs. (1) and (2), we can write

$$|\dot{U}_p(t)|^2_{\max} = \left| \dot{U}_0(t)\ (1;\ 0)\ \tilde{L}_0\ S\ L_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2 \tag{4}$$

with

$$S_0 = \tilde{L}_0\ S\ L_0 = \begin{pmatrix} \dot{\lambda}_1 & 0 \\ 0 & \dot{\lambda}_2 \end{pmatrix}, \tag{5}$$

where S_0 is the representation of the symmetrical BPM in the polarization eigenbasis. Detailed description and proof of the aforementioned statements were given in Refs. 2 and 3.

Reduction of the BPM to the diagonal form by means of the congruent transformation given by Eq. (5) is impossible for the asymmetrical BPM (nonreciprocal media). The question of optimal parametrization of these media (or their BPM) remains open (I failed to find papers in which such parametrization has been developed).

Let us consider the general form of representation of the Cartesian asymmetrical BPM in various polarization bases. Let us assume that in general the Cartesian BPM of an arbitrary medium is specified by the four complex numbers

$$S_g = \begin{pmatrix} \dot S_{11} & \dot S_{12} \\ \dot S_{21} & \dot S_{22} \end{pmatrix}, \tag{6}$$

and $\dot S_{21} \neq \dot S_{12}$ (consequence of nonreciprocal properties of the media). Let us expand the operator S_g in a system of the orthogonal Pauli matrices completed by the unit matrix

$$\delta_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \delta_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \delta_3 = j \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{7}$$

Then the operator S_g takes the form

$$S_g = A_0\,\delta_0 + A_1\,\delta_1 + A_2\,\delta_2 + A_3\,\delta_3 = \sum_{i=0}^{3} A_i\,\delta_i\,, \tag{8}$$

with the expansion coefficients

$$A_i = 0.5\ \mathrm{Sp}\,\{S_g\,\delta_i\}\,, \tag{9}$$

where Sp is the spur of the operator enclosed in the brackets in Eq. (9).

Using Eqs. (6) and (7) in Eq. (8), we write down

$$S_g = 0.5\,\{(\dot S_{11} + \dot S_{22})\,\delta_0 + (\dot S_{11} - \dot S_{22})\,\delta_1 + $$
$$+ (\dot S_{12} + \dot S_{21})\,\delta_2 + (\dot S_{12} - \dot S_{21})\,\delta_3\}\,. \tag{10}$$

It is obvious that the first three terms of Eq. (10) form the symmetrical operator

$$S_g^s = \begin{pmatrix} \dot S_{11} & 0.5(\dot S_{12} + \dot S_{21}) \\ 0.5(\dot S_{12} + \dot S_{21}) & \dot S_{22} \end{pmatrix},$$

$$S_g = S_g^s + \dot\Delta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{11}$$

where $\dot\Delta = 0.5\,(\dot S_{12} - \dot S_{21})$, and the fourth term is antisymmetrical operator with the weighting coefficient $\dot\Delta$. Let us find the representation of Cartesian operator S_g in arbitrary basis with the parameters ε and θ:

$$S_\varepsilon = \tilde L\, S_g\, L = \tilde F_\varepsilon\, \tilde R_\theta\, S_g\, R_\theta\, F_\varepsilon\,. \tag{12}$$

Using the expressions for the operators R_θ and F_ε (see Eq. (3)), we obtain

$$S_\varepsilon = \tilde L\, S_g\, L = \tilde L \left\{ S_g^s + \dot\Delta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} L =$$
$$= \tilde L\, S_g^s L + \dot\Delta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{13}$$

since the relation

$$\tilde L \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \to \mathrm{inv}\,, \tag{14}$$

is valid for any L, i.e., the operator δ_3 in Eq. (8) is invariant under rotation (transformation) specified by the operator L and remains unchanged in any basis of the representation. We can draw a very important conclusion from Eq. (13): the difference between nondiagonal elements of BPM is invariant for the basis parameters, is caused only by nonreciprocal properties of the medium, and is its objective characteristic.

Since the first term in Eq. (13) describes the unitary congruent transformation of the symmetrical operator S_g^s, for some $\varepsilon = \varepsilon_0$ and $\theta = \theta_0$, the relation

$$\tilde L_0\, S_g^s\, L_0 = \begin{pmatrix} \dot\lambda_1 & 0 \\ 0 & \dot\lambda_2 \end{pmatrix} \tag{15}$$

is valid and hence the asymmetrical Cartesian operator S_g takes the form

$$S_0 = \tilde L_0\, S_g\, L_0 = \begin{pmatrix} \dot\lambda_1 & 0 \\ 0 & \dot\lambda_2 \end{pmatrix} + \dot\Delta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{16}$$

in the basis with the parameters ε_0 and θ_0 (see Eq. (6)).

