DETERMINATION OF THE INSTRUMENTAL FUNCTION IN MEASUREMENTS OF SMALL-ANGLE SCATTERING PHASE FUNCTIONS

V.V. Veretennikov

Institute of Atmospheric Optics, Siberian Branch of the Russian Academy of Sciences, Tomsk
Received June 10, 1998

An exact analytical solution of the problem on determining the instrumental function from measurement data on scattering phase function is obtained in a small-angle approximation taking into account finite angular dimensions of a light source and a receiver. It is shown that the instrumental function can be described in terms of the integral of a product of Bessel functions. A method is proposed for representing instrumental function by elementary functions. The influence of the instrumental function on the accuracy of measurements of small-angle scattering phase functions is being discussed for the case of varying particle size.

1. INTRODUCTION

The scattering phase function of large particles in the region of small scattering angles is highly sensitive to variations in the microstructure of scattering particles' ensemble. This has motivated the development of optical methods for diagnostics of coarse disperse media by solving the inverse problem using data on small-angle scattering phase functions. The investigations into this problem have been initiated by authors of Refs. 1 and 2. The account for multiple scattering effects, when solving the problems of diagnostics of coarse disperse media, made necessary the development of methods for inverting the correlation function of particle shadows.[3,4] The latter function is the Hankel transformation of the diffraction component of the small-angle scattering phase function.

In practice, the measurements of scattering phase functions in the region of small scattering angles are usually hindered because of a finite angular width of a light beam coming from a source and receiver's field of view angle. The joint effect of these factors, that, in the final result, determines the instrumental function of a measuring device, increases with increasing size of scattering particles. As a consequence, this effect restricts the value of the maximum particle size above which the small-angle measurements are no longer informative. To quantify the influence of the instrumental function on the precision of small-angle methods of diagnostics of scattering media, rigorous treatment of light field nearby the incidence direction is needed. The technique and calculated results on the instrumental function of a sun photometer have been discussed in Ref. 5 based on the direct numerical integration, within the frameworks of single-scattering approximation, over the photometer field of view and the Sun disc.

In this paper, the formalism of small-angle radiative transfer equation (RTE) in the single-scattering approximation is used to establish the relation between the signal at a receiver input and the scattering phase function. This approach provides for obtaining simple and versatile analytical expressions that enable easy and straightforward calculations of the instrumental functions at a variety of angular characteristics of the light sources and receivers.

2. THE INITIAL EQUATIONS

We assume that the scattering medium occupies the region $z > 0$, and its optical characteristics are functions of the spatial coordinate z alone. Then we assume that the light flux with an infinite cross section, emitted by an external stationary source, propagates along the OZ axis and produce, on the boundary of the medium in the $z = 0$ plane, the intensity pattern

$$I(\mathbf{n}_\perp, z = 0) = I_0(\mathbf{n}_\perp), \tag{1}$$

where $\mathbf{n}_\perp = (n_x, n_y)$ is the projection of the direction vector $\mathbf{n}$ onto the plane XOY, and $I_0(\mathbf{n}_\perp)$ is only significant when $\gamma = |\mathbf{n}_\perp| \ll 1$. In this case, the light field in the medium is described by the small-angle RTE,[6,7] whose solution in the Fourier-domain can be presented by the product

$$\tilde{I}(\mathbf{p}, z) = \tilde{I}_0(\mathbf{p})\, F(\mathbf{p}), \tag{2}$$

where the tilde is used to denote Fourier transforms with respect to variable $\mathbf{n}_\perp$; and $\mathbf{p}$ is the angular frequency. The factor $F(\mathbf{p})$, after the change of variables, $\boldsymbol{\rho} = \mathbf{p}/k$, $k = 2\pi/\lambda$, where λ is the wavelength of light, becomes a transverse function of the field cross-coherence for the plane incident wave $I_0(\mathbf{n}_\perp) = \delta(\mathbf{n}_\perp)$, where $\delta(\mathbf{n}_\perp)$ is the two-dimensional delta function. In the single-scattering approximation,

