CALCULATION OF LIGHT FIELD CHARACTERISTICS IN TURBID MEDIA

M.D. Aleksandrov and V.S. Remizovich

Engeneering–Physics Institute, Moscow
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The depth behavior of light scattering in turbid media with slowly decreasing phase functions (i.e., when the scattering probability decreases with increasing scattering angle γ more slowly than γ^{-4}) is studied. The Fokker–Planck approximation is not used. Postulating the angular spectrum shape in the depth regime we have proposed and realized a regular algorithm for determining the spectrum parameters. The depth variance and extinction coefficient are described analytically. The results of our study are in good agreement with numerical data and in limiting case they coincide with the results of the diffusion approach.

INTRODUCTION

The study of light propagation through absorbing media with a strongly pronounced anisotropic scattering on individual centres for $1 - \langle\cos\gamma\rangle \ll 1$, where $\langle\cos\gamma\rangle$ is the mean cosine of a single scattering angle, is of practical and theoretical importance for many problems in hydro–optics and atmospheric physics as well as for different artificial optical media with the size of scattering centres much larger than the light wavelength ($a \gg \lambda$) and with their relative index of refraction ~ 1. Under these conditions, the effective angle of single scattering is small $\gamma_{eff} \sim \lambda/a \ll 1$ (Refs. 1–4). A comprehensive information on the law of photon scattering on an isolated scattering centre is contained in the scattering phase function $\chi(\gamma)$ normalized according to the condition

$$2\pi\int\limits_0^\pi \sin\gamma\chi(\gamma)\,d\gamma = 1\ , \quad \text{so that} \quad \langle\cos\gamma\rangle = \pi\int\limits_0^\pi \sin2\gamma\chi(\gamma)\,d\gamma\ . \quad (1)$$

In strongly absorbing (turbid) media with large scattering centres the double inequalities

$$\sigma \gg \kappa \gg D\ , \text{ i.e., } \bar{l} \ll l_a \ll l_{tr}\ , \quad (2)$$

are fulfilled. Here σ is the scattering coefficient, κ is the coefficient of true absorption, D is the angular diffusion coefficient, $\bar{l} = \sigma^{-1}$ is the length of elastic scattering, $l_a = \kappa^{-1}$ is the length of absorption, $l_{tr} = (2D)^{-1}$ is the transport length of scattering, i.e., the length of a propagation path after passing which in a substance the originally monodirectional flux of photons becomes almost completely isotropic due to a purely elastic scattering.

In this case by the moment when absorption of light becomes essential in forming the light field, the photon have already undergone a large number of elastic collisions ($n_{\rm col} \sim l_a/\bar{l} \gg 1$). In the depth region of the greatest interest (z l_a), the scattered light field is formed due to multiple scattering of photons. However, because of a strong absorption ($l_a \ll l_{tr}$) the light beam does not become isotropic since the photons undergone a relatively stronger scattering propagate along a more bent trajectories and are absorbed before they reach the depth $z \sim l_{tr}$.

Therefore, even at a comparatively large depth $z \gtrsim \sqrt{l_a l_{tr}}$ the light is scattered at small angles

$$\langle\theta^2\rangle_z \ll 1\ , \quad (3)$$

where $\langle\theta^2\rangle_z$ is the mean square of a multiple scattering angle at the depth z. Therefore, to describe the photon propagation one can make use of a small–angle approach.[3-7]

With a small–angle scattering, the mean path passed by photons in a layer of substance of the thickness z (at normal incidence of the beam on the medium surface) is determined from the relation

$$\langle s\rangle_z = \int\limits_0^z dz' \left\langle \frac{1}{\cos\theta} \right\rangle_{z'} \approx z + \frac{1}{2}\int\limits_0^z \langle\theta^2\rangle_{z'}\, dz'\ . \quad (4)$$

The absorption and scattering processes become competitive in the region of depths where $(\langle s\rangle_z - z)$ l_a.

Although the radiation transfer equation in a small–angle approximation (SA) is quite simple its analytical solution can hardly be obtained in the case of a medium with an abitrary sharply anisotropic scattering phase function. The principle problem is to correctly account for the mutual effect of strong absorption and multiple scattering which result in fluctuations of a photon free path. There are no serious problems but in the case of small depths $(\langle s\rangle_z - z)$ l_a, where the light absorption does not affect the formation of the radiation angular spectrum[4,8,9] (standard small–angle approach (SSA)).

