Alternative method of calculation of circular parameters of horizontal wind velocity in sodar measurements

V.A. Fedorov

Institute of Atmospheric Optics,
Siberian Branch of the Russian Academy of Sciences, Tomsk

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The paper describes the algorithms of determination of circular averages, standard deviations, skewness, and kurtosis of the direction φ of the horizontal wind velocity $\mathbf{V}_h$ obtained from measurements with a Volna-3 sodar. In contrast to the standard approach, they do not require preliminary calculation of current $\varphi(i)$ values themselves. The trigonometric φ moments are determined through the corresponding functions of the orthogonal components of the vector $\mathbf{V}_h$. The usefulness of circular standard deviation as a measure of the angular variance of the horizontal wind velocity in acoustic sounding of the atmosphere is addressed.

Introduction

In Ref. 1 Fedorov describes the use of the methods of circular statistics in the processing software system of Volna-3 sodar for estimation of angular parameters of horizontal wind velocity $\mathbf{V}_h$. These methods are based on definitions of cosine- and sine-moments of the random angle ϕ with respect to a given direction ϕ_0[2]:

$$\begin{cases}\alpha_p(\phi_0) = M\{\cos p(\phi - \phi_0)\},\\ \beta_p(\phi_0) = M\{\sin p(\phi - \phi_0)\},\end{cases} \tag{1}$$

where M denotes the mean; and p is the order of trigonometric moment (in the case considered here, $p = 1, 2, 3, 4$ is used). These relations are presented as follows

$$\begin{cases}\alpha_p(\phi_0) = \alpha_p \cos p\phi_0 + \beta_p \sin p\phi_0,\\ \beta_p(\phi_0) = \beta_p \cos p\phi_0 - \alpha_p \sin p\phi_0,\end{cases} \tag{2}$$

where

$$\alpha_p = \alpha_p(0) = M\{\cos p\phi\};\quad \beta_p = \beta_p(0) = M\{\sin p\phi\}$$

are cosine- and sine-moments with respect to zero direction $\phi_0 = 0$. Uniting α_p and β_p into complex plane, we can write:

$$\tau_p = \tau_p(0) = \alpha_p + j\beta_p = \rho_p \exp(j\mu_p),$$

where

$$\rho_p = \sqrt{\alpha_p^2 + \beta_p^2};\quad \mu_p = \arg\tau_p$$

is the absolute value and polar angle τ_p. When $|\tau_1| \neq 0$, the circular average and circular standard deviation of directions of the random angle ϕ[2] are uniquely determined quantities

$$\mu = \mu_1 = M_c(\phi) = \arg\tau_1,\quad \sigma_c(\phi) = \sqrt{-2\ln\rho},$$

where

$$\rho = \rho_1 = |\tau_1|$$

is the absolute value of the average vector with the current coordinates $\{\cos\phi(i), \sin\phi(i)\}$. If in formula (1) we take $\phi_0 = \mu$, then it is possible to pass to the central trigonometric moments, and instead of Eqs. (2) we can write:

$$\begin{cases}\alpha_p(\mu) = \alpha_p \cos p\mu + \beta_p \sin p\mu,\\ \beta_p(\mu) = \beta_p \cos p\mu - \alpha_p \sin p\mu.\end{cases} \tag{3}$$

At the same time, it is valid that $\alpha_1(\mu) = \rho$ and $\beta_1(\mu) = 0$.[2] Taking into account the results from Ref. 2, Fedorov[1] uses as the circular skewness $\gamma_c(\phi)$ and kurtosis $\varepsilon_c(\phi)$ the following quantities: $\gamma_c(\phi) = -\beta_2(\mu)/[2\sqrt{2}(1-\rho)^{3/2}]$ and $\varepsilon_c(\phi) = [\alpha_2(\mu) - \rho^4]/[2(1-\rho)^2]$.