$$F(\mathbf{p}) = \exp[-\tau(z)] + B\, \tilde{x}(p), \tag{3}$$

0235-6880/98/10 889-05 $02.00

where $\tau(z) = \int_0^z \sigma_{ext}(t)\, dt$ is the optical depth of the layer along the path $[0, z]$; $B = \Lambda\tau(z) \exp[-\tau(z)]$, $\Lambda = \sigma_{sc}/\sigma_{ext}$ is the single scattering albedo, σ_{ext} and σ_{sc} are the extinction and scattering coefficients; and $\tilde{x}(p)$ is the Hankel transform of a small-angle scattering phase function $x(\gamma)$

$$\tilde{x}(p) = 2\pi \int_0^\infty \gamma\, J_0(p\gamma)\, x(\gamma)\, d\gamma, \quad p = |\mathbf{p}| \tag{4}$$

with the normalization condition being $\iint_{-\infty}^{\infty} x(\mathbf{n}_\perp)\, d\mathbf{n}_\perp = 1$.

When writing the expression (3) it was assumed that the scattering phase function $x(\gamma)$ does not depend on spatial coordinates, and that its individual terms describe either incident or scattered radiation separately.

The inverse Fourier transform of the functions $F(\mathbf{p})$ yields the solution $J(\mathbf{n}_\perp, z)$ to the small-angle RTE for a plane incident wave; in the single scattering approximation, this solution being coincident, within the accuracy to the normalization factor, with the scattering phase function $x(\mathbf{n}_\perp)$ for $\mathbf{n}_\perp \neq 0$:

$$J(\mathbf{n}_\perp, z) = e^{-\tau}\, \delta(\mathbf{n}_\perp) + Bx(\mathbf{n}_\perp). \tag{5}$$

According to expression (2) and convolution theorem, the solution of a small-angle RTE $I(\mathbf{n}_\perp, z)$ can be represented as a two-dimensional convolution of the radiation intensity in free space $I_0(\mathbf{n}_\perp)$ with the function $J(\mathbf{n}_\perp, z)$ from the equation (5):

$$I(\mathbf{n}_\perp, z) = e^{-\tau}\, I_0(\mathbf{n}_\perp) + B(I_0{**}x)(\mathbf{n}_\perp). \tag{6}$$

The first term in expression (6) describes the intensity of attenuated incident radiation, while the second term

$$I_{sc}(\mathbf{n}_\perp, z) = B(I_0{**}x)(\mathbf{n}_\perp) \tag{7}$$

is the intensity of singly scattered radiation which is sought here in the form of a two-dimensional convolution of the scattering phase function $x(\mathbf{n}_\perp)$ with the beam intensity in free space $I_0(\mathbf{n}_\perp)$. Formula (7) can be interpreted in two ways. First, it may be understood as the one describing how a linear system, therein as a scattering medium with a pulse response $x(\mathbf{n}_\perp)$, influences the input signal $I_0(\mathbf{n}_\perp)$; or, secondly, as the expression of the law of transformation of scattering phase function $x(\mathbf{n}_\perp)$ in the measuring device, with the account for properties of the light beam illuminating the medium. Let us concentrate on the second aspect of this interpretation.

3. INSTRUMENTAL FUNCTION IN MEASUREMENTS OF THE BEAM INTENSITY

Now return to the initial expression (2) and take into account the circular symmetry of the system. In that case formula (7) can be written in the form of a Hankel transform

$$I_{sc}(\gamma, z) = B(2\pi)^{-1} \int_0^\infty p\, J_0(\gamma p)\, \tilde{I}_0(p)\, \tilde{x}(p)\, dp. \tag{8}$$

Let us assume that the intensity of incident radiation is distributed according to the following law:

$$I_0(\mathbf{n}_\perp) = I_0\, U_{\gamma_s}(\mathbf{n}_\perp), \tag{9}$$

where $I_0 = \text{const}$;

$$U_{\gamma_s}(\mathbf{n}_\perp) = \begin{cases} 1, & 0 < \gamma < \gamma_s, \quad \gamma = |\mathbf{n}_\perp|, \\ 0, & \gamma > \gamma_s \end{cases} \tag{10}$$

is the unit-step function defined in the plane. The Fourier transform of this function is

$$\tilde{U}_{\gamma_s}(p) = 2\pi \frac{\gamma_s\, J_1(\gamma_s\, p)}{p}. \tag{11}$$

