ON THE ACCURACY OF SOLUTIONS OF THE SCALAR TRANSFER EQUATION

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The question of the accuracy and the domain of applicability of the solutions of the transfer equation for the intensity of the scattered radiation which do not take account of the polarization state of the radiation is considered. Previously obtained numerical estimates of the differences of the solutions of the scalar and vector transfer equations are generalized and analyzed in the case of optically isotropic media and sources of wide unpolarized beams. A number of problems are clarified which pertain to sources of linearly polarized beams, where the question may be posed of using the scalar transfer equation instead of the vector equation.

Beginning with the earliest works, the question has been raised in the theory of the multiple scattering of radiation of determining the domain of applicability and the accuracy of the solutions of the scalar transfer equation (TE) for the intensity which do not take the polarization state of the radiation into account. Earlier investigations addressed primarily the classical problems of the theory of radiation transfer in atmospheres and are bound up with a consideration of natural sources of unpolarized light.[1–18] In these studies it was shown via numerical calculations that from the point of view of determining the energetic characteristics of the light fields, the solutions of the scalar TE differ little from the more accurate solutions of the vector TE (for the Stokes parameters), which take the polarization into account. This observation served as the basis for the extensive use of the scalar transfer theory (as an approximation) for the solution of a wide range of geophysical, astrophysical, and a number of other problems. At present the question of the accuracy of the scalar TE solutions is still of current interest due to the continuing development of the vector transfer theory, e.g., in application to laser sources.

The present work is devoted to general aspects of the problem. The results of previous investigations of the accuracy of the solutions of the scalar TE are generalized and analyzed; the domain of applicability of the scalar TE in the theory of polarized radiation transfer is determined. The treatment is based on the Stokes parameters and Green's matrix formalism.

First recall that in the vector transfer theory the response of the scattering medium is described by the 4x4 Green's matrix of the (here assumed to be stationary) problem:

$$\vec{S}(\vec{r},\vec{n}) = \hat{G}(\vec{r},\vec{n})\vec{S}_{0\vec{r}_0\vec{n}_0} \qquad (1)$$

where $\vec{S}_0$ and $\vec{S}$ are the Stokes vector parameters of the incident and scattered radiation, and the vectors $\vec{r}(\vec{r}_0)$ and $\vec{n}(\vec{n}_0)$ define the point and the direction of observation (illumination).

The matrix $\hat{G}(\vec{r},\vec{n})$ satisfies the equation

$$\vec{n}\vec{\nabla}_{\vec{r}}\hat{G}(\vec{r},\vec{n}) = \hat{\varepsilon}(\vec{n})\hat{G}(\vec{r},\vec{n}) - \frac{1}{4\pi}\iint_{(4\pi)}\hat{L}(\pi-\chi)\hat{\sigma}(\vec{n},\vec{n}')\hat{L}(-\chi')\times$$

$$\times\ \hat{G}(\vec{r},\vec{n}')d\Omega' = 0,$$

where the boundary conditions for G_{ik} are defined by the problem. Here $\hat{\nabla}_r$ is the gradient with respect to $\vec{r}$; $\hat{\varepsilon}(\vec{n})$ and $\hat{\sigma}(\vec{n},\vec{n}')$ are the extinction and single-scattering matrices of the medium (in this case, a homogeneous one); χ and χ' are the angles between the scattering plane $\vec{n}\times\vec{n}'$ and the reference planes for $\hat{G}$ in the directions n and n',[1,9] and the matrix $L(-\chi')$ has the following nonvanishing elements: $L_{11} = L_{44} = 1$, $L_{22} = L_{23} = \cos\chi'$, and $L_{23} = -L_{32} = \sin 2\chi'$.