Estimates of these circular parameters are based on sampling trigonometric moments with respect to zero direction[1,2]:

$$a_p = \hat{\alpha}_p = \frac{1}{N}\sum_{i=1}^{N}\cos p\phi(i),\quad b_p = \hat{\beta}_p = \frac{1}{N}\sum_{i=1}^{N}\sin p\phi(i), \tag{4}$$

where N is the number of current values $\phi(i)$. Then, the estimates of the average direction and standard deviation ϕ have the form

$$\hat{\mu} = \hat{M}_c(\phi) = \arg(a_1 + jb_1);\quad \hat{\sigma}_c(\phi) = \sqrt{-2\ln r},$$

where $r = \hat{\rho} = \sqrt{a_1^2 + b_1^2}$. The sampling central moments $a_p(\hat{\mu}) = \hat{\alpha}_p(\hat{\mu})$ and $b_p(\hat{\mu}) = \hat{\beta}_p(\hat{\mu})$ can be obtained with the use of formulas (3). At the same time, estimates of circular skewness and kurtosis take the form of

$$\hat{\gamma}_c(\phi) = -b_2(\hat{\mu})/[2\sqrt{2}(1-r)^{3/2}],$$

$$\hat{\varepsilon}_c(\phi) = [a_2(\hat{\mu}) - r^4]/[2(1-r)^2].$$

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Expression for the standard error of estimate of circularly average direction $\sigma[\hat{M}_c(\phi)]$ is presented in Ref. 2, while expressions for $\sigma[\hat{\sigma}_c(\phi)]$, $\sigma[\hat{\gamma}_c(\phi)]$, and $\sigma[\hat{\varepsilon}_c(\phi)]$ were obtained in Ref. 1. They are determined by N and corresponding combinations of the moments α_p and β_p up to the fourth order inclusive.

Fedorov[1] notes that in calculation of the circular parameters of horizontal wind velocity one can use two equally valuable variants of $\phi(i)$ determination. In the first, standard approach, we use as $\phi(i)$ the usual meteorological direction of wind velocity $\varphi(i)$, calculated from current values of the orthogonal components $V_x(i)$ and $V_y(i)$ of the vector $\mathbf{V}_h$:

$$\varphi(i) = \arctan[V_y(i)/V_x(i)] + C_\varphi, \quad (0 \le \varphi(i) < 2\pi), \tag{5}$$

where C_φ is the multiple of $\pi/2$, determined by positions of $V_x(i)$ and $V_y(i)$ on the coordinate plane. It is usually assumed that the X- (Y-) axis points due north (east), while angles are counted off the northern direction clockwise.[3] In the second approach, we use as $\phi(i)$ the deviations $\theta'(i)$ from the current values $\varphi(i)$ of the "instant" vectors $\mathbf{V}_h(i)$ from meteorological direction θ of the average vector $M(\mathbf{V}_h)$:

$$\begin{cases} \theta'(i) = \varphi(i) - \theta = \arctan[v(i)/u(i)] + C_\theta, \\ (-\pi \le \theta'(i) < \pi), \end{cases} \tag{6}$$

where u, v are the longitudinal and latitudinal components of the horizontal velocity $\mathbf{V}_h$ and C_θ minimizes $\theta'(i)$ in absolute value. Note that positive (negative) directions of the angles are measured clockwise (counterclockwise). The calculated values of the angular parameters and their standard errors due to the use of statistics φ and θ' coincide. It only should be kept in mind that $M_c(\theta') = \mu' = M_c(\varphi) - \theta = \mu - \theta$. (In the general case, $\mu' \ne 0$, see Ref. 1).

Thus, traditional use of the methods of circular statistics for estimating angular parameters of horizontal wind velocity necessitates preliminary calculation of N values of $\varphi(i)$ Eq. (5) or $\theta'(i)$ Eq. (6) at each chosen height. Moreover, in addition to immediate calculation of function arctan, for correct retrieval of $\varphi(i)$ (or $\theta'(i)$) it is necessary to choose C_φ (or C_θ) value N times out of its five possible variants.

On the other hand, the determination of circular parameters themselves and their accuracy characteristics calls for calculation of $8N$ values of trigonometric functions $\cos p\varphi(i)$, $\sin p\varphi(i)$ (or $\cos p\theta'(i)$, $\sin p\theta'(i)$), $p = 1, 2, 3, 4$ more. As a result, the time required to accomplish these calculations throughout the altitude range may be quite long, substantially slowing down sodar measurements of the entire set of wind parameters and other characteristics. Therefore, presently the processing software system of Volna-3 sodar incorporates another, simpler and faster, method of determination of circular parameters of the horizontal wind velocity. It does not assume calculation of current angular $\varphi(i)$ (or $\theta'(i)$) values and their cosine and sine coordinates (4), it is rather based on equivalent replacement of the latter by certain functions of the corresponding orthogonal components of the vector $\mathbf{V}_h$. Moreover, it is more suitable for a microprocessor implementation.

