The equation for mean field of a wave in statistically anisotropic random medium

E.Z. Gribova and A.I. Saichev

N.I. Lobachevskii State University, Nizhnii Novgorod

Received September 17, 2001

The closed integrodifferential equation for the mean field in a medium with spindle-shaped inhomogeneities oriented along the direction of propagation of an incident wave is derived using local Chernov's method. Diffraction by the inhomogeneities and the change of the scattering efficiency with small changes in the wave propagation angles are taken into account. The derived equation is analyzed for the cases of weak and strong anisotropy. It is proposed to use the results of this study for analysis of data of remote sensing of randomly inhomogeneous media.

The statistics of waves of various natures in large-scale randomly inhomogeneous media is now studied rather thoroughly. As a rule, analysis is based on equations for moment functions of waves. Such equations derived in the small-angle approximation of quasioptics and the Markovian approximation have become classic and can be found in many books (see, e.g., Refs. 1–3). However, these equations are inapplicable in the media, where random inhomogeneities are oriented along the direction of wave propagation, because they do not describe a significant change in scattering at minor variations of the wave propagation angle, as well as weakening of scattering due to diffraction by the inhomogeneities of the medium. At the same time, the situation of a statistically anisotropic medium is quite widespread. Thus, ice crystals in clouds have a shape of cylinders oriented randomly in space.[4] An example important for the ocean physics is random inner waves, because the horizontal scale of the spatial correlation of sonic speed fluctuations is known to be always much larger than the vertical one.[5]

When deriving equations for the moment functions of waves in such media, it is necessary to reject the Markovian approximation ignoring the finite longitudinal scale of correlation of medium inhomogeneities. The equations for the moment functions of waves in underwater acoustic channels (the case of cake-shaped inhomogeneities) were considered in Ref. 6.

In this paper, we derive and analyze equation for the mean field in the medium with spindle-shaped inhomogeneities oriented along the direction of wave propagation. This model corresponds to the structure of clouds with high content of ice crystals and the obtained results can be used in remote sensing of such clouds.

1. Equation for the mean field

In the small-angle approximation, the propagation of a wave is described by a parabolic equation

$$\frac{\partial u}{\partial x} - \frac{i}{2k}\Delta_\perp u = ik\tilde{n}(x, \boldsymbol{\rho})\, u, \qquad (1)$$

where $u(x, \boldsymbol{\rho})$ is the complex amplitude of the wave; x is the longitudinal coordinate; $\boldsymbol{\rho} = \{y, z\}$ are cross coordinates; $\Delta_\perp$ is the Laplacian in the cross plane; $k = 2\pi/\lambda$ is the wave number; λ is the wavelength; $\tilde{n}(x, \boldsymbol{\rho})$ describes fluctuations of the medium refractive index with the given correlation function

$$\langle \tilde{n}(x, \boldsymbol{\rho})\, \tilde{n}(x + \tau_\circ, \boldsymbol{\rho} + \tau_\perp)\rangle = B_n(\boldsymbol{\tau})$$

(the angle brackets denote averaging over an ensemble of realizations of random inhomogeneities in the medium).

Using Eq. (1), we can derive equation for the mean field $\langle u(x, \boldsymbol{\rho})\rangle$ of the wave in the case of medium inhomogeneities oriented along the axis x. Averaging Eq. (1) gives

$$\frac{\partial \langle u\rangle}{\partial x} - \frac{i}{2k}\Delta_\perp \langle u\rangle = ik\,\langle \tilde{n}\, u\rangle. \qquad (2)$$

The mixed moment $\langle \tilde{n}(x, \boldsymbol{\rho})\, u(x, \boldsymbol{\rho})\rangle$ entering into Eq. (2) can be expressed through $\langle u\rangle$ and B_n using the Chernov's local method of small perturbations.[2,7] Following this method, we assume that in the layer $[x - l, x]$, where $l \ge l_\circ$, $l_\circ$ is the scale of correlation of $\tilde{n}$ along the axis x, medium inhomogeneities weakly affect the wave. Therefore, we can write the field in the mixed moment in the Born approximation:

$$u(x, \boldsymbol{\rho}) = u_0(x, \boldsymbol{\rho}) + u_1(x, \boldsymbol{\rho}), \qquad (3)$$