Representation (16) makes it possible to introduce the following parameters of asymmetrical BPM: ε_0 and θ_0, the parameters of the eigenbasis of the "symmetrical" part of BPM; $\dot\lambda_1$ and $\dot\lambda_2$, the eigenvalues of the "symmetrical" part of BPM; $\xi = \dfrac{\sqrt{2}\,\dot\Delta}{\|S_g\|}$, the complex coefficient of the nonreciprocity of the medium, where $\|\ \|$ denotes the Euclidean norm.

Let us determine the signal $\dot U_p(t)$ observed in the single–channel single–point system on irradiation of the nonreciprocal medium described by the operator S_g (see Eq. (11). After replacing of the operator S in Eq. (1) by the operator S_g, we obtain

$$\dot U_p(t) = \dot U_0(t)\ \tilde{\mathbf h}\ S_g\ \mathbf h \tag{17}$$

and, using Eq. (2) for **h**, we obtain

$$\dot U_p(t) = \dot U_0(t)\ (1;\,0)\ \tilde L \left\{ S_g^s + \dot\Delta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} L \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$
$$= \dot U_0(t)\ (1;\,0)\ \tilde L\ S_g^s\ L \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{18}$$

since the relation

$$(1;\,0)\ \tilde L \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} L \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \tag{19}$$

is valid for any L.

Thus, signal (18) observed in the single–channel system in the case of the nonreciprocal medium depends only on the "symmetrical" part of BPM and goes to its extreme (in power) values when the medium is irradiated by the eigenstates of polarization of its "symmetrical" part, i.e., by the field described by the vector

$$\mathbf h_0 = L_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = R_{\theta_0}\, F_{\varepsilon_0} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{20}$$

where ε_0 and θ_0 are the parameters of the eigenbasis of the "symmetrical" part of BPM. It is the reason why the parameters ε_0, θ_0, λ_1, and λ_2 specified for asymmetrical operator (11) have practically the same meaning that the corresponding operators of the symmetrical BPM considered above.

Physical meaning of the nonreciprocity parameter ξ of the medium is the following. Obviously, the square norm of the operator S_g in Eq. (11) is equal to the sum of the square norms of its symmetrical and asymmetrical parts, because

$$\|S_g\|^2 = \sum_{\substack{i=1\\j=1}}^{2} |\dot{S}_{ij}|^2 = |\dot{S}_{11}|^2 + |\dot{S}_{22}|^2 + |\dot{S}_{12}|^2 + |\dot{S}_{21}|^2 =$$

$$= |\dot{S}_{11}|^2 + |\dot{S}_{22}|^2 + 0.5|\dot{S}_{12} + \dot{S}_{21}|^2 + 0.5|(\dot{S}_{12} - \dot{S}_{21})|^2, \quad (21)$$

and since the first three terms in the right–hand side of Eq. (21) specify the square norm of the operator S_g^s (see Eq. (11)) and the fourth term is the square norm of the antisymmetrical operator in the right–hand side of Eq. (11), we can write

$$\|S_g\|^2 = \|S_g^s\|^2 + 2\,|\dot{\Delta}|^2\,, \quad (22)$$

where $\dot{\Delta} = 0.5\,(\dot{S}_{12} - \dot{S}_{21})$. It follows from Eq. (22) that the ratio of the BPM square norm to its asymmetrical part is equal to

$$\frac{2\,|\dot{\Delta}|^2}{\|S_g\|^2} = |\xi|^2\,, \quad (23)$$

and hence the absolute value of the nonreciprocity coefficient of the medium carries the information about the relation between the reciprocal and nonreciprocal parts of the total effective scattering cross section. Obviously, the nonreciprocity coefficient is equal to zero for all reciprocal media, and the absolute value of ξ is within the interval (0, 1) for an arbitrary medium. The argument of the coefficient ξ determines the difference between the absolute phases of the operators of symmetrical and asymmetrical parts of BPM and, evidently, is indicative of the separation of its reciprocal and nonreciprocal parts in space, like in the case in which the phase difference between the eigenvalues $\dot{\lambda}_1$ and $\dot{\lambda}_2$ determines the separation of the model elements (for example, vibrators in the two–vibrator model of scatterer[2]) of a medium along the line of sight.

Let us note in conclusion that the proposed parametrization of BPM of an arbitrary medium is the optimal description of its "internal" scattering properties. The optimum is ensured by sufficient and nonredundant nature of these parameters as well as by the strict physical interpretation of each of them.

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