1. Relation between the trigonometric moments of the vector $\mathbf{V}_h$ and its orthogonal components

First, we shall consider the variant, which uses ordinary Cartesian V_x, and V_y-components. We assume that $V_x(i) < 0$ and $V_y(i) > 0$. Then, in formula (5) $C_\varphi = 2\pi$ (see Ref. 3) and $3\pi/2 < \varphi(i) < 2\pi$. Introducing the variable $z = V_y/V_x < 0$ and using well known trigonometric identites[4] we obtain:

$$\cos\varphi = \cos[\arctan(z) + 2\pi] =$$

$$= \cos\left(-\arccos\frac{1}{\sqrt{1+z^2}}\right) = \frac{1}{\sqrt{1+z^2}} = \frac{|V_x|}{V_\mathrm{m}} = -\frac{V_x}{V_\mathrm{m}};$$

$$\sin\varphi = \sin[\arctan(z) + 2\pi] =$$

$$= \sin\left(\arcsin\frac{z}{\sqrt{1+z^2}}\right) = \frac{z}{\sqrt{1+z^2}} = \frac{V_y\,|V_x|}{V_\mathrm{m}\,V_x} = -\frac{V_y}{V_\mathrm{m}},$$

where

$$V_\mathrm{m}(i) = |\mathbf{V}_h(i)| = \sqrt{V_x^2(i) + V_y^2(i)}$$

is the absolute value of the current horizontal wind velocity. Analogously, we can show the validity of the obtained relations for all other possible positions of $V_x(i)$, $V_y(i)$ on the considered coordinate plane, including the case with $V_x(i) = 0$. Then, taking into account trigonometric identities,[4] the above-mentioned definitions of cosine- (α_p) and sine- (β_p) moments of the random angle φ can be written in terms of variables $z_x = V_x/V_\mathrm{m}$ and $z_y = V_y/V_\mathrm{m}$ in equivalent form as:

$$\begin{cases} \alpha_1 = M(\cos\varphi) = -M(z_x), \\ \alpha_2 = M(\cos 2\varphi) = 1 - 2M(z_y^2), \\ \alpha_3 = M(\cos 3\varphi) = -3\alpha_1 - 4M(z_x^3), \\ \alpha_4 = M(\cos 4\varphi) = 1 - 8M(z_x^2 z_y^2), \\ \beta_1 = M(\sin\varphi) = -M(z_y), \\ \beta_2 = M(\sin 2\varphi) = 2M(z_x z_y), \\ \beta_3 = M(\sin 3\varphi) = 3\beta_1 + 4M(z_y^3), \\ \beta_4 = M(\sin 4\varphi) = 2\beta_2 - 8M(z_x z_y^3). \end{cases} \tag{7}$$

From this, the ensuing analogs of expressions (4) for the corresponding sampling trigonometric moments $a_p = \hat{\alpha}_p$ and $b_p = \hat{\beta}_p$ of directions $\varphi(i)$ are:

$$\begin{cases} a_1 = -\dfrac{1}{N}\sum_{i=1}^{N} z_x(i), \\ a_2 = 1-\dfrac{2}{N}\sum_{i=1}^{N} z_y^2(i), \\ a_3 = -3a_1-\dfrac{4}{N}\sum_{i=1}^{N} z_x^3(i), \\ a_4 = 1-\dfrac{8}{N}\sum_{i=1}^{N} z_x^2(i)z_y^2(i), \\ b_1 = -\dfrac{1}{N}\sum_{i=1}^{N} z_y(i), \\ b_2 = \dfrac{2}{N}\sum_{i=1}^{N} z_x(i)z_y(i), \\ b_3 = 3b_1+\dfrac{4}{N}\sum_{i=1}^{N} z_y^3(i), \\ b_4 = 2b_2-\dfrac{8}{N}\sum_{i=1}^{N} z_x(i)z_y^3(i). \end{cases} \quad (8)$$