In recent years a large number of papers has been published[3–7,10–15] which dealt with the analytical calculations of radiation intensity using SA for wide and narrow, stationary and nonstationary light beams incident both normally and obliquely on a medium. However, in all these works the Fokker–Planck approach (small–angle diffuse approach (SADA) is used, in which the integral of elastic collisions entering into the transfer equation is written in a differential form. The possibility of using the SADA are associated with rather rigourous restrictions imposed on the shape of the scattering phase function $\chi(\gamma)$: it must decrease with increasing γ quicker than γ^{-4}, i.e., quicker than the Rutherford phase function.[16,17] However, it is well known that just the opposite situation is observed in the majority of real turbid media[1–3,17,18]. Therefore, it becomes of vital

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importance to develop a technique for calculating light fields just in such media.

In the present paper we propose a method for studying light fluxes at large depths in media with the scattering phase functions relatively slowly decreasing with the γ increase, $\chi_\nu(\gamma \gg \gamma_{eff}) \sim \gamma^{-2(1+\nu)}$, where $\nu < 1$ taking into account the joint effect of the photon absorption and scattering in turbid media.

FORMULATION OF THE PROBLEM

Let a wide stationary light flux with the intensity I_0 be incident on a plane boundary of a turbid medium occupying the half–space $z > 0$ (the z axis is directed into a substance along the normal to its surface). Then the intensity of radiation $I(z, \mu)$ ($\mu = \cos\theta$) propagating at an angle θ to the z axis will be described by the equation

$$\mu \frac{\partial I}{\partial z} + \mathrm{k}I(z;\, \mu) = \hat{B}I \,, \tag{5}$$

where $\hat{B}I$ is the Boltzmann integral of elastic collisions.[1–4] The boundary condition for Eq. (5) is

$$I(z = 0;\, \mu > 0) = \frac{1}{2\pi} I_0\, \delta(1 - \mu) \,. \tag{6}$$

The small–angle approach allows an essential simplification of Eqs. (5) and (6) to be made. As is usual for the SA, all angular variables can be assumed to vary within the infinite limits. Moreover, the angle γ of a single scattering causing the transition from the state (θ', φ') to the state (θ, φ) is determined by the expression

$$\gamma^2 \approx \theta^2 + \theta'^2 - 2\theta\theta' \cos(\varphi - \varphi') \,. \tag{7}$$

The total light flux at the depth z is

$$E(z) = 2\pi \int_0^\infty \theta I(z;\, \theta)\, \mathrm{d}q \tag{8}$$

and, taking Eq. (4) into account, it is determined by the expression[6]

$$E(z) = I_0 \exp\{-\mathrm{k}\langle s\rangle_z\} =$$

$$= I_0 \exp\left\{ -\mathrm{k} \left[z + \frac{1}{2} \int_0^z \langle\theta^2\rangle_{z'}\, \mathrm{d}z' \right] \right\} . \tag{9}$$

Since the photons propagate along "bent" trajectories it seems to be logical to separate out the Bouguer exponent in the intensity not as a function of the depth z, but of the mean path $\langle s\rangle_z$ by representing $I(z, \theta)$ in the form

$$I(z, \theta) = E(z)\, \Phi(z, \theta) \,, \tag{10}$$

where $E(z)$ is the unknown value of the radiation flux (Eqs. (8) and (9)). The angular function $\Phi(z, \theta)$ is normalized on the interval $(0 \le z < \infty)$ according to the condition

$$2\pi \int_0^\infty \Phi(z, \theta)\, \theta\, \mathrm{d}\theta = 1 \,. \tag{11}$$

Thus, the value $2\pi\Phi(z, \theta)\,\theta\, \mathrm{d}\theta$ implies that at the depth z a photon propagates within the angular interval from θ to $\theta + \mathrm{d}\theta$. The value $\langle\theta^2\rangle_z$ is related to $\Phi(z, \theta)$ as follows:

$$\langle\theta^2\rangle_z = \frac{2\pi}{E(z)} \int_0^\infty \theta^3 I(z, \theta)\, \mathrm{d}\theta = 2\pi \int_0^\infty \theta^3 \Phi(z, \theta)\, \mathrm{d}\theta \,. \tag{12}$$

By substituting Eq. (10), on account of Eq. (9), into Eq. (5), and assuming that $\mu \approx 1 - \frac{1}{2}\theta^2$, and holding only the first nonvanishing terms, one obtains the equation for the angular function $\Phi(z, \theta)$ in the SA

$$\frac{\partial \Phi}{\partial z} + \frac{\mathrm{k}}{2} [\theta^2 - \langle\theta^2\rangle_z]\, \Phi(z, \theta) = \hat{B}\Phi \,, \tag{13}$$

where, according to the small–angle approach

$$\hat{B}\Phi = -\sigma \int_0^{2\pi} \mathrm{d}\varphi' \int_0^\infty \theta'\, d\theta'\, \chi(\gamma)\, [\Phi(z, \theta) - \Phi(z, \theta')] \,, \tag{14}$$

and γ^2 is determined from Eq. (7). Corresponding boundary condition has the form

$$\Phi(z = 0;\, \theta) = \frac{1}{2\pi} \frac{\delta(\theta)}{\theta} \,. \tag{15}$$

The value $\langle\theta^2\rangle_z$ is unknown and should sought be after in the process of solving the problem.

