CALCULATIONS OF REFRACTION ANGLES BASED ON THE HOMOGENEOUS MODEL OF THE ATMOSPHERE

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An approximate formula for calculating refraction angles for arbitrarily long slant paths is derived within the framework of the spherically homogeneous model of the atmosphere. In order to take refraction into account when viewing extraterrestrial objects (H > 50 km), an exact formula is derived, which enables one to calculate the refraction angles to within the error in the astronomical refraction, either measured or calculated. The applicability limits of the formulas obtained are discussed and estimates of their accuracy are given.

Within the framework of the spherically symmetric model of the atmosphere the refraction angle is calculated for arbitrary altitude H and for zenith angle ξ by the exact formula[1]

$$\mathrm{tg}\, r = \frac{\sin(\zeta - \theta) - G \sin\zeta}{-\cos(\zeta - \theta) + G \cos\zeta}, \tag{1}$$

where $G = R_0/(R_0 + H)$ and

$$\theta = \int_0^H \frac{dh}{R\sqrt{(Rn/A)^2 - 1}}. \tag{2}$$

Here R_0 is the Earth's radius, $A = R_0\, n_0 \sin\xi$, $n \equiv n(h)$ and n_0 are the refractive indices at the current altitude along the ray and at the observation point, and $R = R_0 + h$. If the observation point is at the altitude H_0, then R_0 and H in formulas (1) and (2) must be replaced by $R_0 + H_0$ and $H - H_0$, respectively. For precise calculations the radius of curvature of the normal cross section of the Earth's ellipsoid must be used instead of R_0 (Ref. 1).

To determine r from these formulas one must have the instantaneous refractive index profile at the observation point and calculate θ by means of numerical integration. This makes it difficult to use exact formulas, and so in most case r is calculated by approximate formulas obtained using some theoretical model of the atmosphere.[1–5] The accuracy of the aforesaid formulas does not satisfy current requirements, and, besides, the use of approximate formulas requires that one know the parameters of the atmospheric models at the observation point. In this connection, the purpose of the present paper is to develop a more accurate and easy-to-use technique for calculating r using atmospheric parameters measured only at ground level.

The technique is based on the assumption that the refractive index profile along the ray can be replaced by its near-ground value, i.e., $n(h) = n_0$. In this technique an equivalent thickness H_e, called below the height of the homogeneous atmosphere corresponding to the atmospheric layer between the initial and final points of the ray trajectory. In this case the integral (2) becomes tabular and its value is equal to[2]

$$\theta = \zeta - \arcsin\frac{A_1}{R_0 + H_e} + \arcsin\frac{A}{R_0 + H_e} - \arcsin\frac{A}{R_0 + H}, \tag{3}$$

where $A_1 = R_0 \sin\xi$ and H_e is the height of the homogeneous atmosphere. In contrast to Ref. 2, we take H_e to be independent of the altitude H of the observed object

$$H_e = \left[H_e^0 + \frac{k_2}{2k_1}(H_e^0)^2\right]/k_1, \tag{4}$$

where

$$H_e^0 = \frac{R_c T_0^{\upsilon}}{g_0}\left[1 - \frac{P(H)}{P_0}\right]. \tag{5}$$

In Eqs. (4) and (5) g_0 is the acceleration due to gravity at the observation point (at altitude H_0), which is equal to

$$g_0 = g_c k_1 (1 - k_2 H_0), \tag{6}$$

where $k_1 = 1 - 0.0026 \cos\varphi$; $k_2 = 3.14 \cdot 10^{-7}\ \mathrm{m}^{-1}$; $g_c = 9.80665\ \mathrm{m/s^2}$ is the acceleration due to gravity at sea level ($H_0 = 0$) and latitude $\varphi = 45°$;

$R_c = 287.05$ $m^2/(deg \cdot s^2)$ is the specific gas constant of dry air; $P(H)$ and P_0 are the atmospheric pressure at the altitude H and at the observation point, respectively; T_0^v is the virtual air temperature at the observation point, which is calculated as follows[7]:

$$T_0^v = T_0(1 + 0.378e_0/P_0), \tag{7}$$

where e is the partial pressure of water vapor and T_0 is the measured temperature value.

Equations (1)–(3) are exact for the spherically homogeneous model of the atmosphere, which we are using here. The only fundamental restriction on the use of these formulas is the impossibility of calculating θ for $H < R_0(n_0 \sin\xi - 1)$, which corresponds to the case with horizontal and grazing slant paths of finite length.