Now let us turn from Eq. (2) to the scalar TE. In the case of illumination of the medium by natural light, the radiation intensity is determined by the first element G_{11} of Green's matrix: $S_1 = G_{11}I_0(1)$. Let us represent the first equation of system (2) in the form:

$$B^{sc}\{G_{11}\} - B(G_{11}) = 0, \quad i=2,3,4. \qquad (3)$$

Equation (3) differs from the scalar TE

$$B^{sc}\{G\} = \vec{n}\vec{\nabla}_{\varepsilon\vec{r}}G(\vec{r},\vec{n}) + G(\vec{r},\vec{n}) - \frac{\Lambda}{4\pi}\iint_{(4\pi)}a_1(\vec{n},\vec{n}')\times$$

$$\times\ G(\vec{r},\vec{n}')d\Omega' = 0 \qquad (4)$$

($\varepsilon = \varepsilon_{11}$, $\Lambda a_1(\vec{n}, \vec{n}') = (1/\varepsilon_{11})\sigma_{11}(\vec{n}, \vec{n}')$, G is the solution of Eq. (4)) by the presence of the source function $B(G_{11})$ ($i \neq 1$). Suppose that the scattering medium is optically isotropic, i.e.,

$$\varepsilon_{1k}(\vec{n}) = \varepsilon\delta_{1k},\ \hat{\sigma}(\vec{n}, n') = \sigma\hat{D}(x),\ x = \cos\xi = (\vec{n}\cdot\vec{n}');$$

$$\hat{D} = \begin{bmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & -b_2 \\ 0 & 0 & b_2 & a_4 \end{bmatrix},\quad \frac{1}{2}\int_{-1}^{1} a_1(x)dx = 1; \tag{5}$$

$$b_j(\xi = 0) = 0,\ |b_j(\xi \ll 1)| \sim \xi^2 a_1(\xi) \ll 1,\ j = 1, 2. \tag{6}$$

Then from Eq. (2) we obtain for the source function

$$B(G_{21}, G_{31}) = \frac{\Lambda}{4\pi}\iint_{(4\pi)} b_1(\vec{n}\cdot\vec{n}')[\cos 2\chi\, G_{21}(\vec{r}, \vec{n}') - \sin 2\chi \times G_{31}(\vec{r}, \vec{n}')]d\Omega'. \tag{7}$$

Let us note the physical meaning of $B(G_{21}, G_{31})$ (7). As is well known, during the scattering process the natural light from the external source becomes partially polarized (in Eqs. (2) and (5) $G_{11} \neq 0$ ($i = 2, 3, 4$) since $\sigma_{21} \neq 0$). The linear polarization characteristics G_{11}($i = 2, 3$) which arise in the medium influence the intensity of each succeeding scattering event, beginning with the second, ($\sigma_{12} \neq 0$ (5)). The source function $B(G_{21}, G_{31})$ (7), (3) reflects these phenomena — it describes the sources of the "polarization distortions" of the intensity.

It follows from what has been said that the intensity approximation in scattering theory that consists in using the solutions of the scalar TE instead of the vector one corresponds to the analogous approximation of the first element G_{11} of Green's matrix. Let us call it for short the "scalar" approximation (SA). The solution $G(\vec{r}, \vec{n})$ of the scalar TE, together with those characteristics that appear in the TE (4) — the extinction and scattering coefficients ε and σ, the photon survival probability $\Lambda = \sigma/\varepsilon$, and the scattering phase function $a_1(\vec{n}\cdot\vec{n}')$ — describe the properties of the scattering of unpolarized radiation. The approximate character of the solution of the scalar TE is a result of the fact that it does not take into account the influence of the polarization of the scattered radiation on the intensity ($G_{11}(\vec{r}, \vec{n}) \neq G(\vec{r}, \vec{n})$, because, strictly speaking, $B(G_1) \neq 0$ in Eq. (3)); at the same time, as estimates have shown (see below), for a large number of problems of practical importance, the difference $(G_{11} - G)/G$ is small. Thus, $G_{11}(\vec{r}, \vec{n}) \approx G(\vec{r}, \vec{n})$.

