EXTINCTION AND SCATTERING CROSS SECTIONS OF RANDOMLY ORIENTED PARTICLES OF ARBITRARY SHAPE

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By the T–matrix method in the Barber–Yeh description [*Appl. Opt.* **14**, 2864 (1975)], *analytical expressions for the extinction and scattering cross sections of randomly oriented particles of arbitrary shape have been derived.*

At present the T–matrix method is widely used for investigation of the characteristics of light scattering by nonspherical particles. This method was developed by Waterman[2,3] both for ideal conductors[2] and dielectric particles.[3] The method is based on the Poincaré–Huygens principle.[4] The alternative justification for the method was found in Ref. 1 with the use of the Shchelkunoff principle, according to which the scattered field was induced by the equivalent system of surface currents. In the above–noted cases incident and scattered fields were expanded into a series in vector spherical harmonics[6] with the respective wave number $k = 2\pi/\lambda$

$$\mathbf{E}^i(\mathbf{r}) = \sum_{\tau mn} D_{mn}\,[a_{\tau mn}\,\mathrm{Rg}\,\mathbf{M}_{\tau mn}(k\,r) + b_{\tau mn}\,\mathrm{Rg}\,\mathbf{N}_{\tau mn}(k\,r)], \quad (1)$$

$$\mathbf{E}^s(\mathbf{r}) = \sum_{\tau mn} D_{mn}\,[p_{\tau mn}\,\mathbf{M}_{\tau mn}(k\,r) + q_{\tau mn}\,\mathbf{N}_{\tau mn}(k\,r)],\ r > r_0,$$

where $D_{mn} = (2 - \delta_{m0})\dfrac{(2n+1)\,(n-m)!}{4n\,(n+1)\,(n+m)!}$, δ_{m0} is the Kronecker symbol; $\mathrm{Rg}\mathbf{M}_{\tau mn}$, $\mathrm{Rg}\mathbf{N}_{\tau mn}$, $\mathbf{M}_{\tau mn}$, and $\mathbf{N}_{\tau mn}$ are the linearly independent solutions of the Helmholtz vector wave equation in the spherical coordinate system[6] that are different in using the spherical Bessel $j_n(kr)$ or Hankel $h^{(1)}_{(n)}(kr)$ functions of the first kind, respectively; r_0 is the radius of the circumscribed sphere of a scattering particle; $\sigma = o, e$.

Coefficients of expansion (1) are related by the linear expression[1]

$$\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix} = \begin{bmatrix}\mathbf{T}^{11} & \mathbf{T}^{12}\\ \mathbf{T}^{21} & \mathbf{T}^{22}\end{bmatrix}\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}. \quad (2)$$

In this paper we present the derivation of the formula for the scattering cross section of an ensemble of randomly oriented particles of an arbitrary shape. The scattering cross section of an arbitrary oriented particle has the form[7]

$$C_{\mathrm{scat}} = \frac{\pi}{k^2}\sum_{\sigma=o,e}\sum_{n=1}^{\infty}\sum_{m=0}^{n} D_{mn}\,\{|p_{\tau mn}|^2 + |q_{\tau mn}|^2\}. \quad (3)$$

Let us specify an arbitrary right–handed coordinate system with the radiation propagation geometry shown in Fig. 1. Let the propagation direction be specified by the wave vector $\mathbf{k}$ with the spherical angles θ and φ in the spherical coordinate system. In this case the scattering cross section $C_{\mathrm{scat}}(\theta, \varphi)$ of unpolarized incident radiation is equal to a half–sum of the respective cross sections for two linear perpendicular polarization states $\mathbf{i}_\theta$ and $\mathbf{i}_\varphi$. Let us note that the unit vector of the direction of propagation of the incident radiation and the unit vectors $\mathbf{i}_\theta$ and $\mathbf{i}_\varphi$ are the basis vectors of the spherical coordinate system. Taking into account the aforementioned, the sought–after value of the scattering cross section of randomly oriented particles can be obtained by integrating over all equally probable directions of the incident radiation propagation. Assuming that the particle position is fixed in the coordinate system, we obtain

$$\langle C_{\mathrm{scat}}\rangle = \frac{1}{4\pi}\int_0^{2\pi} d\,\varphi \int_0^{\pi} \sin\theta\, d\theta\, C_{\mathrm{scat}}(\theta, \varphi). \quad (4)$$

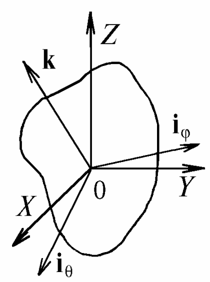

FIG. 1. Geometry of incident radiation propagation.