Comparison with the results of Ref. 2 shows that application of Eq. (4) to the calculation of H_e makes it possible to increase the systematic error in Eqs. (1) and (2) with respect to the altitude H (See Table I in Ref. 2). However, the systematic error in the zenith angle still remains considerable. Table I presents the values of the $\delta r = r_c - r$ for two extreme atmospheric states, where r_c is the exact refractive angle calculated using Eqs. (1) and (2) using actual refractive index profiles and r is the refractive angle calculated by Eqs. (1) and (3) taking Eq. (4) into account. The accuracy of the obtained formulas is seen to be quite satisfactory for zenith angles $\xi \le 80°$. Since the variation δr depends systematically on ξ and, to some extent, depends residually on the altitude H, it is possible to minimize δr in some way or another.

It will be shown below that the value δr can be eliminated completely for all intents and purposes over the altitude range $H \ge H_{ef}$, provided that the instantaneous astronomical refraction r_∞ is known (H_{ef} is the atmospheric altitude above which the effect of beam refraction can be neglected, i.e., $n(h) = 1$ for $H \ge H_{ef}$).

Indeed, let us represent Eq. (2) as follows:

$$\theta = \theta_1(H_{ef}) + \theta_2(H) = \int_0^{H_{ef}} f(h)dh + \int_{H_{ef}}^{H} f(h)dh, \tag{8}$$

where $f(h)$ is the integrand in Eq. (2). Since $n(h) = 1$ for $h \ge H_{ef}$, the integral $\theta_2(H)$ in Eq. (8) integrates exactly to give

$$\theta_2(H) = \arcsin\frac{A}{R_0 + H_{ef}} - \arcsin\frac{A}{R_0 + H}, \tag{9}$$

and for astronomical objects ($H = \infty$) the angle $\theta_2(\infty)$ is equal to

$$\theta_2(\infty) = \arcsin\frac{A}{R_0 + H_{ef}}. \tag{10}$$

For $H = \infty$ the angle θ is equal to

$$\theta_\infty = \theta_1(H_{ef}) + \theta_2(\infty) = \theta_1(H_{ef}) + \arcsin\frac{A}{R_0 + H_{ef}}. \tag{11}$$

On the other hand, it follows from Eq. (1) that at $H = \infty$ we have the following formula for calculating the astronomical refraction r_∞

$$r_\infty = \theta_\infty - \zeta, \tag{12}$$

and taking into account Eq. (11) we obtain the following formula for θ_1 (H_{ef}):

$$\theta_1(H_{ef}) = r_\infty + \zeta - \arcsin\frac{A}{R_0 + H_{ef}}. \tag{13}$$

Substituting Eq. (13) into Eq. (8) and taking into account Eq. (9), we obtain a simple and exact formula for θ

$$\theta = r_\infty + \zeta - \arcsin\frac{A}{R_0+H}, \quad H \ge H_{ef}. \tag{14}$$

Thus, Eqs. (1) and (14) allow one to calculate the refraction angles for any zenith angle and any altitude $H \ge H_{ef}$ with the same accuracy as that of the astronomical refraction r_∞. The value of H_{ef} depends of the accuracy required in the determination of r, so it should be taken to be $H_{ef} = 60$ km in most cases of practical interest.

As for the value of r_∞, one can use either the measured values of the astronomical refraction or the values calculated by one way or another with the required accuracy. For example, the formula obtained in Ref. 6 provides quite sufficient accuracy in the zenith angle calculations over a practical range of angles. This formula has since been made more accurate, i. e., the dependence of the correction term δr_∞ on the wave length has been obtained and a correction has been introduced to take the air humidity into account. As the results of these changes one can calculate r_∞ by the following formula:

$$r_\infty = \rho''\left[\arcsin\frac{A}{R_0 + H_e} - \arcsin\frac{A_1}{R_0 + H_e}\right] \tag{15}$$

where $\rho'' = 206265$, and

$$\delta r_\infty = \exp\left\{\left[a_1 + b_1T_0^v + c_1P_0\right] + \left[a_2 + b_2T_0^v + c_2P_0\right] \times \operatorname{tg}\left[\zeta - \left[a_3 + b_3T_0^v + c_3P_0\right]\right]\right\}. \tag{16}$$

Within the wavelength range 0.4–10 μm the value k_λ is equal to

$$k_{\lambda} = N_0(\lambda)/N_0(0.6943), \tag{17}$$

where $N_0(\lambda)$ and $N_0(0.6943)$ are the near-ground values of the refractive index at the required wavelength and at the ruby laser wavelength $\lambda = 0.6943$ μm, respectively. The value of H_e in Eq. (15) is calculated by Eqs. (4) and (5) with $P(H) = 0$. Values of the coefficients a, b, and c in Eq. (16) are presented in Ref. 6 (the coefficient b_2 in Ref. 6 is written wrongly with the sign "–" instead of the correct sign " + "). Bellow, the error σ_r of the determination of the refraction angles using Eqs. (1), (14), and (15) is given for all altitudes $H \geq 60$ km (including $H = \infty$).