We now proceed to discussion of the variant of using uv-components of the vector $\mathbf{V}_h$, which determines $\theta'(i)$. In this case, for all possible current values of $u(i)$ and $v(i)$ it is fulfilled that $\cos\theta'(i) = z_u(i)$, $\sin\theta'(i) = z_v(i)$, where $z_u(i) = u(i)/V_m(i)$, $z_v(i) = v(i)/V_m(i)$, and the absolute value of $\mathbf{V}_h(i)$ is written as $V_m(i) = \sqrt{u^2(i)+v^2(i)}$. As a verifying example, we assume that $u(i) < 0$ and $v(i) > 0$. For this, the corresponding value $C_\theta = \pi$, in expression (6), and $\pi/2 < \theta'(i) < \pi$. Using the variable $z = v/u < 0$, we obtain

$$\cos\theta' = \cos[\arctan(z)+\pi] = -\cos\left(-\arccos\frac{1}{\sqrt{1+z^2}}\right) = = -\frac{1}{\sqrt{1+z^2}} = -\frac{|u|}{V_m} = \frac{u}{V_m},$$

$$\sin\theta' = \sin[\arctan(z)+\pi] = -\sin\left(\arcsin\frac{z}{\sqrt{1+z^2}}\right) = = -\frac{z}{\sqrt{1+z^2}} = -\frac{v|u|}{V_m u} = \frac{v}{V_m}.$$

Then, the equivalent relations for cosine- (α'_p) and sine- (β'_p) moments of the random angle θ' can be written as

$$\begin{cases} \alpha'_1 = M(\cos\theta') = M(z_u), \\ \alpha'_2 = M(\cos 2\theta') = 1-2M(z_v^2), \\ \alpha'_3 = M(\cos 3\theta') = -3\alpha'_1+4M(z_u^3), \\ \alpha'_4 = M(\cos 4\theta') = 1-8M(z_u^2 z_v^2), \\ \beta'_1 = M(\sin\theta') = M(z_v), \\ \beta'_2 = M(\sin 2\theta') = 2M(z_u z_v), \\ \beta'_3 = M(\sin 3\theta') = 3\beta'_1-4M(z_v^3), \\ \beta'_4 = M(\sin 4\theta') = 2\beta'_2-8M(z_u z_v^3). \end{cases} \quad (9)$$

From this, the ensuing expressions for considered sampling trigonometric moments a'_p and b'_p of directions $\theta'(i)$ are:

$$\begin{cases} a'_1 = \dfrac{1}{N}\sum_{i=1}^{N} z_u(i), \\ a'_2 = 1-\dfrac{2}{N}\sum_{i=1}^{N} z_v^2(i), \\ a'_3 = -3a'_1+\dfrac{4}{N}\sum_{i=1}^{N} z_u^3(i), \\ a'_4 = 1-\dfrac{8}{N}\sum_{i=1}^{N} z_u^2(i)z_v^2(i), \\ b'_1 = \dfrac{1}{N}\sum_{i=1}^{N} z_v(i), \\ b'_2 = \dfrac{2}{N}\sum_{i=1}^{N} z_u(i)z_v(i), \\ b'_3 = 3b'_1-\dfrac{4}{N}\sum_{i=1}^{N} z_v^3(i), \\ b'_4 = 2b'_2-\dfrac{8}{N}\sum_{i=1}^{N} z_u(i)z_v^3(i). \end{cases} \quad (10)$$

Thus, in estimating circular parameters of the horizontal wind velocity, the use of formulas (7)–(10) allows us, without loss of accuracy, to avoid direct calculation of the current angular values $\varphi(i)$ (or $\theta'(i)$) and their cosine or sine coordinates (4). At the same time, for introduced trigonometric moments, the important relations following from initial definitions (1)–(3) are yet valid:

$$\alpha'_p = \alpha_p(\theta), \quad \beta'_p = \beta_p(\theta), \quad (11)$$

$$\alpha'_p(\mu') = \alpha_p(\mu), \quad \beta'_p(\mu') = \beta_p(\mu). \quad (12)$$

In particular, from the expression (12) it follows that circular characteristics of scattering, namely ρ, $\sigma_c(\phi)$, skewness $\gamma_c(\phi)$, and kurtosis $\varepsilon_c(\phi)$, are invariant with respect to the change of origin of angles ϕ, consistent with calculations by Fedorov.[1] Analogously, in the classical linear statistics, the characteristics of scattering, skewness, and kurtosis, expressed in terms of the corresponding central moments, do not depend on the position of origin on the scale of values of the observed random quantity.