where the zero approximation $u_0(x, \boldsymbol{\rho})$ is the complex amplitude of the wave in the case when no inhomogeneities in the layer $[x - l, x]$ occur;

$$u_1(x, \boldsymbol{\rho}) = \frac{k^2}{2\pi}\int\limits_{x-l}^{x}\frac{\mathrm{d}x'}{x - x'}\int\limits_{-\infty}^{\infty}\mathrm{d}\boldsymbol{\rho}'\,\tilde{n}(x', \boldsymbol{\rho}')\, u_0(x', \boldsymbol{\rho}')\times$$
$$\times\exp\left[\frac{ik\,(\boldsymbol{\rho} - \boldsymbol{\rho}')^2}{2(x - x')}\right]. \qquad (4)$$

Let us represent $u_0(x, \boldsymbol{\rho})$ as an expansion into plane waves

$$u_0(x, \boldsymbol{\rho}) = \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{\vartheta}\psi_0(x, \boldsymbol{\vartheta}) \exp(ik\boldsymbol{\vartheta\rho}), \quad \boldsymbol{\vartheta} = (\alpha, \beta)$$

(the angle α is measured in the horizontal plane; β is the vertical component).

Since the complex amplitudes of plane waves in a homogeneous medium obey the law

$$\psi_0(x, \boldsymbol{\vartheta}) = \psi_0(x', \boldsymbol{\vartheta}) \exp[-(ik\vartheta^2/2)(x - x')],$$

expressing u_0 through ψ_0 in Eq. (4) and taking into account the last equality, we obtain

$$u_1(x, \boldsymbol{\rho}) = ik \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{\vartheta}\psi_0(x, \boldsymbol{\vartheta}) \exp(ik\boldsymbol{\vartheta\rho}) \int_0^l \mathrm{d}\tau_\diamond \times$$

$$\times \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{\tau}_\perp \tilde{n}(x - \tau_\diamond, \boldsymbol{\rho} - \boldsymbol{\tau}_\perp - \boldsymbol{\vartheta}\tau_\diamond) \frac{k}{2\pi i\tau_\diamond} \exp\left(\frac{ik\,\tau_\perp^2}{2\tau_\diamond}\right). \quad (5)$$

Let $l_\perp$ be the smallest scale of the inhomogeneities $\tilde{n}$ along the cross coordinates. Then the inequality

$$\sqrt{\tau_\diamond \lambda}/l_\perp \ll 1 \quad (6)$$

corresponds to the ray-optics approximation: the characteristic size of the “aperture” $l_\perp$ is much larger than the radius of the first Fresnel zone $\sqrt{l_\diamond \lambda}$. In this case

$$u_1(x, \boldsymbol{\rho}) = ik \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{\vartheta}\psi_0(x, \boldsymbol{\vartheta}) \exp(ik\boldsymbol{\vartheta\rho}) \times$$

$$\times \int_0^l \mathrm{d}\tau_\diamond \tilde{n}(x - \tau_\diamond, \boldsymbol{\rho} - \boldsymbol{\vartheta}\tau_\diamond). \quad (7)$$

Note that the condition (6) can be rewritten as

$$kl_\perp^2/l_\diamond \gg 1 \quad \text{or} \quad \vartheta_\mathrm{a} \gg \vartheta_\mathrm{sc}, \quad (8)$$

where $\vartheta_\mathrm{a} = l_\perp/l_\diamond$ is referred to as the angle of anisotropy, and $\vartheta_\mathrm{sc} = 1/kl_\perp$ is the scattering angle. Let us introduce the parameter of anisotropy as $m = \vartheta_\mathrm{sc}/\vartheta_\mathrm{a} = l_\diamond/kl_\perp^2$. We speak about weak anisotropy, if the condition $m \ll 1$ is satisfied. Thus, representation of the field in the form (7) is possible just in the case of weak anisotropy. For example, if $l_\perp = l_\diamond$, then the condition (8) transforms into $kl_\diamond \gg 1$ or $\vartheta_\mathrm{sc} \ll 1$, i.e., the condition of large-scale medium inhomogeneities. This condition must be satisfied for the parabolic equation (1) to be valid. Therefore, in the case of isotropic large-scale inhomogeneities, when Eq. (1) is valid, the approximation (7) is valid too. It is obvious that condition (8) and the approximation (7) are true for the inhomogeneities that are “oblate” ($l_\diamond < l_\perp$) along the axis. It should be noted that if ϑ^*, is the largest (in the absolute value) wave propagation angle measured from the axis x, satisfy the inequality