ζ, deg.	≤ 75	80	85	86	87	88	89	90
σ_r, ang. sec.	<0.01	0.01	0.11	0.25	0.7	1.7	31	390

The accuracy of the calculations of r by the presented technique has been evaluated based on a comparison with the exact values r_c for near-ground temperatures within the interval from –60° to +60°C, atmospheric pressure P_0 from 500 to 1100 mbar and e_0 from 0 to 50 mbar. About 40 actual profiles of the meteorological elements have been used to calculate the refractive index profiles within spectral range from 0.4 to 10 μm.

To calculate the refraction angles r for $H < 60$ km, one can use Eqs. (1) and (3). The extreme error values are presented in Table I. The exact value of the atmospheric pressure at the altitude H is assumed to be known. The additional error $\sigma_r(P)$ due to pressure errors can be estimated with the help of a formula obtained by taking the derivatives of Eqs. (1) and (3) with respect to P. Because of the cumbersomeness of the obtained expression we do not present it here, and give only the results of calculations of $\sigma_r(P)$ obtained using only annual average profiles for a given season and region as $P(H)$. The results of calculating $\sigma_r(P)$ for the Balkhash station are presented in Table II, from which it can be seen that more accurate techniques for determining the pressure should be used for calculating r for $H \leq 10$ km. Asterisks at $\xi = 89°$ and 90° in Table II correspond to the region where Eq. (3) is not valid.

TABLE I

Values of δr (ang. sec) for near-ground temperature –60°C (numerator) and +60°C(denominator).

Hight, km	Zenith angle, deg. 45	70	75	80	85	88*
1	0.015/0.001	0.01/–0.01	–0.06/–0.02	–0.40/–0.07	–4.43/–0.57	–96.2/–9.3
5	0.091/0.013	0.33/0.09	0.55/0.18	1.28/0.54	7.04/3.72	73.1/34.5
10	0.085/0.018	0.52/0.23	1.09/0.54	3.24/1.78	21.8/12.0	197.0/83.1
25	0.066/0.031	0.57/0.42	1.29/1.01	4.13/3.30	29.5/21.4	259.5/128.5
50	0.038/0.020	0.35/0.27	0.82/0.66	2.79/2.29	23.1/17.0	241.2/119.4
100	0.019/0.010	0.19/0.15	0.47/0.38	1.77/1.45	18.0/13.2	222.6/109.2
1000	0.003/0.001	0.04/0.03	0.14/0.11	0.75/0.62	12.0/8.7	197.0/95.2
∞	0.001/0	0.02/0.01	0.08/0.07	0.54/0.44	10.3/7.5	187.6/90.1

* For $\xi \geq 89°$ the values of δr reach 10 or more angle min.

TABLE II

Values of $\sigma_r(P)$ (ang.sec) obtained using seasonal profiles of the pressure.

Hight, km	Zenith angle, deg. 45	70	75	80	85	86	87	88	89	90
1	2.2	6.0	8.2	12.5	25.0	33.3	46.9	84.0	*	*
5	0.40	1.1	1.5	2.2	4.4	5.5	7.1	9.9	14.8	16.3
10	0.18	0.48	0.66	1.0	2.0	2.5	3.2	4.4	5.9	5.9
15	0.05	0.14	0.19	0.29	0.62	0.78	1.0	1.4	2.0	2.0
25	0.01	0.04	0.05	0.08	0.19	0.25	0.35	0.53	0.76	0.82
50	0.	0.01	0.01	0.01	0.04	0.05	0.08	0.13	0.20	0.23

Analysis of the data presented in Tables I and II shows that Eqs. (1) and (3) can be used to calculate the refraction angles at zenith angles $\xi \leq 80°$. The error sharply increases with further increase of ξ.

Thus, within the framework of the spherically-symmetrical model of the atmosphere simple exact formulas have been obtained for the refraction angles at large altitudes. The advantages of the formulas lie in their accuracy and simplicity and the fact that they do not require aerological measurements. Besides, these formulas make it possible to easily take into account the additional influence of horizontal gradients of the refractive index on the value of the refraction in the vertical plane, provided the value of r_∞ in Eq. (14) has been obtained from observations. This is especially important when calculating r for large zenith angles, when horizontal gradients have a more significant effect on the results of the calculations.[8] The accuracy of the thusly obtained refraction angles will depend on the measurement errors of both the astronomical refraction and the near-ground refractive index or the meteorological parameters which determine it.

For altitudes $5 \le H < 60$ km one can calculate the refraction angles using Eqs. (1) and (3) and taking into account the dependence of H_e on altitude (Eq. (4)) with 'the error $\sigma \simeq 1''$ for $\xi = 70°$ and $3''$–$4''$ for $\xi = 80°$. This error can be decreased and the ranges of ξ and H can be widened, provided that the residual systematic error δr (see Table I) is eliminated and the accuracy of the pressure $P(H)$ determination at the altitudes $H \le 10$ km is increased.

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