Using a representation of affinore of a plane electromagnetic wave in terms of vector wave spherical harmonics,[6] we obtain the expressions for the coefficients of expansion (1) of the incident plane wave[8] for the $\mathbf{i}_\theta$ polarization

$$a_{\substack{o\\e}mn} = \mp\, 4\, i^n\, m\, Q_n^m(\cos\theta)\,\begin{matrix}\sin m\varphi\\ \cos m\varphi\end{matrix}, \quad (5)$$

$$b_{\substack{o\\e}mn} = 4\, i^{n-1}\, S_n^m(\cos\theta)\,\begin{matrix}\cos m\varphi\\ \sin m\varphi\end{matrix},$$

and for the $\mathbf{i}_\varphi$ polarization

$$a_{\substack{o\\e}mn} = -\, 4\, i^n\, S_n^m(\cos\theta)\,\begin{matrix}\cos m\varphi\\ \sin m\varphi\end{matrix}, \quad (6)$$

$$b_{\substack{o\\e}mn} = \mp\, 4\, i^{n-1}\, m\, Q_n^m(\cos\theta)\,\begin{matrix}\sin m\varphi\\ \cos m\varphi\end{matrix},$$

where $S_n^m = \mathrm{d}P_n^m(\cos\theta)/\mathrm{d}\theta$, $Q_n^m = P_n^m(\cos\theta)/\mathrm{in}\theta$, and P_n^m are the associated Legendre functions.

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Calculating the value of the integrand of Eq. (4) being equal to a half–sum of the scattering cross sections for $\mathbf{i}_\theta$ and $\mathbf{i}_\varphi$ polarization states of the incident radiation, using Eqs. (2) and (3), and after integrating with orthogonality of the system of functions sin $m\varphi$ and cos $m\varphi$ as well as the relationships

$$\int_0^\pi \sin\theta \, d\theta \, [S_n^m \, Q_{n'}^m + Q_n^m \, S_{n'}^m] = 0 \, , \tag{7}$$

$$\int_0^{2\pi} d\varphi \int_0^\pi \sin\theta \, d\theta \left[\begin{matrix} \cos^2 m\varphi \\ \sin^2 m\varphi \end{matrix} S_n^m S_{n'}^m + m^2 \begin{matrix} \sin^2 m\varphi \\ \cos^2 m\varphi \end{matrix} Q_n^m \, Q_{n'}^m \right] = \begin{matrix} \pi \, \delta_{nn'} \, D_{mn}^{-1} \\ \pi \, (1 - \delta_{m0}) \, \delta_{nn'} \, D_{mn}^{-1} \end{matrix} \, ,$$

being used, we derive

$$<C_{\text{scat}}>= \frac{2\pi}{k^2} \sum_{\sigma,\tau=o,e} \sum_{i,j=1}^{2} \sum_{mnm'n'} (1 - \delta_{\tau\sigma} \delta_{m'0}) \, D_{mn} \, D_{m'n'}^{-1} \, | T_{\text{r}mn sm'n'}^{ij} |^2. \tag{8}$$

For axisymmetric particles Eq. (4) is simplified to a one–dimensional integral. Assuming $\varphi = 0$ and taking into account that $T_{\text{r}mnsm'n'}^{ij} = \delta_{mm'} \, T_{\text{r}mnsm'n'}^{ij}$ and

$$\int_0^\pi \sin\theta \, d\theta \, [S_n^m \, S_{n'}^m + m^2 \, Q_n^m \, Q_{n'}^m] = (2 - \delta_{m0}) \, \delta_{nn'} \, D_{mn}^{-1} \, / \, 2 \, ,$$

we obtain[9]

$$<C_{\text{scat}}>= \frac{\pi}{k^2} \sum_{\sigma,\tau=o,e} \sum_{i,j=1}^{2} \sum_{mnn'} (2 - \delta_{m0}) \, D_{mn} \, D_{m'n'}^{-1} \, | T_{\text{r}mnsmn'}^{ij} |^2. \tag{9}$$

It should be noted that expansion (1) is not unique. Various systems of vector spherical harmonics (as a rule, they are the linear combination of linearly independent solutions of the Helmholtz vector wave equation[6]) used for expansion of incident and scattered fields yield different representations of the T–matrices and, therefore, different calculation formulas. For example, spherical harmonics used in Ref.10 differ from that considered in this paper by the factor $D_{mn}^{1/2}$. In this case the normalization factors D_{mn} are omitted from formulas (1), (3), (8), and (9). The most convenient representation of the T–matrix method was given in Ref.11 where the systems of harmonics $\mathbf{M}_{\tau mn}$ and $\mathbf{N}_{\tau mn}$ were used. Each of these harmonics was transformed independently when rotating the coordinate system.[12] Using the T–matrix method in the Tsang–Kong–Shin description,[11] Mishchenko[13] derived formula analogous to Eq. (8) by integrating the intensity of scattered radiation over the solid angle 4π.

Using the formula

$$C_{\text{ext}} = -\frac{\pi}{k^2} \, \text{Re} \sum_{\sigma,\tau=o,e} \sum_{n=1}^{\infty} \sum_{m=0}^{n} D_{mn} \, [a_{\text{r}mn}^* \, p_{\tau mn} + b_{\text{r}mn}^* \, q_{\tau mn}] \tag{10}$$

analogous to Eq. (3) and following the above–described procedure, we derive for the extinction cross section[7]

$$<C_{\text{ext}}>= -\frac{\pi}{k^2} \sum_{\sigma,\tau=o,e} \sum_{mn} (1 - \delta_{\sigma\nu} \delta_{m0}) \, [T_{\text{r}mn\text{r}mn}^{11} + T_{\text{r}mn\text{r}mn}^{22}]. \tag{11}$$

We note that analogous formula was given in Ref.14 where the optical theorem and the T–matrix method in the Tsang–Kong–Shin description were used.

The formulas derived above can be used for estimating the light scattering cross sections of randomly oriented ellipsoid particles.[15]

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