$$\vartheta^* \ll \vartheta_\mathrm{a}, \quad (9)$$

what is equivalent to the condition $\vartheta\tau_\diamond \sim \vartheta^* l_\diamond \ll l_\perp$, then the approximation (7) reduces to a simpler form:

$$u_1(x, \boldsymbol{\rho}) = ik\, u_0(x, \boldsymbol{\rho}) \int_{x-l}^{x} \mathrm{d}x'\, \tilde{n}(x', \boldsymbol{\rho}').$$

Substituting this equation into Eqs. (3) and (2) yields equation for the mean field in the Markovian approximation. The situation becomes different in principle, if random inhomogeneities are oriented along the direction of wave propagation, that is, if $l_\diamond > l_\perp$ and, consequently, the anisotropy angle $\vartheta_\mathrm{a} \ll 1$. In this case, the conditions (8) and (9) may be violated, even if the parabolic equation (1) is applicable. Therefore, in deriving equation for the mean field in this case, one should use equation (5) that allows for diffraction by the inhomogeneities and variation of scattering at small variations of the wave propagation angles, rather than the approximation (7). Then, under condition that extinction of the mean field due to scattering is low in the layer with the thickness $l \sim l_\diamond$, we obtain closed integrodifferential equation for the mean field

$$\frac{\partial \langle u \rangle}{\partial x} - \frac{i}{2k} \Delta_\perp \langle u \rangle + \frac{k^2}{8} \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{\rho}' \langle u(x, \boldsymbol{\rho} - \boldsymbol{\rho}') \rangle\, a(\boldsymbol{\rho}') = 0, \quad (10)$$

where

$$a(\boldsymbol{\rho}') = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{\vartheta}\, A(\boldsymbol{\vartheta}) \exp(ik\,\boldsymbol{\vartheta\rho});$$

$$A(\boldsymbol{\vartheta}) = 8\int_0^{\infty} \mathrm{d}\tau_\diamond \int_{-\infty}^{\infty} \mathrm{d}\tau_\perp B_n(\tau_\diamond, \tau_\perp + \boldsymbol{\vartheta}\tau_\diamond) \frac{k}{2\pi i\tau_\diamond} \exp\left(\frac{ik\,\tau_\perp^2}{2\tau_\diamond}\right).$$

2. Analysis of aspect sensitivity

The form of Eq. (10) for the mean field is different in the cases of weak and strong anisotropy. To demonstrate this, let us consider the case, that can be practically interesting, of spindle-shaped inhomogeneities having the scale $l_\diamond$ along the axis x and the scale $l_\perp < l_\diamond$ in the cross plane yz. For definiteness, let us take the Gaussian correlation function for the inhomogeneities

$$B_n(\tau_\diamond, \tau_\perp) = \sigma_n^2 \exp[-\tau_\diamond^2/2l_\diamond^2 - \tau_\perp^2/2l_\perp^2].$$

This gives

$$A(\vartheta) = 8\,\sigma_n^2\, l_\diamond \int_0^{\infty} \frac{\mathrm{d}\tau}{1 + im\tau} \times$$

$$\times \exp\left[-\frac{\tau^2}{2}\left(1 + \frac{\vartheta^2}{\vartheta_\mathrm{a}^2(1 + im\tau)}\right)\right]; \quad (11)$$

$$a(\rho) = \frac{k}{2\pi} 8\,\sigma_n^2 \int_0^{\infty} \frac{\mathrm{d}\tau}{m\tau^2} \exp\left[-\frac{\tau^2}{2}\left(1 + \frac{\rho^2(1 + im\tau)}{l_\perp^2 m^2 \tau^4}\right)\right]. \quad (12)$$

Let us consider the asymptotic behavior of the coefficient $A(\vartheta)$ in the case of weak and strong anisotropy. As follows from Eq. (11), at $m \ll 1$ we have

$$A(\vartheta) = 4\sqrt{2\pi}\,\sigma_n^2\, l_\diamond / \sqrt{1 + \vartheta^2/\vartheta_a^2}\,.$$