With the account of (4) and (11), the intensity $I_{sc}(\gamma, z)$ in the equation (8) can be expressed via the scattering phase function $x(\omega)$ as

$$I_{sc}(\gamma, z) = 2\pi\, I_0\, B\, \gamma_s \int_0^\infty \omega\, A(\omega, \gamma, \gamma_s)\, x(\omega)\, d\omega, \tag{12}$$

where the weighting function $A(\omega, \gamma, \gamma_s)$ is represented by the integral of the product of Bessel functions, namely

$$A(\omega, \gamma, \gamma_s) = \int_0^\infty J_0(\omega s)\, J_0(\gamma s)\, J_1(\gamma_s s)\, ds. \tag{13}$$

Mathematically, formula (12) expresses the rule of change to one-dimensional integral when calculating the two-dimensional convolution (7) of the scattering phase function $x(\mathbf{n}_\perp)$ with the unit-step function $U_{\gamma_s}(\mathbf{n}_\perp)$ (see expression (10)) in the plane. Reference 8 gives a simple representation of the integral (13) via the elementary functions, which is as follows:

$$\gamma_s\, A(\omega, \gamma, \gamma_s) = \begin{cases} 0, & \omega \geq \gamma + \gamma_s, \\ 0, & \omega \leq |\gamma - \gamma_s|, \quad \gamma > \gamma_s, \\ 1, & \omega \leq |\gamma - \gamma_s|, \quad \gamma_s > \gamma, \\ \dfrac{\alpha}{\pi}, & 0 < |\gamma - \gamma_s| \leq \omega \leq \gamma + \gamma_s, \end{cases} \tag{14}$$

where α is the angle subtending γ in the triangle with the sides ω, γ and γ_s:

$$\alpha = \arccos \frac{\omega^2 + \gamma^2 - \gamma_s^2}{2\omega\gamma}. \tag{15}$$

Thus, in the final result, the role of the weighting function in this scheme is determined by the angle α, described by expression (15). Figure 1 presents examples of the dependences $\alpha(\xi, \eta)/\pi$ as functions of the ratio $\xi = \omega/\gamma_s$ for different values of the ratio $\eta = \gamma/\gamma_s$.