Clearly, the SA as an approximation to G_{11} can be used to determine those energetic characteristics of the radiation that are described by the matrix element G_{11}. For illumination of a medium by circularly polarized light ($S_0 = I_0(1, 0, 0, S_{04})$ the total intensity S_1, as in the case $S_{04} = 0$, is such a characteristic. Indeed, in Eq. (5) $\varepsilon_{14} = 0$ and $\sigma_{14} = 0$, and Eq. (2) gives practically vanishing solutions for G_{14}. Assuming in Eq. (1) that $G_{14} = 0$, we have $S_{11} \approx G_{11}I_0$, i.e., circular polarization does not affect the intensity. In the case of sources of linearly or elliptically polarized radiation ($S_0 = I_0(1, S_{02}, S_{03}, S_{04})$) SA should be thought of as applying to the first term ($\sim G_{11}$) in the expression for $S_1(1)$. Introducing $\Delta G = G_{11} - G$ and substituting $G_{11} = G + \Delta G$ in Eq. (1), we can write

$$S_1 = GI_0 + \Delta GI_0 + \Delta S_1,\ \Delta S_1 = (G_{12}S_{02} + G_{13}S_{03})I_0. \tag{8}$$

Here in SA $\Delta G \approx 0$. As to the term ΔS_1 it is necessary, as a rule, to take it into account. Only in individual cases, to be considered below, can the contributions of ΔS_1 to S_1 be discounted. Thus, it can be seen that for light scattering in optically isotropic media the contributions to the intensity are governed only by the linear polarization characteristics: G_{11} (7) and S_{01} (8) ($i = 2, 3$).

Let us now analyze the data on the SA errors available in the literature. They were obtained for media of type (5) for the case of illumination by natural light. Only infinitely extended sources and media were considered. Error estimates for these various cases are summarized in Table I. As can be seen from Table I, the SA error is small (we will denote it by δ). In addition, for media with a Rayleigh scattering law the error reaches values of the order of 10%, but for all other cases (larger-scale dispersed media) it is significantly less: $|\delta| \leq 1$–3%. Let us stop at this point and discuss the causes of the smallness of $|\delta|$.

The extent to which the polarization of the radiation influences the intensity in the scattering process depends on the conditions of irradiation of the scatterer. Note that the maximum ("one hundred per cent") effect takes place in the case of scattering by a dipole at an angle of $\pi/2$ if the incident beam is completely polarized and unidirectional. If the medium is illuminated by unpolarized light, then the first correction to the intensity comes from the polarization of the singly scattered light, which is equal to $P^{(1)} = (b_1/a_1)$ (5). In addition, the given polarization is, first of all, not total: $|P^{(1)}| < 1$ ($|P^{(1)}| = 1$ only in isolated scattering directions), and $|P^{(1)}| \ll 1$ (6) is always the case inside the regions $\xi \ll 1$ and $\pi - \xi \ll 1$; second, the scattered radiation distribution is more or less diffuse. As a result of these two properties in combination with the vector nature of the polarization $\vec{P}^{(1)}$, its influence on the intensity of double scattering turns out to be small. Thus, the error $|\delta|^{(2)} = |\Delta G/G_{11}|^{(2)}$ for $G_{11}^{(2)}$ (the double scattering contribution) is not large, and the magnitude of the error $|\delta|$ for G_{11} that takes both of the first two multiplicities of scattering into account is even less:

$$|\delta| = \left|\frac{\Delta G^{(2)}}{G_{11}^{(1)} + G_{11}^{(2)}}\right| < |\sigma|^{(2)}.$$

Here account has been taken of the fact that $\Delta G^{(1)} = 0$. The SA error also remains small under E conditions of multiple scattering. Its accumulation from one scattering event to the next is "hindered" by the scattering properties themselves, namely, the quasidiffuseness of the radiation distribution and the conservation of zero polarization during forward scattering.

Let us consider the dependence of the error on the scattering characteristics of the medium. As follows from Eq. (7) regarding the property $G_{11} = G_{11}(b_{11})$ (i = 2, 3), the function $B(G_{21}, G_{31})$ is characterized by a quadratic dependence on the matrix element $b_1(x)$ (5) and, hence, on $P^{(1)}$. The values of $|\delta|$ for various media differ because they depend on the properties of the polarization I and intensity distributions attendant to each scattering event. The largest values of $|\delta|$ are typical of media with the Rayleigh scattering law:

$$a_1(x) = \frac{3}{4}(1+x^2), \quad P^{(1)}(x) = \frac{1-x^2}{1+x^2}.$$

In this case a substantial part of the radiation is strongly polarized during each scattering event. For example, the range of angles $\xi \in [70°, 110°]$, where $P^{(1)} > 0.8$, contains 53% of the radiant energy.