If the system of equations (13)–(15) is solved and $\langle\theta^2\rangle_z$ is determined, it is possible to determine the light flux $E(z)$ and, hence, the intensity $I(z, \theta)$ using Eq. (9).

Using the summation theorem for the Bessel functions[19] it is possible to represent collisional integral (14) in the form

$$\hat{B}\Phi = -\frac{\sigma}{2\pi} \int_0^\infty \omega\, \mathrm{d}\omega\, J_0(\omega\, \theta)\, \Phi_\omega(z)\, [1 - \chi(\omega)] \,, \tag{16}$$

where $J_0(x)$ is the Bessel function, and $\Phi_\omega(z)$ and $\chi(\omega)$ are the Bessel images of the angular and the scattering phase function, respectively.

The analysis of experimental data and numerical calculations[1-3,18] shows that the scattering phase functions of large scattering centres in sea water, clouds, fogs, aerosols, and so on, within the scattering angles $\gamma_{eff} \ll \gamma \ll 1$ have a power–law (or close to it) shape $\chi(\gamma) \sim \gamma^{-2(1+\nu)}$, where the value of the parameter ν usually lies in the interval $0.25 \le \nu \le 0.75$. The scattering phase function for a turbulent medium has the same view[2] ($\nu = 5/6$ is the characteristic of the Kolmogorov–Obukhov spectrum). Taking into account this fact, we shall use below a two–parameter relation for $\chi(\gamma)$ which in a small–angle approach ($\gamma \ll 1$) has the form

$$\chi_\nu(\gamma) = \frac{\nu\, \gamma_{eff}^{2\nu}}{\pi[\gamma_{eff}^2 + \gamma^2]^{(1+\nu)}} \,. \tag{17}$$

Value (17) satisfies the condition of normalization (1). The values $\gamma_{eff} \lesssim 5°$ and 6–15° are characteristic of sea water and clouds.[1–13,18], respectively. The value $\nu = 1$ agrees with the

Rutherford scattering law, and $\nu = 1/2$ leads to the Henny–Greenstein scattering phase function.[1,2,11]

It is not difficult to calculate the Bessel image of scattering phase function (17)

$$\chi_\nu(\omega) = \frac{2^{1-\nu}}{\Gamma(\nu)} (\omega\gamma_{eff})^\nu K_\nu(\omega\gamma_{eff}) , \tag{18}$$

where $\Gamma(\nu)$ is the γ–function, $K_\nu x)$ is the Macdonald function. In the depth region, where $\langle\theta^2\rangle_z \gg \gamma_{eff}^2$ the main contribution to integral (16) comes from the values $\omega \lesssim \frac{1}{\theta} \ll \gamma_{eff}^{-1}$ and Eq. (18) can be expanded in a series over the small parameter $\omega\gamma_{eff} \ll 1$. While making this expansion one can calculate the coefficient of angular diffusion $D_\nu = \frac{1}{2}\sigma\langle 1 - \cos\gamma\rangle$. This expansion has different form for $\nu > 1$ and $\nu < 1$ (see Refs. 9 and 20)

$$1 - \chi_\nu(\gamma_{eff}\,\omega \ll 1) \approx \begin{cases} \dfrac{D_\nu}{\sigma}\,\omega^2 , & \text{for } \nu - 1 \gg \gamma_{eff}^2; \\ \dfrac{\Gamma(2-\nu)}{\nu\Gamma(1+\nu)}\,\dfrac{D_\nu}{\sigma}\,\omega^{2\nu} , & \text{for } 1 - \nu \gg \gamma_{eff}^2 . \end{cases} \tag{19}$$

It can be seen from Eq. (19) that when $\nu > 1$ the value $(1 - \chi_\nu(\omega)) \sim \omega^2$ and when $\nu < 1$ $(1 - \chi_\nu(\omega)) \sim \omega^{2\nu}$, i.e., it strongly depends on ν. If we formally assume $\nu = 1$ in Eq. (19), then both expansions are joined.