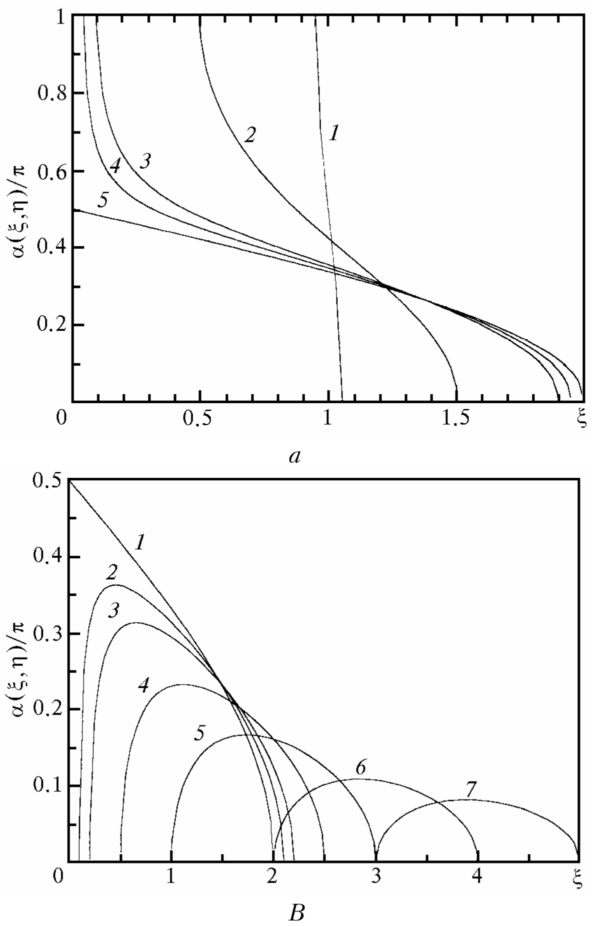

FIG. 1. Family of the dependences $\alpha(\xi, \eta)/\pi$ *as functions of the ratio* $\xi = \omega/\gamma_s$ *at different ratios* $\eta = \gamma/\gamma_s$ *(a)* $\eta = 0.05$ *(curve 1),* 0.5 *(curve 2),* 0.9 *(curve 3),* 0.95 *(curve 4), and* 1.0 *(curve 5); and (b)* $\eta = 1.0$ *(curve 1),* 1.1 *(curve 2),* 1.2 *(curve 3),* 1.5 *(curve 4),* 2.0 *(curve 5),* 3.0 *(curve 6), and* 4.0 *(curve 7).*

4. INSTRUMENTAL FUNCTION IN THE CASE OF A RECEIVER WITH A FINITE FIELD OF VIEW ANGLE

In the above discussion we have analyzed the procedure of transformation of the scattering phase function in the beam intensity measurements. Now we shall extend our analysis to the case of measurements of the light flux power within a finite solid angle. The receiver's sensitivity function $U_{\gamma_r}(\mathbf{n}_\perp)$, in angular coordinates, is assumed to be a step-wise function of a form similar to that presented by the expression (10), where γ_r is the receiver's field of view angle. If the optical axis of the receiver is oriented along the direction $\mathbf{n}'_\perp(|\mathbf{n}|'_\perp \ll 1)$, then the power of a light flux per unit area incident on the receiver is determined by the convolution

$$P(\mathbf{n}'_\perp) = (I{**}U_{\gamma_r})(\mathbf{n}'_\perp), \qquad (16)$$

where the intensity $I(\mathbf{n}_\perp, z)$ is given by formula (6). Substituting expression (6) into the equation (16), and taking formula (9) into account, we obtain

$$P(\mathbf{n}'_\perp) = I_0\,[\mathrm{e}^{-\tau}\,H(\mathbf{n}'_\perp) + B(x{**}H)(\mathbf{n}'_\perp)], \qquad (17)$$

where the instrumental function

$$H(\mathbf{n}'_\perp) = (U_{\gamma_s}{**}U_{\gamma_r})(\mathbf{n}'_\perp) \qquad (18)$$

is a two-dimensional convolution of circles on the plane, which have radii γ_s and γ_r and are separated at the distance $\gamma = |\mathbf{n}'_\perp|$; this instrumental function has the frequency response

$$\tilde{H}(p) = \tilde{U}_{\gamma_s}(p)\,\tilde{U}_{\gamma_r}(p). \qquad (19)$$

From geometric considerations it follows[8] that

$$H(\gamma) = \begin{cases} 0, & \gamma \ge \gamma_s + \gamma_r, \\ \pi\gamma_s^2, & \gamma \le |\gamma_s - \gamma_r|, \quad \gamma_s < \gamma_r, \\ \pi\gamma_r^2, & \gamma \le |\gamma_s - \gamma_r|, \quad \gamma_s > \gamma_r, \\ \gamma_s^2\beta + \gamma_r^2\alpha - \gamma_s\gamma_r\sin\delta, & 0 < |\gamma_s - \gamma_r| \le \gamma \le \gamma_s + \gamma_r, \end{cases} \qquad (20)$$

where α, β, and δ are the angles that subtend, γ_s, γ_r, and γ, of the triangle sides, respectively. The relations (20) allow us to calculate the signal component due to the directly transmitted incident radiation. The results of calculating $H(\gamma)$ may be found in Ref. 8.