The SA error for "non-Rayleigh" media is considerably smaller. This is caused by several factors, first of all, by the fact that the radiant energy is maximal inside the forward scattering region, where $P^{(1)}$ is small (relation (6)) for all optically isotropic media. For those media with scattering phase functions that are strongly elongated in the forward direction (haze, clouds, seawater, etc.) the error $|\delta|$ is a negligibly small quantity of the order of the fourth small-angle moment of the highly elongated function of the "brightness body" type.[19]

TABLE I.

Data on the error of the scalar approximation of the energetic characteristics of the scattered radiation for illumination of the medium by a wide unpolarized beam.

Medium	Problem	Characteristic	SA error	Reference	Remark
Rayleigh scattering	Reflection, transmision $\tau_0 \sim 1$, $\Lambda \sim 1$	$G_{11}(\mu_0=1, \lvert\mu\rvert=1)$ $G_{11}(\mu_0=1, \lvert\mu\rvert\approx 0)$ albedo	0.08÷0.1 -0.1÷-0.12 0.005	[2, 3, 7, 10]	$\frac{\lvert\Delta G\rvert}{G_{11}} \le 12\%$
	Reflection $\tau_0 \to \infty$ $\Lambda \sim 1$	$G_{11}(\mu_0=1, \lvert\mu\rvert=1)$ $G_{11}(\mu_0=1, \lvert\mu\rvert\approx 0)$ albedo	0.04 -0.06÷-0.07 0	[1, 3, 10]	for $\Lambda \neq 1$ see Ref. 11
	Depth mode	$\gamma(\Lambda = 0.06÷0.8)$ $\sigma(\Lambda > 0.95)$ $\Lambda(\gamma \approx 0.99)$ $\Lambda(\gamma \ll 1)$ $\varphi_1(\Lambda = 0.5;\ \mu = 1)$ $\varphi_1(\Lambda = 0.5;\ \mu = 0)$ $\varphi\ (\Lambda = 0.5;\ \mu = -1)$	-0.01 $\sim -\sqrt{1-\Lambda}$ -0.04 $\sim -\gamma^2$ 0.03 -0.05 0.08	[4, 5, 6, 12, 13, 17]	$\frac{\lvert\Delta\gamma\rvert}{\gamma} \le 1\%$ $\frac{\lvert\Delta\Lambda\rvert}{\Lambda} \le 4\%$ $\frac{\lvert\Delta\varphi\rvert}{\varphi} \le 10\%$
Particules $2\pi a/\lambda = 2$ (a-radius λ-wavelength)	Depth mode	σ Λ	< 0.005 < 0.002	[6, 9]	
Haze L	Reflection, transmission	G_{11}	< 0.001	[10]	
Cloud (visible and IR)	Reflection	G_{11}	< 0.01	[8, 18, 22]	by absolute value
	Transmission $\tau_0 \gg 1$	G_{11}	<0.01÷0.02	[17, 22]	
	Depth mode	$\sigma(\Lambda > 0.98)$	$< 10^{-5}$		
Seawater (visible)	Reflection	G_{11}	<0.01	[18]	
	Depth mode	σ	<0.001	[23]	
Cloudless atmosphere (visible)	Reflection, Transmission	G_{11}	0.02÷0.04	[14, 15, 16]	

For a cloudless atmosphere, including both a Rayleigh and a large-scale dispersed scattering component, and for other media of such type, dependences of $P^{(1)}(x)$ close to the Rayleigh dependence are typical, but the magnitudes $\left|P^{(1)}_{\max}\right| < 1$. The less is the quantity $(P^{(1)}_{\max})^2$, the less is the magnitude of the error $|\delta|$. For example, for $P^{(1)}_{\max} = 0.7$ ($(P^{(1)}_{\max})^2 = 0.49$) the magnitude of $|\delta|_{\max}$ for the Earth's atmosphere is less than half the magnitude of $|\delta|_{\max}$ for a Rayleigh atmosphere of the same optical thickness τ_0 or, rather, effective thickness $\tau_0^* = (1 - \Lambda\alpha)\tau_0$, where α is the integral of the small-angle part of the scattering phase function (for the orders of magnitude of $|\delta|_{\max}$ consult Table I.

