APPLICABILITY LIMITS OF THE METHOD OF PHYSICAL OPTICS IN THE PROBLEMS OF LIGHT SCATTERING BY LARGE CRYSTALS. III. SCATTERING BY AN INFINITELY LONG PLATE

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Using the physical optics method, asymptotic expressions have been obtained for the cross sections and efficiency factors of extinction and scattering when one of the linear dimensions of a rectangular plate tends to infinity. It has been found that for this scattering problem the error of the physical optics method depends on the values of two definite integrals. Each of them is an asymptotic expression of corresponding double integral obtained for rectangular plate. It is shown that estimate of the systematic error is essentially simplified with the use of asymptotic expressions instead of double integrals.

An estimate of the error of the physical optics method was given in Ref. 1 in the form of inequality the right–hand side of which was a linear combination of two double integrals. Each of them depended on two diffraction parameters corresponding to two linear dimensions of a rectangular plate. It was shown[1] that the increase of one of the diffraction parameters led to the decrease of the error of the physical optics method. But in this case, the systematic error was limited by a horizontal asymptote. To determine its position, we need either to integrate numerically fast oscillating functions of two variables or to proceed to the limit under condition that one of the two diffraction parameters tends to infinity.

However, preliminary prepared tables of horizontal asymptotic values would essentially simplify the estimate of the error of physical optics method in solving one or another scattering problem. Then it is of interest to simplify the statement of the problem of scattering by the rectangular plate. To do this, it is necessary to tend one of the linear dimensions of the plate to infinity and then to reduce the problem to a linear case.

Let a semitransparent plate with the refractive index n and the absorption index $\kappa = 0$ be infinite along the y axis and have the width $2a$ along the x axis and the thickness d along the z axis. Let the plane wave $\mathbf{E}_i$ be incident on the plate normally, i.e., $\mathbf{E}_i = \mathbf{E}e^{ikz}$. We define the amplitude of the electric component of the incident field in the form

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{x}'_0 E_1 + \mathbf{y}'_0 E_2 = \mathbf{x}_0 E_{p_1} + \mathbf{y}_0 E_{p_2} , \qquad (1)$$

where

$$E_{p_1} = E_1 \cos\xi - E_2 \sin\xi , \quad E_{p_2} = E_1 \sin\xi + E_2 \cos\xi .$$

For such definition of the amplitude $\mathbf{E}$, its components $\mathbf{E}_1$ and $\mathbf{E}_2$ are at an angle ξ relative to the x and y axes.

Transition from a body of finite size to an infinitely long body results in the qualitative change of the Green's function for the problem of scattering. In fact, the Green's function of a scatterer of finite size in the form of a spherical wave is transformed to the Hankel function[2] upon integrating over one of the transverse coordinates between infinite limits. In this case, the scattered field for the infinitely long plate has the structure of a cylindrical wave. Considering the cylindrical coordinate system (ρ, φ, y) whose axis coincides with the y axis and using the asymptotic representation of the zero order Hankel function of the first kind for large argument,[3] we write down the following expression for the electric component of the scattered field:

$$\mathbf{E}_s = \mathbf{A}\sqrt{\frac{2\pi}{k\rho}} \exp[i(k\rho - \pi/4)] , \qquad (2)$$

where

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 = (\boldsymbol{\varphi}_0 E_{p_1} + \mathbf{y}_0 E_{p_2}) S(\varphi) . \qquad (3)$$

The unit vectors of the Cartesian coordinate system $\mathbf{x}_0$, $\mathbf{y}_0$, and $\mathbf{z}_0$ affixed to the plate are related with the unit vectors of the cylindrical system of coordinates $\boldsymbol{\rho}_0$, $\boldsymbol{\varphi}_0$, and $\mathbf{y}_0$ by the following expression:

$$\mathbf{x}_0 = \boldsymbol{\rho}_0 \sin\varphi + \boldsymbol{\varphi}_0 \cos\varphi ,$$
$$\mathbf{y}_0 = \mathbf{y}_0 , \qquad (4)$$
$$\mathbf{z}_0 = \boldsymbol{\rho}_0 \cos\varphi - \boldsymbol{\varphi}_0 \sin\varphi .$$