2. Relation between circular standard deviation of the vector $\mathbf{V}_h$ and parameters of its uv-components

The equivalent relations presented are useful in that they provide better insight into physical meaning of the circular parameters considered. The introduction of the latter relies on the fact that for small variations δ of the random angle ϕ, its distribution $W(\phi)$ over the circle is close to the distribution over the corresponding short interval of a straight line.[2] Next, based on definitions of statistical characteristics of $W(\phi)$ on a straight line, their circular analogs are obtained. More complicated is the situation with circular standard deviation $\sigma_c(\phi) = \sqrt{-2\ln\rho}$. Mardia[2] argues that "informative interpretation of this characteristic, based on analogies with scattering characteristics on the straight line, is only possible in the case of distributions close to wound normal". If there is not such a confidence, the circular variance $D_c(\phi) = 1 - \rho$ is recommended for use as a measure of variance of the random directions ϕ. However, analysis of large amount of experimental data, obtained using a Volna-3 sodar, allows us to conclude that it is the possible to use $\sigma_c(\varphi) = \sigma_c(\theta')$ as a characteristic of angular standard deviation $\sigma(\phi)$ of horizontal wind velocity. This is supported by Fig. 6 from Ref. 1, which presents altitude profiles of $\sigma(\phi)$, obtained using $\hat{\sigma}_c(\varphi)$ and three other $\hat{\sigma}(\theta')$ estimates. One of them, namely, $\hat{\sigma}_1(\theta')$, corresponds to the linear part of expansion into Taylor series about the means $M(u)$ and $M(v) = 0$ of the initial relation (6) for θ', that is we use the formula

$$\sigma_1(\varphi) = \sigma_1(\theta') = I_v = \sigma(v)/M(u),$$

where I_v is intensity of turbulence for the v-component; and $\sigma(v)$ is the corresponding standard deviation. This approximation is widely used in practice[5–7] and considered to be valid if pulsations of wind direction do not exceed 20–30°. The second estimate, $\hat{\sigma}_2(\theta')$, is based on accounting for the quadratic terms in expansion (6), i.e., on the relationship

$$\sigma_2(\theta') = I_v\sqrt{1 + I_u^2},$$

where $I_u = \sigma(v)/M(u)$ is the intensity of turbulence for the u-component. The third approach consists in transformation of the initial angular distributions $W(\theta')$ from the circle to the interval of a straight line $-\pi \le \theta' < \pi$ and subsequent use of the methods of linear statistics (the estimate $\hat{\sigma}_{cl}(\theta')$). This picture shows high correlation of standard angular deviations $\sigma(\phi)$, obtained by all the four methods. It is noteworthy, that the maximum values (up to 60° for $\hat{\sigma}_c(\varphi)$) and largest variance $\hat{\sigma}(\theta')$ are observed near underlying surface at low wind velocities. At other altitudes, where the increase of wind velocity and decrease of $\sigma(\phi)$ down to approximately 25° are observed, the differences between all estimates $\hat{\sigma}(\phi)$ substantially decrease.

Now we shall obtain the relation of circular standard deviation $\sigma_c(\phi)$ of the vector $\mathbf{V}_h$ with the parameters of its uv-components. For this, taking into account the definitions introduced above, we express the square of $\sigma_c(\phi)$ through the trigonometric first-order cosine- and sine-moments of directions θ':

$$\sigma_c^2(\phi) = \sigma_c^2(\theta') = -2\ln\sqrt{\alpha_1'^2 + \beta_1'^2}.$$

Next, expanding into Taylor series about $\alpha_1' = 1$ and $\beta_1' = 0$ up to quadratic terms inclusive we obtain

$$\sigma_c^2(\phi) = 2(1-\alpha_1') + (1-\alpha_1')^2 - \beta_1'^2.$$

At the same time, from the equivalent relations (9) it follows that:

$$\alpha_1' = M\left\{u/\sqrt{u^2+v^2}\right\} \text{ and } \beta_1' = M\left\{v/\sqrt{u^2+v^2}\right\}.$$