The second term in expression (17)

$$P_{sc}(\mathbf{n}'_\perp) = I_0\,B(x{**}H)(\mathbf{n}'_\perp) \qquad (21)$$

represents the power of singly-scattered signal. Taking into account the expression (19) and the convolution theorem, this expression can be reduced to

$$P_{sc}(\gamma) = I_0\,B(2\pi)^{-1}\int_0^\infty p\,J_0(\gamma p)\,\tilde{H}(p)\,\tilde{x}(p)\,\mathrm{d}p \qquad (22)$$

or to

$$P_{sc}(\gamma) = I_0\,B\,(2\pi)^2\gamma_s\gamma_r\int_0^\infty \omega C(\omega, \gamma, \gamma_s, \gamma_r)\,x(\omega)\,\mathrm{d}\omega. \qquad (23)$$

when using the scattering phase-function in the expression (22).

The factor $C(\omega, \gamma, \gamma_s, \gamma_r)$ in the expression (23) plays the role of the instrumental function, when two-dimensional integral (21) is replaced by the two-dimensional integral (23); this factor, in its turn, is defined as the integral of the product of four Bessel functions:

$$C(\omega, \gamma, \gamma_s, \gamma_r) = \int_0^\infty J_0(\omega p)\, J_0(\gamma p) \times$$

$$\times J_1(\gamma_s p)\, J_1(\gamma_r p)\, p^{-1}\, dp. \quad (24)$$

Let us recast integral (24) into a more convenient, for computations, form. To do this, let us write the expression (24) in the following form:

$$P_{sc}(\mathbf{n}'_\perp) = (I_{sc}{**}U_{\gamma_r})(\mathbf{n}'_\perp) \quad (25)$$

and making use of the formula for the transition to a one-dimensional integral when calculating the two-dimensional convolution with a unit-step function on the plane, as was done when deriving expression (12), what yields the following expression:

$$P_{sc}(\gamma) = 2\pi\, \gamma_r \int_0^\infty \gamma' A(\gamma', \gamma, \gamma_r)\, I_{sc}(\gamma')\, d\gamma'. \quad (26)$$

The product $\gamma_r A(\gamma', \gamma, \gamma_r)$ in the equation (26) is determined by formula (14), while $I_{sc}(\gamma')$ follows from formula (12). By substituting $I_{sc}(\gamma')$, as given by the expression (12), into the equation (26) and changing the order of integration, we obtain

$$C(\omega, \gamma, \gamma_s, \gamma_r) = \int_0^{\gamma_s+\gamma_r} A(\omega, \gamma', \gamma_s)\, A(\gamma', \gamma, \gamma_r) \gamma'\, d\gamma'. \quad (27)$$

The instrumental function $C(\omega, \gamma, \gamma_s, \gamma_r)$ satisfies the normalization condition

$$\int_0^\infty C(\omega, \gamma, \gamma_s, \gamma_r)\, \omega\, d\omega = \Omega_s\, \Omega_r, \quad (28)$$

where $\Omega_s = \pi\gamma_s^2$ and $\Omega_r = \pi\gamma_r^2$. Since the integrand in the equation (27) factors into elementary functions the function $C(\omega, \gamma, \gamma_s, \gamma_r)$ can be calculated by this formula much easier than by equation (24).

Figure 2 presents an example of normalized function $Q(\omega, \gamma) = \omega C(\omega, \gamma, \gamma_s, \gamma_r)/(\Omega_s\, \Omega_r)$, calculated for the parameters $\gamma_s = 0.0047$ and $\gamma_r = 0.0058$, which correspond to the scheme of measurements of aureole scattering phase functions with a sun photometer.[9] Two curves, *4* and *5*, in the right-hand side of Fig. 2 correspond to the viewing angles of 1 and 2°, respectively. As is seen from this figure the value $Q(\omega, \gamma)$, as a function of ω, is unimodal in shape and its skewness grows at $\omega \to 0$. The function $Q(\omega, \gamma)$ tends, with the increasing ω, to take a symmetric shape, at $\omega > \gamma_s + \gamma_r$, about the axis $\omega = \gamma$ and, while vanishing outside the interval $|\omega - \gamma| \le \gamma_s + \gamma_r$.