For this relation between the coordinate systems, the angle φ is counted off from the z axis in the OXZ coordinate plane. The function $S(\varphi)$ is the integral characteristic of scattering in the far zone of the beams coming from the plate in forward and back directions. It is determined by the expression

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$$S(\varphi)=\frac{k}{2\pi}\left[\frac{1+\cos\varphi}{2}F_1(\varphi)(1-T)+\frac{1+\cos(\pi-\varphi)}{2}F_2(\varphi)R\right]. \quad (5)$$

The complex values T and R are here the Fresnel transmittance and reflectance for a plane wave incident normally on a semitransparent layer of thickness d. They have the same form as in the problems of scattering by rectangular[1] and round[4] plates. The functions $F_1(\varphi)$ and $F_2(\varphi)$ are the Fraunhofer integrals of the phase functions from $-a$ to a. Upon integrating, the functions $F_1(\varphi)$ and $F_2(\varphi)$ are transformed to the form

$$F_1(\varphi) = 2a\,\frac{\sin(p\sin\varphi)}{p\sin\varphi},$$

$$F_2(\varphi) = 2a\,\frac{\sin(p\sin(\text{p}-\varphi))}{p\sin(\text{p}-\varphi)}, \quad (6)$$

where $p = ka$ is the diffraction parameter, $k = 2\pi/\lambda$ is the wave number, and λ is the wavelength. Taking into account that $F_1(\varphi) = F_2(\varphi)$ and designating them by $F(\varphi)$, we write down the expression for the angular function $S(\varphi)$ in the form

$$S(\varphi) = \frac{k}{2\pi}F(\varphi)\left(\frac{1+\cos\varphi}{2}(1-T)+\frac{1-\cos\varphi}{2}R\right). \quad (7)$$

It has been shown in Ref. 1 that when a plane wave is incident normally on the plate, the scattering and extinction characteristics do not depend on the wave polarization state. So let us choose such polarization state for which the solution of the scattering problem is simplest. Then we generalize the derived formulas to the case of arbitrary polarization. Let $E_{p_1} = 0$ and $E_{p_2} = 1$, which corresponds to the linear polarization of the incident wave whose electric component is directed along the y axis. Then we can write for the electric component of the total field

$$\mathbf{E}_t = \mathbf{E}_i - \mathbf{E}_s = \mathbf{y}_0(\psi_i - \psi_s), \quad (8)$$

where

$$\psi_i = \exp(ikz),\ \psi_s = \sqrt{\frac{2\pi}{k\rho}}\exp[i(k\rho - \pi/4)]\,S(\varphi), \quad (9)$$

i.e., the scattering problem is reduced to a scalar case.

Following the general conception that was used in derivation of the formula for the extinction cross section,[5] we can write the expression for this characteristic in the case of a plane wave

$$\sigma_{ext} = \frac{4\pi}{k}\,\mathrm{Re}(S(0)). \quad (10)$$

When the wave is incident normally on the plate, the geometric shadow is formed with linear size $2a$ in the OXZ plane. Taking into account that $S(0) = \frac{p}{\pi}\times(1-T)$, we obtain for the extinction efficiency factor

$$Q_{ext} = \frac{\sigma_{ext}}{2a} = 2(1-\mathrm{Re}(T)). \quad (11)$$

For this statement of the problem, i.e., for the normal incidence of the wave on the infinitely long plate, the formula for the extinction efficiency factor has the same form as for round[4] and rectangular[1] plates.