If $\vartheta/\vartheta_a \ll 1$, i.e., the wave propagates at a small angle to the axis x, then the extinction coefficient proves to be constant:

$$A(\vartheta) = A_0 \approx 4\sqrt{2\pi}\,\sigma_n^2\, l_\diamond.$$

Then from Eq. (12) we obtain

$$a(\boldsymbol{\rho}) = A_0\, \delta(\boldsymbol{\rho}).$$

The resulting equation for the mean field is the same as in the Markovian approximation:

$$\frac{\partial\langle u\rangle}{\partial x} - \frac{i}{2k}\Delta_\perp\langle u\rangle + \frac{k^2}{8} A_0 \langle u\rangle = 0. \qquad (13)$$

In the other limiting case ($m \gg 1$), we can find from Eq. (11) that the coefficient A no longer depends on the scattering angle:

$$A(\vartheta) = A(0) = 8\,\sigma_n^2\, l_\diamond \int\limits_0^\infty \frac{\mathrm{d}\tau}{1 + im\tau} \exp\left(-\frac{\tau^2}{2}\right),$$

wherefrom it follows that equation for the mean field has the form similar to that of Eq. (13), but different extinction coefficient, because it decreases due to diffraction by strongly oblong medium inhomogeneities. The above-said is illustrated in Fig. 1, the plots in which correspond to different values of the parameter m.

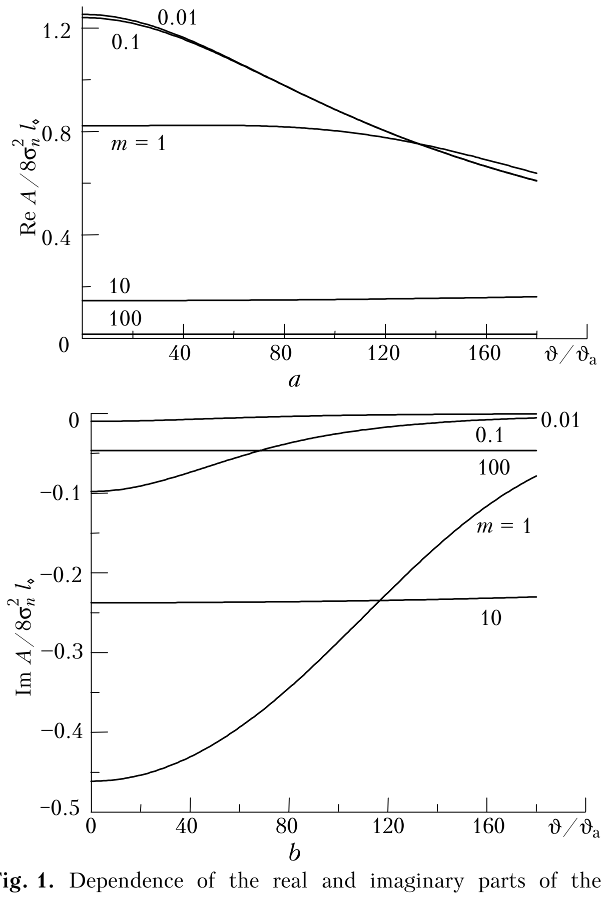

Fig. 1. Dependence of the real and imaginary parts of the extinction coefficient on the angle ϑ at different values of the anisotropy parameter.

Let us now analyze the behavior of the extinction coefficient A at $\vartheta = 0$ as a function of the anisotropy parameter. It is seen from Fig. 2 that at strong anisotropy the coefficient A decreases sharply tending to zero in the direction of incident radiation. Since this coefficient accounts for scattering by medium inhomogeneities, this means that the field in the direction $\vartheta = 0$ decreases – rays deflect from their initial direction.