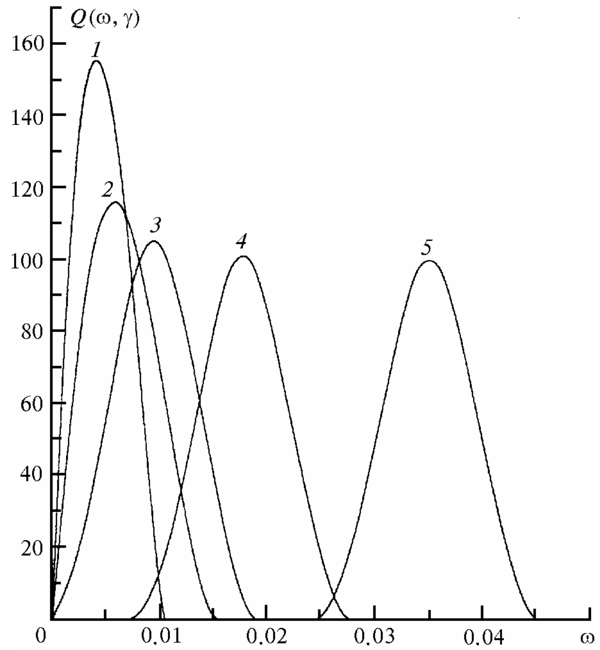

FIG. 2. Normalized instrumental function $Q(\omega, \gamma)$ as a function of ω for $\gamma_s = 0.0047$, $\gamma_r = 0.058$ and different values of γ: 0 (curve 1), 0.005 (curve 2), 0.0087 (curve 3), 0.175 (curve 4), and 0.0349 (curve 5).

The calculated results shown in Fig. 3 illustrate how the instrumental function $C(\omega, \gamma, \gamma_s, \gamma_r)$, as given by the expression (27), influences the results of measuring the small-angle scattering phase function which, in the approximation of Fraunhofer diffraction on a spherical particle of radius r, is defined by the following formula:

$$x(\gamma) = J_1^2(kr\gamma)/(\pi\gamma^2). \quad (29)$$

The normalized power of a signal due to single scattering is as follows:

$$\bar{x}(\gamma) = \int_{(L)} Q(\omega, \gamma)\, x(\omega)\, d\omega. \quad (30)$$

The integral in the equation (30) is taken over a finite interval $L \le 2(\gamma_s + \gamma_r)$. The curves in Fig. 3 have been calculated for the wavelength $\lambda = 0.55$ μm, and the function $Q(\omega, \gamma)$ is calculated for the same values of the parameters γ_s and γ_r used to calculate the curves depicted in Fig. 2. The period of oscillations of the scattering phase function $x(\gamma)$, defined by formula (29), is approximately estimated as $\Delta\gamma = \pi/kr$.