Taking into account the quadratic terms in expansions of these expressions into Taylor series about the means $M(u)$ and $M(v) = 0$ and taking the required averages, we obtain

$$\alpha_1' = 1 - \sigma^2(v)/2M^2(u) = 1 - I_v^2/2;$$

$$\beta_1' = -\mathrm{cov}(u,v)/M^2(u),$$

where $\mathrm{cov}(u, v)$ is the correlation moment between uv-components of the vector $\mathbf{V}_h$. Therefore, the first-order cosine-moment for θ' is related with the intensity of turbulence for the v-component. At the same time, the relation for β_1' is identical to the expression obtained in Ref. 1 for $M(\theta')$. Thus, the sine-moment β_1' characterizes the shift of the mean direction of the "instant" vectors $\mathbf{V}_h(i)$ from θ direction of the mean vector $M(\mathbf{V}_h)$. As noted in Ref. 1, in most cases this shift is not large; it is significant mainly at low wind velocities near underlying surface. In any case, the contribution of $\beta_1'^2$ to $\sigma_c^2(\phi)$ as compared with other terms can be neglected and we may assume the validity of the relation

$$\sigma_c^2(\phi) \cong 2(1-\alpha_1') + (1-\alpha_1')^2.$$

Taking into account the expansion for α_1' we finally obtain

$$\sigma_c(\phi) \cong \frac{\sigma(v)}{M(u)}\sqrt{1 + \frac{\sigma^2(v)}{4M^2(u)}} = \sigma_1(\theta')\sqrt{1 + \frac{\sigma_1^2(\theta')}{4}}. \quad (13)$$

Note that in derivation of Eq. (13) we did not use the assumption on functional form of the angular distribution $W(\phi)$ of vector $\mathbf{V}_h$. Therefore, they also must be valid for $W(\phi)$ other than wound normal distribution and Mises distribution close to it.

Thus, the circular standard deviation $\sigma_c(\phi)$ has clear physical meaning. For small angular fluctuations δ of vector $\mathbf{V}_h$ it practically corresponds to usual standard deviation $\sigma_1(\theta')$ arising from the linear part of the expansion of the initial relation (6) for θ' into Taylor series. For large δ, the standard deviation $\sigma_c(\phi)$ much better corresponds to $\sigma_2(\theta')$, where the quadratic terms of the expansion (6) are also taken into account. These conclusions totally agree with the data of real measurements presented in Ref. 1, Fig. 6. Moreover, from Ref. 1, Figs. 7 and 8, it also follows that almost throughout the entire altitude range considered, the angular distributions of vector $\mathbf{V}_h$ significantly differ from wound normal and Mises distributions. Only above 370 m we can expect the validity of these distributions. Summarizing the above said, we can conclude that in acoustic sounding of the atmosphere, with its characteristic spatiotemporal averaging and data sampling scales, the use of circular standard deviation $\sigma_c(\phi)$ as a measure of angular variance of the horizontal wind velocity is well substantiated and can be successful in practical applications.

3. Standard errors of estimates of circular parameters of the vector $\mathbf{V}_h$

For completeness of presentation, we would like to show the identity of standard errors of estimation of the above-mentioned circular parameters, independent of the form of the angular statistics of φ or θ' used. Although this fact was already discussed elsewhere,[1] it also followed from the calculations performed, and did not directly follow from analytical relations which were mostly presented in terms of cosine- (α_p) and sine- (β_p) moments of the random angle φ with respect to zero direction $\varphi_0 = 0$. At the same time, the relation between α_p, β_p and α'_p, β'_p of angles φ and θ' is determined by expressions (11), which complicates the comparative analysis of the indicated errors. Therefore, we will write these relations in terms of central trigonometric moments. For this, we shall use well known relations $(\cos\mu = \alpha_1/\rho,\ \sin\mu = \beta_1/\rho)^2$ and their corollaries $(\cos 2\mu = (\alpha_1^2 - \beta_1^2)/\rho^2,\ \sin 2\mu = 2\alpha_1\beta_1/\rho^2)$. Next, taking into account formulas (3), from Ref. 1 we obtain expressions for standard errors of estimates of circular average and standard deviation of the directions φ in the new form sought:

$$\sigma[\hat{M}_c(\varphi)] = \sigma[\hat{\mu}] = \sqrt{[1-\alpha_2(\mu)]/2N\rho^2},$$

$$\sigma[\hat{\sigma}_c(\varphi)] = \sigma(r)/\rho\sigma_c(\varphi),$$

where

$$D[r] = \sigma^2(r) = [1 - 2\rho^2 + \alpha_2(\mu)]/2N$$

is the variance of the absolute value of sampling average vector with the current coordinates $\{\cos\varphi(i), \sin\varphi(i)\}$.