The dependence of $P^{(1)}(x)$ for aerosol formations consisting of non-small and "non-soft" particles is, as a rule, alternating in sign. This means that there are more zeros of the polarization and that the polarization "azimuth" distribution is more diffuse than in the case of Rayleigh scattering. For such media the values of $|\delta|$ are particularly small (haze, clouds) (see Table I).

The comparatively small values of $|\delta|$ in the case of Rayleigh scattering (in the table $|\delta|_{\max} \sim 12\%$) are due, as was noted earlier, to the quasi diffuseness in $\vec{n}$ of the scattered radiation field. This is typical in most cases. However, if, as a result of the specifics of the geometry of the problem, the radiation field in the medium possesses a directionality toward the region of larger values of $P^{(1)}$, then the SA error may cease to be small. Let us consider such a situation. Let a Rayleigh layer be illuminated by a unidirectional point source directed normal to the layer (along the $\vec{z}$ axis). In this case, as is well known, the singly scattered radiation at any off-axis point of the medium (one for which $\vec{r}_\perp \neq 0$ where $\vec{r}_\perp$ is the distance from the beam axis) is concentrated within the radial plane passing through the point and the beam axis. In addition, at small depths τ the energy density of the field is maximal inside the region of mean scattering angles in the vicinity of the direction $\vec{n}_\perp = \frac{\vec{r}_\perp}{r_\perp}$ perpendicular to the beam axis, i.e., where $P^{(1)}$ is large. For such properties of the single-scattering field the double-scattering intensity correction $\left(\frac{\Delta G}{G}\right)^{(2)}$ is considerable in two directions, namely the direction perpendicular to $\vec{n}_\perp$: $\vec{n}_\perp(|\mu| = 1)$ and the direction perpendicular to $\vec{n}_2$ ($\mu = 0$, $\left(\mu = 0,\ \Phi = \frac{\pi}{2} \pm k\pi\right)$ ($\mu = \cos\theta$, θ is reckoned from the $\vec{z}$ direction at the point $\vec{r}_\perp$, the azimuth Φ is reckoned from the radial plane). This is shown in Fig. 1, which presents results of a calculation of $\left(\frac{\Delta G}{G}\right)^{(2)}$ at $\tau = 0$ (for the reflection problem). In the directions $\vec{n}_1$ and $\vec{n}_2$ the contribution of the singly scattered radiation vanishes ($G^{(1)} = 0$), therefore the SA error for the total intensity in this case is not small, either.

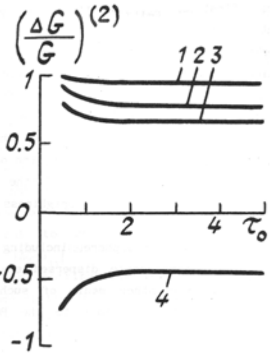

FIG. 1. The double-scattering contribution to the polarization correction $\left(\frac{\Delta G}{G}\right)^{(2)}$ *to the intensity in the problem of reflection of the radiation from a thin Rayleigh layer illuminated in the direction of its normal by a directed point source: 1)* $\rho = \varepsilon\vec{r}_\perp = 0.5$ *, 2) – 1; and 3) 5 for* $|\mu| = 1$*; 4)* $\rho = 1$ *for* $\mu = 0$, $\Phi = -\frac{\pi}{2}$

Let us analyze the estimates of $|\delta|$ for a Rayleigh medium (Table I) and note the physical peculiarities of the formation of the SA error. In the problem of reflection of radiation from a layer of arbitrary thickness the values of G_{21} and G_{21} are determined mainly by the values of $G^{(1)}_{21}$ and $G^{(1)}_{31}$ (the single-scattering values),[1] and the difference ΔG due to them is determined mainly by the difference $\Delta G^{(2)}$ due to double scattering. In the transmission problem this property is fulfilled for $\tau_0 < 0.5$. It is important to stress the alternating sign property of the SA error for the normalized angle intensity distributions. It is a general feature of media and radiative transfer problems, and it implies that the error for the integral characteristics is less than that for intensity and less than for diffuse sources than for directed ones.