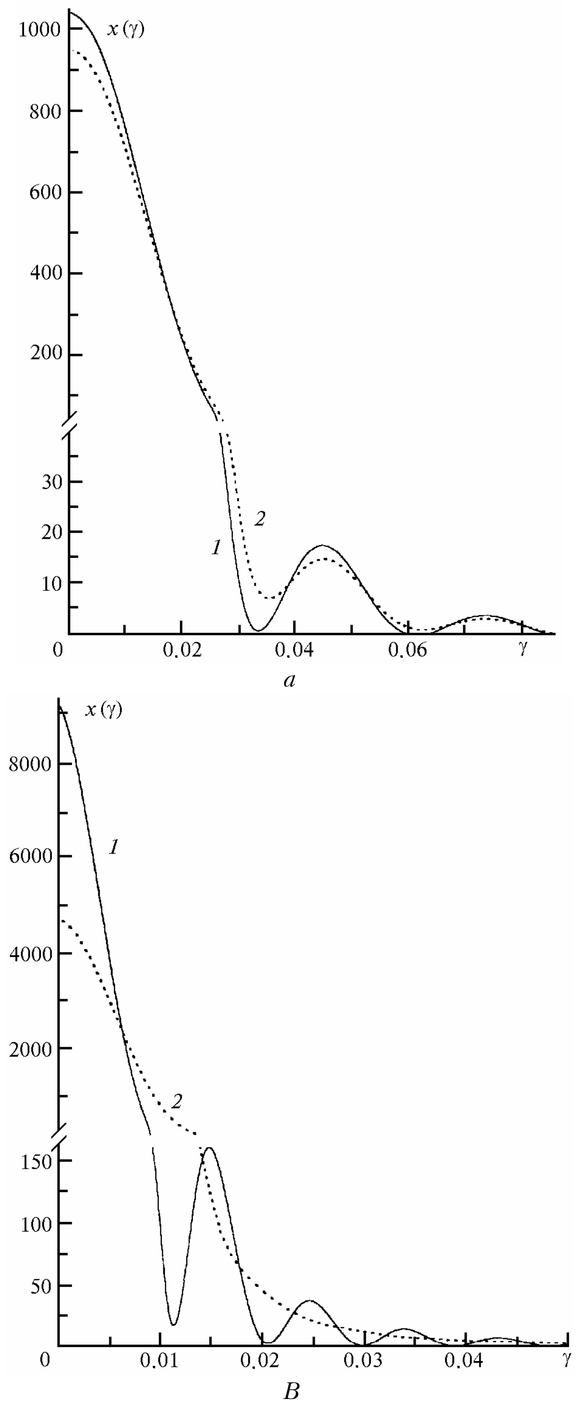

FIG. 3. Small-angle scattering phase function (1) and normalized power of a signal (2) due to single scattering at the wavelength $\lambda = 0.55$ μm *for spherical particles with radii* $r = 10$ *(a) and* 30 μm *(b).*

As the calculated results show, the smoothing effect of the instrumental function is most strong when the "window" formed by it has the width $L > \Delta\gamma$. In the example considered here, this condition is satisfied for particles with radii $r > 13–14$ μm. In particular, one can see from Fig. 3 that, for particles with radius $r = 30$ μm, the effect of the instrumental function of a measuring device on the shape of a scattering phase function is in a spread of its principal maximum that leads to a two-fold reduction of its amplitude, and, thus, to smoothing of the oscillations.

5. CONCLUSION

Thus the results presented here clearly show that measured values of the small-angle scattering phase function can be strongly distorted by the finite divergence of a light beam incident on the medium. For the measuring devices like sun photometers, these factors become important in the case of scattering particles with the radii larger than 10 μm (at $\lambda = 0.55$ μm). The degree of such distortions can readily be estimated using the expressions proposed in this paper for the instrumental function of a measuring device. The effect of instrumental function increases with the increasing size of a particle. Therefore, the small-angle methods, developed for diagnostics of disperse composition of scattering media, must take this fact into account.

REFERENCES

1. K.S. Shifrin, Trudy VZLTI, No. 2, 108–121 (1956).
2. J.H. Chin, C.M. Sliepcevich, and M. Tribus, Phys. Chem. **59**, 841–844 (1955).
3. A.G. Borovoi, N.I. Vagin, and V.V. Veretennikov, Opt. Spektrosk. **61**, No. 6, 1326–1330 (1986).
4. V.V. Veretennikov, Atmos. Oceanic Opt. **7**, Nos. 11–12, 811–815 (1994).
5. A.A. Isakov, Atmos. Oceanic Opt. **10**, No. 7, 447–454 (1997).
6. L.S. Dolin, Izv. Vyssh. Uchebn. Zaved. SSSR, Ser. Radiofizika **7**, No. 2, 380–382 (1964).
7. E.P. Zege, A.P. Ivanov, and I.L. Katsev, *Image Transfer in a Scattering Medium* (Nauka i Tekhnika, Minsk, 1985), 327pp.
8. V.V. Veretennikov, Atmos. Oceanic Opt. **11**, No. 9, 858–862 (1998).
9. M.V. Kabanov, ed., *Regional Monitoring of the Atmosphere. Part II. New Devices and Measurement Techniques* (Spectr Publishing House, Tomsk, 1997), 295 pp.