THE DEPTH REGIME

Not pretending to be able of determining the angular function $\Phi(z, \theta)$ at any depth, we should like first to consider the case of relatively large depths, where

$$\langle s\rangle_z - z \gtrsim l_a \text{ , i.e., } \frac{k}{2}\int\limits_0 \langle\theta^2\rangle_{z'}\,dz' \gtrsim 1 . \tag{20}$$

At large depths the factorization of the radiation angular spectrum[1–7] is feasible

$$I_{as}(z; q) = a\exp(-kz)\,F_{as}(q) . \tag{21}$$

Taking the condition of normalization (11) into account the factorization means, in fact, that the angular function $\Phi_{as}(\theta) = \Phi(z \to \infty; \theta)$ is independent of the depth. The value k is the extinction coefficient of the medium at large depths. Following Ref. 12, the variance $\langle\theta^2\rangle_\infty$ of the angular spectrum at large depths is determined as

$$\langle\theta^2\rangle_\infty = 2\pi\int\limits_0^\infty \theta^3\Phi_{as}(\theta)\,d\theta \text{ , and } k = k\left[1 + \frac{1}{2}\langle\theta^2\rangle_\infty\right] . \tag{22}$$

In the case of large depths Eq. (13) takes the form

$$\frac{k}{2}[\theta^2 - \langle\theta^2\rangle_\infty]\,\Phi_{as}(\theta) = \hat{B}\Phi_{as} , \tag{23}$$

where $\hat{B}\Phi_{as}$ is determined by Eq. (16). To solve Eq. (23) approximately in the case of $\nu < 1$, we are interested in, let the sought–after angular function $\Phi_{as}(\theta)$ be of the form which is similar to scattering phase function (17) but decreasing with increasing θ more rapidly than θ^{-4}

$$\Phi_{as}(\theta, \alpha, \langle\theta^2\rangle_\infty) = \frac{1+\alpha}{\pi}\,\frac{(\alpha\langle\theta^2\rangle_\infty)^{1+\alpha}}{[\alpha\langle\theta^2\rangle_\infty + \theta^2]^{2+\alpha}} , \tag{24}$$

where $\alpha > 0$. Equation (24) satisfies the condition of normalization (11) and formula (22) for $\langle\theta^2\rangle_\infty$. The values α and $\langle\theta^2\rangle_\infty$ are free parameters. To determine them one should construct a relevant system of two equations.

The first equation of such a system follows from Eq. (23), in which we assume that $\theta^2 = \langle\theta^2\rangle_\infty$,

$$\hat{B}\Phi_{as}\big|_{\theta^2 = \langle\theta^2\rangle_\infty} = 0 . \tag{25}$$

The Bessel image of the angular function is determined by the expression similar to Eq. (18)

$$\Phi_{as}(\omega, \alpha, \langle\theta^2\rangle_\infty) = \frac{2^{-\alpha}}{\Gamma(1+\alpha)}\left(\omega\sqrt{\alpha\langle\theta^2\rangle_\infty}\right)^{1+\alpha} \times$$

$$\times K_{1+\alpha}\left(\omega\sqrt{\alpha\langle\theta^2\rangle_\infty}\right) . \tag{26}$$

Substitution of Eqs. (24) and (26) into Eq. (25) in which the collisional integral is taken in the form of Eq. (16) and calculation of corresponding integrals reduces condition (25) to an equivalent relation

$${}_2F_1\left(2 + \nu + \alpha,\ 1 + \nu;\ 1;\ -\frac{1}{\alpha}\right) = 0 , \tag{27}$$

where ${}_2F_1(a, b, c, z)$ is the hypergeometric Gaussian function.[19]

Thus, in the first approach the value of the parameter α can be determined using the small parameter $\gamma_{\rm eff}^2/\langle\theta^2\rangle_\infty \ll 1$ independently of $\langle\theta^2\rangle_\infty$, being only a function of ν, i.e., $\alpha = \alpha(\nu)$. Equation (27) was solved numerically (the function $\alpha(\nu)$ is shown in Fig. 1). At the same time in some particular cases the study can be carried out analytically, yielding $\alpha(\nu \to 0) \approx \nu$, and $\alpha(\nu \to 1) \approx 5.14/(1 - \nu)$. When $\alpha = 1$, Eq. (27) is reduced to a simpler transcendental equation for determining the value ν

$$\frac{2}{\nu^2}\cot\frac{\pi\nu}{2} = \frac{2+\nu}{1+\nu}\left[\frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}\right]^2 . \tag{28}$$

It follows from this equation that $\nu \approx 1/3$. This agrees with the result obtained by the direct solution of general equation (27).