For large depths τ in a semi-infinite medium (in the depth regime, as $\tau \to \infty$) the intensity S_1 is given by the expression[1,4,23]

$$S_1(\tau,\mu,\mu_0) = [G_{11,0}(\tau,\mu,\mu_0) + G_{12,0}(\tau,\mu,\mu_0) s_{02}]\, I_0;$$
$$G_{1k,0}(\tau,\mu,\mu_0) = \varphi_1(\mu) u_k(\mu_0)\, e^{-\gamma\tau},\ k=1,2, \qquad (9)$$

where $\mu_0 = \vec{n}\cdot\vec{Z}$ (Z is the medium axis), the functions $G_{1k,0}(\tau,\ \mu,\ \mu_0)$ are the solutions of Eq. (2) and (5) for the zeroth azimutal harmonic, and the characteristics

$\varphi_1(\mu)$ $\left(\frac{1}{2}\int_{-1}^{1}\varphi_1(\mu)d\mu = 1\right)$ and γ for Rayleigh scattering and the solution of the Milne problem $u(u_1, u_2, 0, 0)$ were considered, e.g., in Ref. 1. The SA error $\delta_0 = \frac{G_{11,0} - G_0}{G_{11,0}}$ is given by the formula

$$\delta_0(\tau, \mu, \mu_0) = e^{-\Delta\gamma\tau} - 1 + \delta_\varphi(\mu) + \delta_u(\mu_0),$$

where $\Delta\gamma = \gamma - \gamma^{sc}$, $\delta_\varphi = \frac{\varphi_1 - \varphi^{sc}}{\varphi_1}$, $\delta_u = \frac{u_1 - u^{sc}}{u_1}$ ("sc" denotes the "scalar" solution). Using data on $\Delta\gamma$ and δ_φ (Refs. 5, 6, 12, and 13) and assuming that $\delta_u(\mu_0) \approx \delta_\varphi(\mu_0)$ (Refs. 1 and 23), it is possible to estimate $\delta_0(\tau, \mu, \mu_0)$. For $\Lambda = 1$, when the radiation field is completely diffuse and unpolarized, $\Delta\gamma = 0$ and $\delta_\varphi = 0$. In this case $|\delta_0| = |\delta_u|$, where $|\delta_u(\Lambda = 1)|$ is a negligibly small quantity.[1] For $\Lambda < 1$ $\Delta\gamma < 0$ (see Table I). This property can be easily understood if one takes into account the fact that in directions $\mu \approx 1$, where the field energy is maximal (for $\Lambda < 1$), the contribution of the Rayleigh-type polarization to the intensity is positive: $G_{11,0}(1) > G_0(1)$. Hence it follows that the polarization correction for the integral angular field characteristics (the flux density) is also positive, i.e., $e^{-\gamma\tau} > e^{-\gamma^{sc}\tau}$. Since $\Delta\gamma < 0$, $|\delta_0| \sim e^{|\Delta\gamma|\tau}$. As can be seen, under conditions of stationarity of the field properties in a medium with absorption the SA error grows ("accumulates") with depth. As $\Lambda \to 0$ the radiation inside the medium becomes directed along the Z axis, where the polarization $P(1) = 0$, hence $|\delta| \to 0$. The values of $|\delta|$ are maximal inside the region $\Lambda \sim 0.7$, where $\Delta\gamma \approx -0.009$. For example, $\delta(\mu = \mu_0 = 1) \approx -0.15$ at the depth $\tau = 10$. Here the intensity values $G_{11,0} \approx 7 \cdot 10^{-3}$. The properties of δ in the problem of transmission through a thick layer are similar.