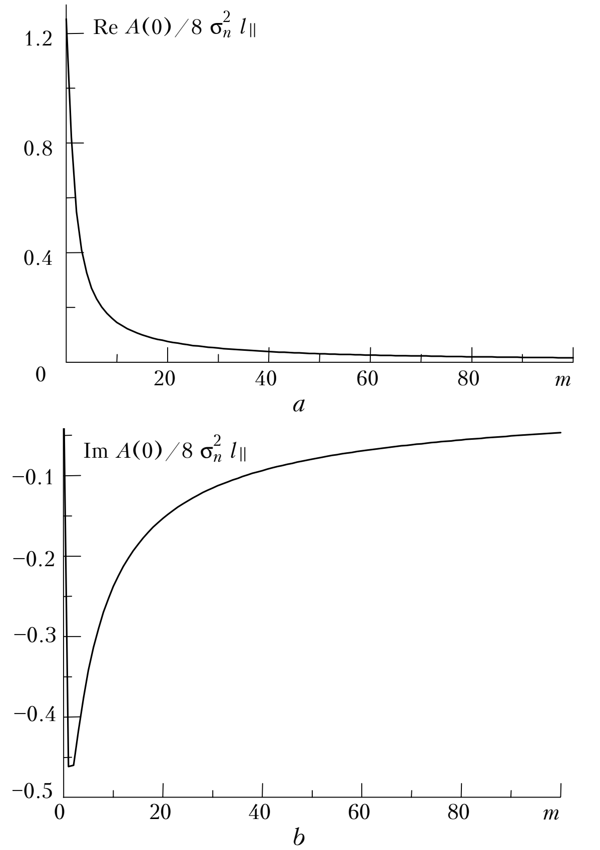

Fig. 2. Dependence of the real and imaginary parts of the extinction coefficient on the anisotropy parameter at $\vartheta = 0$.

The appearance of a local minimum in the angular distribution of the ray intensity in the direction of the longest correlation length of inhomogeneities was actually found at statistical simulation of light scattering by oblong inhomogeneities[8,9] (note that the effect is not observed in the case of isotropic or oblate inhomogeneities). A more detailed analysis based on the transfer equation for the ray intensity is beyond the scope of this paper. However, it can be shown that the obtained angular dependence of the extinction coefficient leads to the appearance of maximum in the ray diffusion coefficient for the direction at $\vartheta = 0$.

The behavior of rays in the medium with such a diffusion coefficient can be interpreted illustratively by using known analogy between optical rays and medium particles.[10] Let us consider Brownian particles being in the medium with the random temperature field. Since the particle diffusion coefficient at a constant pressure is directly proportional to temperature, particles are pushed out from warmer areas to colder ones and stay there for rather a long time. As a result, the mean concentration of an admixture is minimum in the areas, where the diffusion coefficient is high.

Similarly, in the problem on wave propagation considered for the case of statistically inhomogeneous medium, in which the diffusion coefficient has a maximum in the direction of wave incidence, rays deflect from this direction and the resulting distribution of the ray intensity has two maxima.

Since the determined angular dependence $A(\vartheta)$ is caused by the oblong shape of random inhomogeneities, the obtained results are applicable to analysis of data on wave scattering at remote sensing of randomly inhomogeneous media.

Acknowledgments

This work was partly supported by the Russian Foundation for Basic Research (Grant No. 00–02–16167) and the Program "Leading Scientific Schools of Russia" (No. 00–15–96619).

References

1. S.M. Rytov, Yu.A. Kravtsov, and V.I. Tatarskii, *Introduction to Statistical Radiophysics.* Part 2. *Random Fields* (Nauka, Moscow, 1978), 463 pp.
2. L.A. Chernov, *Waves in Randomly Inhomogeneous Media* (Nauka, Moscow, 1975), 174 pp.
3. V.I. Klyatskin, *Stochastic Equations and Waves in Randomly Inhomogeneous Media* (Nauka, Moscow, 1980), 366 pp.
4. A.G. Petrushin, in: *Scattering and Absorption of Optical Radiation in Ice Crystal Cloud Medium. Problems of Cloud Physics*, ed. by L.P. Semenov (Gidrometeoizdat, St. Petersburg, 1998), pp. 118–149.
5. S.M. Flatte, ed., *Sound Transmission through a Fluctuating Ocean* (Cambridge Univ. Press, 1979).
6. A.I. Saichev and M.M. Slavinskii, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **28**, No. 1, 75–83 (1985).
7. L.A. Chernov, Akust. Zh. **15**, No. 4, 594–603 (1969).
8. A.V. Aistov and V.G. Gavrilenko, Opt. Spektrosk. **83**, No. 3, 427–432 (1997).
9. A.I. Saichev, V.G. Gavrilenko, E.Z. Gribova, and A.V. Aistov, Proc. SPIE **4338**, 17–26 (2000).
10. S.N. Gurbatov, A.N. Malakhov, and A.I. Saichev, *Nonlinear Random Waves in Dispersion-Free Media* (Nauka, Moscow, 1990), 216 pp.