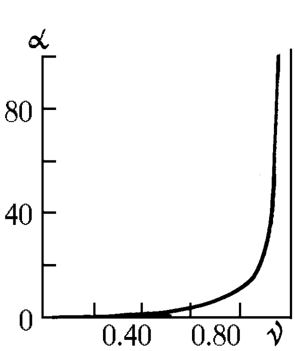

FIG. 1.

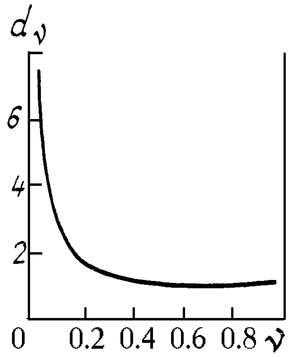

FIG. 2..

The second equation for determining $<\theta^2>_\infty$ is constructed using the condition of correct normalizing the angular function of the second approach obtained at the iterations of Eq. (23), provided that 2(ν) has been already determined from Eq. (27)

$$2\pi \int_0^\infty \theta\, d\theta \left\{ \frac{2}{k} \frac{\hat{B}\Phi_{as}}{\theta^2 - <\theta^2>_\infty} \right\} = 1 . \quad (29)$$

After simple transformations one can find from Eq. (29) the value of angular variance in the depth regime

$$<\theta^2>_\infty = \left[8d_\nu \frac{D_\nu}{k} \right]^{1/1+\nu} , \quad (30)$$

where D_ν is the coefficient of angular diffusion, the function $d_\nu \equiv d(\nu)$ is represented by the expression

$$d_\nu = \frac{4^{\nu-1}}{\nu} \frac{\Gamma(2 + \nu + \alpha(\nu))\, \Gamma(2 - \nu)}{\alpha^{1+\mu}\Gamma(1 + \alpha(\nu))}\, b_\nu , \quad (31)$$

where

$$b_\nu = \int_0^\infty \frac{dx}{x - 1}\, {}_2F_1\left(2 + \nu + \alpha(\nu),\ 1 + \nu;\ 1;\ -\frac{x}{\alpha(\nu)} \right) . \quad (32)$$

It follows from Eq. (32) that we can represent the expression for b_ν with a sufficient accuracy (not lower than 5%) in the form

$$b_\nu \approx \frac{\alpha(\nu)}{1.125 + \alpha(\nu) + \frac{\pi}{4}\nu} . \quad (33)$$

In the limiting case of $\nu = 1$ the value $\alpha \to \infty$ and $d_\nu = 1$. As a result from Eqs. (24) and (30) as $\nu \to 1$, one obtains

$$\Phi_{as}(\theta;\ \nu \to 1) = \frac{\exp\left(-\theta^2/<\theta^2>_\infty^D \right)}{\theta\, <\theta^2>_\infty^D} ,$$

where

$$<\theta^2>_\infty^D = \sqrt{\frac{8D}{\kappa}} , \quad (34)$$

that well agrees with the results obtained using a diffusion approach.[5–7]

The value d_ν, as a function of ν is presented in Fig. 2. As can be seen in this figure in the interval 1 > ν 1/3, $d_\nu \approx 1$. The value determined using formulas (30)–(32) were compared with the analogous results obtained using the Monte Carlo method.[4] The results coincided accurately to the ratio D/κ that is directly associated with conditions of applicability of the small–angle approach (the results of comparison are given in Table I).

TABLE I

<cosγ> (<γ²>)	Photon servival probability $\Lambda = \frac{\sigma}{\sigma + k}$	D/κ	Correction to the index of depth attenuation $\Delta = \frac{k - k}{k + s}$	Δ (numerical calculation[4])
	0.2	$6.25\cdot10^{-3}$	0.0542	0.0518
0.95	0.4	$1.67\cdot10^{-2}$	0.0783	0.0750
(0.1)	0.8	$1.00\cdot10^{-1}$	0.0862	0.0812
	0.2	$3.75\cdot10^{-3}$	0.0386	0.0379
0.97	0.4	$1.00\cdot10^{-2}$	0.0557	0.0545
(0.06)	0.8	$6.00\cdot10^{-2}$	0.0613	0.0587
0.98	0.2	$2.50\cdot10^{-3}$	0.0295	0.0294
	0.4	$6.67\cdot10^{-3}$	0.0425	0.0422
(0.04)	0.8	$4.00\cdot10^{-2}$	0.0468	0.0453

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