Generalizing the results of the above analysis, we may note the following. The use of the scalar TE solutions in transfer theory problems is justified, first of all, by the consideration of unpolarized radiation sources, which are, as a rule, extended or diffuse point sources, and second, by the consideration of optically isotropic media. The polarized radiation "born" during scattering in such media constitutes a relatively small part of the energetics and is quasidiffuse in its distribution character, whose properties guarantee comparatively small values of the SA error.

In connection with the development of scattering theory applied to laser sources, it should be interesting to investigate the problem of the applicability of the scalar theory to the description of light fields created in a medium by linearly polarized light. It should be clarified, first of all, in which cases it is possible to ignore the polarization of the incident beam, i.e., when it is possible to neglect the small terms in the solutions that depend on the matrix elements G_{12} and G_{13}, in particular, the term $\Delta S_1(G_{12}, G_{13})$ in the solutions for (Eq. 8). This question is considered below. It is assumed that the medium is illuminated by a directed beam of arbitrary width and that the beam polarization $P_0 = \sqrt{S_{02}^2 + S_{03}^2} = 1$.

Let us first consider problems of axially symmetric type. Here the symmetry characterizes the geometry: the form of the medium and the direction of the illumination. In particular, let the layer be illuminated along its normal ($\mu_0 = 1$). For $\mu_0 = 1$ the solutions of the vector TE (2) for the matrix elements G_{1k}, $k = 2, 3$, as well as for their sources, the elements G_{ik} ($i, k = 2, 3$) are the TE solutions for the second azimuthal harmonics, i.e., $\begin{pmatrix} G_{12} \\ G_{13} \end{pmatrix} = \begin{pmatrix} G_{12,2} \\ G_{3,2} \end{pmatrix} \sim \begin{pmatrix} \cos 2\psi \\ \sin 2\psi \end{pmatrix}$ (the azimuth ψ is defined with respect to the beam axis).[20,21] The contributions to the defined quantities due to those matrix elements vanish or are small in the following cases:

1. When determining azimuthally averaged characteristics: average intensity, fluxes, illumination, albedo, field density, etc.

2. When determining the intensity at such depths where because of scattering the light from the source has become depolarized, i.e., where $\max\{G_{ik}\} \ll G_{11}$, $i, k = 2, 3$. Knowing the decay coefficient γ_2 for the second harmonics $G_{ik,2}(i, k = 2, 3)$ and the coefficient γ for $G_{11,0}$ as $\tau \to \infty$ (9), one can estimate the order of magnitude of such depths via the condition $P_0 e^{-(\gamma_2 - \gamma)\tau} < \beta$, where β is some fixed number. For $P_0 = 1$ and $\beta = 0.1$ we have: for media with Rayleigh scattering $\tau > 3$ for $\Lambda = 1$ ($\gamma = 0$, $\gamma_2 = 0.826$), but $\tau > 38$ for $\Lambda = 0.58$ ($\gamma = 0.903$, $\gamma_2 \approx 0.96$). For a cloud (the Deirmenjian *C1 cloud* model) the values $\tau > 14$ for $\Lambda = 1$ ($\gamma_2 \approx 0.165$) and $\tau > 25$ for $\Lambda = 0.98$ ($\gamma = 0.095$, $\gamma_2 \approx 0.187$). Here we used the results of calculations of γ_2 which were based on the work in Refs. 19, 20, and 21. In those media, where the "brightness body" is highly elongated in the forward direction, the light becomes depolarized (if $P_0 = 1$) only at large depths, where the field energy is very small.[20]

3. When determining the field characteristics at small scattering angles in media with elongated scattering phase functions. In this region the matrix elements G_{12} and G_{13} are small because the component $b_1(\zeta \ll 1)$ (6) is small.[20] The error can be estimated using the formula

$$\frac{|\Delta S_1|}{S_1} < P_0 P^{(1)}(\mu)\, e^{-\Lambda(\alpha_1 - \alpha_2)\tau},$$

where μ is close to 1 and $P^{(1)}(\mu) \ll 1$ (6); α_1 and α_2 are the small-angle integrals of the functions $\alpha_1(\zeta)$ and $\frac{1}{2}[\alpha_2(\zeta) + \alpha_3(\zeta)]$(5) (Ref. 20). The values of G_{12} and G_{13} may also be small inside the angle region ~180°.