The relation for standard measurement error of circular skewness, given in Ref. 1, can be presented as follows

$$\sigma[\hat{\gamma}_c(\varphi)] = \left\{\frac{1}{8(1-\rho)^3}D[b_2(\hat{\mu})] + \frac{9\gamma_c^2(\varphi)}{4(1-\rho)^2}D[r] - \right.$$

$$\left. -\frac{3\gamma_c(\varphi)}{2\sqrt{2}(1-\rho)^{5/2}}\mathrm{cov}[b_2(\hat{\mu}),r]\right\}^{1/2},$$

where

$$D[b_2(\hat{\mu})] = D[b_2(\mu)] + 4\alpha_2(\mu)\times$$

$$\times[\alpha_2(\mu)D(\hat{\mu}) + \alpha_3(\mu)/2N\rho - 1/2N]$$

is the sampling variance of the second-order central sine-moment [taken from Ref. 1 after a series of trigonometric transformations with the use of the above-mentioned relations for $\cos\mu$, $\sin\mu$, and Eq. (3)],

$$D[b_2(\mu)] = [1 - \alpha_4(\mu) - 2\beta_2^2(\mu)]/2N$$

is the part of $D[b_2(\hat{\mu})]$, accounting for the neglect of fluctuations of sampling $\hat{\mu}$ value with respect to the true value $\mu = M_c(\varphi)$. It is also valid that:

$$\mathrm{cov}[b_2(\hat{\mu}),r] = \mathrm{cov}[b_2(\mu),r] - 2\alpha_2(\mu)\mathrm{cov}[\hat{\mu},r],$$

where

$$\mathrm{cov}[b_2(\mu),r] = [\beta_3(\mu) - 2\rho\beta_2(\mu)]/2N,$$

$$\mathrm{cov}[\hat{\mu},r] = \beta_2(\mu)/2N\rho.$$

Expression for the standard measurement error in circular skewness in terms of central trigonometric moments has the form

$$\sigma[\hat{\varepsilon}_c(\varphi)] = \left\{D[a_2(\hat{\mu})] + 16[\varepsilon_c(1-\rho) - \rho^3]2D[r] + \right.$$

$$\left. +8[\varepsilon_c(1-\rho) - \rho^3]\mathrm{cov}[a_2(\hat{\mu}),r]\right\}^{1/2} \Big/ \left[2(1-\rho)2\right],$$

where

$$D[a_2(\hat{\mu})] = D[a_2(\mu)] + 4\beta_2(\mu)[\beta_2(\mu)D(\hat{\mu}) + \beta_3(\mu)/2N\rho],$$

$$D[a_2(\mu)] = [1 + \alpha_4(\mu) - 2\alpha_2^2(\mu)]/2N,$$

$$\mathrm{cov}[a_2(\hat{\mu}),r] = \mathrm{cov}[a_2(\mu),r] + 2\beta_2(\mu)\mathrm{cov}[\hat{\mu},r],$$

$$\mathrm{cov}[a_2(\mu),r] = [\rho + \alpha_3(\mu) - 2\rho\alpha_2(\mu)]/2N.$$

Note that for symmetric and unimodal angular distributions, the central sine-moments equal zero, that is, $\beta_p(\mu) = 0$ (Ref. 2). Then, $D[a_2(\hat{\mu})] = D[a_2(\mu)]$ and $\mathrm{cov}[a_2(\hat{\mu}),r] = \mathrm{cov}[a_2(\mu),r]$. Therefore, in this case, the contribution of fluctuations of sampling circular average $\hat{\mu}$ to $\sigma[\hat{\varepsilon}_c(\varphi)]$ can be neglected.

From analysis of the relations presented here and by virtue of expression (12) we deduce the sought identity of the standard errors of estimation of all considered circular parameters independent of the form of the initial angular statistics of φ or θ'.

In conclusion, we should like to note that the method of calculation of circular parameters presented can be used in other technical applications, where the analyzed angular characteristics are based on the corresponding orthogonal components.

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