In the case of slanted illumination of a layer by a polarized beam, items 1, 2, and 3 apply to the characteristics of the radiation at large depths since under such conditions the field properties conform to the axial symmetry properties of the medium. Account should be taken of the fact that for $\mu_0 < 1$ the matrix elements G_{12} and G_{13} in the region of such depths are determined not only by the second harmonics $G_{ik,2}(k = 2, 3)$ but also by the azimuthally independent component $G_{12,0}$. .This can be seen from the expression for S_1 in the limit $\tau \to \infty$ (9). In Eq. (9), by virtue of the property $D_{12}(1) = 0$ (5), (6), the following equalities hold: $u_2(1) = 0$, $G_{12,0}(\mu_0 = 1) = 0$. However, $G_{12,0} \neq 0$ for $\mu_0 \neq 0$. The polarization correction $\Delta S_1(\cos 2\psi, \sin 2\psi)$ is governed by the decay of the source polarization with depth, and the correction $\Delta S_1(G_{12,0})$ is controlled by the light source parameter at the layer boundary. The relative value $K = K = \dfrac{G_{12,0}}{G_{11,0}} S_{02}$ does not depend on ψ, μ, or τ. It remains invariant for integral values and is conserved as $\tau \to \infty$. Thus, for $\mu_0 < 1$ inside the region of moderate-to-large depths one can neglect the effect of source polarization on the energetics of the field in the cases 1, 2, and 3 under the condition that

$$|K(\mu_0)| = \left|\frac{u_2(\mu_0)}{u_1(\mu_0)} \cdot s_{02}\right| \ll 1.$$

Note that K = 0 if the polarization angle of the incident light is 45° ($S_{02} = 0$), is negligibly small for $1 - \mu_0 \ll 1$ $\left(\dfrac{u_2}{u_1} \sim \left(1-\mu_0^2\right)\right)$, and for weakly absorbing media with highly elongated phase functions, for the *Cloud C1* for example, $K \leq 0.1\%$.[17] For sea water, using results from Ref. 23, one gets $K_{max} \sim 3\%$, 8%, and 11% for $\Lambda = 0.9$, 0.7, and 0.5, respectively. Here $K_{max} = K(\mu_{0min})$, and, because of the light refraction at the water surface, $\mu_{0min} = 0.66$. For a Rayleigh medium the value of $K(\mu_0 \approx 0)$ is $\sim 10\%$ for $\Lambda = 1$ and grows as Λ decreases.

Inside the small-angle scattering region, for the case of slanted illumination of the medium a simplification is also possible. In the cases, in which the local beam axis in the medium can be approximately viewed as a local symmetry axis, the small-angle and azimuthally averaged (about the axis) energetic characteristics in the near zone can be determined without taking account of the polarization state of the beam. For media with absorption at $\tau \gg 1$ (after the beam turns in the medium) an additional error $\sim K(\mu_0; s_{02})$ needs to be evaluated.

The correction $\dfrac{\Delta S_1}{S_1}$ (8) is small also in the trivial situation when $b_1(x) \ll a_1(x)$ (5) (for some kinds of large-scale dispersed formations).

The domain of applicability of the solutions of the scalar TE in the transfer theory of linearly polarized light practically coincides with the above-considered domain of applicability of the approximation $\dfrac{\Delta S_1}{S_1} \approx 0$ (8). The additional error of the SA, i.e., the approximation $\dfrac{\Delta G}{S_1} \approx 0$ in Eq. (8) is small in almost all of the cases noted. As estimates have shown (see Fig. 1), an exception may be the SA error in the problem of determining the intensity of the scattered radiation azimuthally averaged about the axis of a narrow beam, including the backward directions (with respect to the source), at small depths or in thin layers of a medium with Rayleigh polarization. Such an intensity is described, according to item 1, by the solution $S_1 = G_{11}I_0$, and the possibility of passage to the solution $S_1 \approx GI_0$ must be justified by special estimates.

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