In the case of a plane wave, the scattering cross section is defined for the unit amplitude of the incident field as the total intensity scattered by the plate in the OXZ plane, i.e.,

$$\sigma_{sca} = \int_0^{2\pi}|\psi_s|^2\rho\,d\varphi = \frac{2\pi}{k}\int_0^{2\pi}|S(\varphi)|^2 d\varphi. \quad (12)$$

Taking into account Eq. (7) for $S(\varphi)$ and introducing a new function

$$f(\varphi) = F(\varphi)/2a = \frac{\sin(p\sin\varphi)}{p\sin\varphi},$$

we transform the formula for the scattering cross section to the form

$$\sigma_{sca} = \frac{2\pi}{k}\frac{p^2}{\pi^2}\Bigg(|1-T|^2\int_0^{2\pi}f^2(\varphi)\left(\frac{1+\cos\varphi}{2}\right)^2 d\varphi + |R|^2\int_0^{2\pi}f^2(\varphi)\left(\frac{1-\cos\varphi}{2}\right)^2 d\varphi + 2\mathrm{Re}([1-T]R^*)\times \int_0^{2\pi}f^2(\varphi)\left(\frac{1+\cos\varphi}{2}\right)\left(\frac{1-\cos\varphi}{2}\right)d\varphi\Bigg). \quad (13)$$

Using the equality

$$\int_0^{2\pi}f^2(\varphi)\left(\frac{1+\cos\varphi}{2}\right)^2 d\varphi = \int_0^{2\pi}f^2(\varphi)\left(\frac{1-\cos\varphi}{2}\right)^2 d\varphi$$

and going to the scattering efficiency factor $Q_{sca} = \sigma_{sca}/(2a)$, we obtain

$$Q_{sca} = (|1-T|^2 + |R|^2)C(p) + 2\mathrm{Re}((1-T)R^*)D(p), \quad (14)$$

where

$$C(p) = \frac{p}{\pi}\int_0^{2\pi}\left(\frac{1+\cos\varphi}{2}\right)^2 f^2(\varphi)d\varphi,$$

$$D(p) = \frac{p}{\pi}\int_0^{2\pi}\left(\frac{1+\cos\varphi}{2}\right)\left(\frac{1-\cos\varphi}{2}\right)f^2(\varphi)d\varphi. \quad (15)$$

Expression (14) for the scattering efficiency factor has the same structure as for the round and rectangular plates, but is different in the form of functions C and D.

Formulas (11) and (14) have been obtained for scalar incident and scattered fields. Let us show that for vector fields, i.e., when it is necessary to consider arbitrary polarization of the incident field, the solution of the problem is reduced to the same formulas (11) and (14). Let us generalize initial formulas (10) and (12) for the extinction and scattering cross sections to the case of arbitrary fields, i.e., let us represent these characteristics in the form

$$\sigma_{\text{ext}} = \frac{4\pi}{k} \frac{\text{Re}(\mathbf{E}^*\mathbf{A}|_{\varphi = 0})}{|\mathbf{E}|^2} , \quad (16)$$

$$\sigma_{\text{sca}} = \frac{1}{|\mathbf{E}|^2} \int\limits_0^{2\pi} \left| \sqrt{\frac{2\pi}{k\rho}} \exp[i(k\rho - \pi/4)]\mathbf{A} \right|^2 \rho \text{d}\varphi =$$

$$= \frac{1}{|\mathbf{E}|^2} \frac{2\pi}{k} \int\limits_0^{2\pi} |\mathbf{A}|^2 \text{d}\varphi . \quad (17)$$

However, for this statement of the scattering problem

$$\mathbf{A}|_{\varphi = 0} = (\mathbf{x}_0 E_{p_1} + \mathbf{y}_0 E_{p_2}) S(0) = \mathbf{E} S(0) ,$$

$$|\mathbf{A}|^2 = \mathbf{A}^* \mathbf{A} = (|E_{p_1}|^2 + |E_{p_2}|^2) |S(\varphi)|^2 .$$

In addition, it should be taken into account that

$$|\mathbf{E}|^2 = \mathbf{E}^* \mathbf{E} = |E_{p_1}|^2 + |E_{p_2}|^2 = |E_1|^2 + |E_2|^2 .$$

As a result, Eqs. (16) and (17) are reduced to aforementioned Eqs. (10) and (12). Hence, the final formulas for scattering characteristics (11) and (14) are valid for arbitrary polarization of the wave incident normally on the plate.

Let us take the value δ, related with the scattering characteristics Q_{ext} and Q_{sca} determined above by the expression $\delta = (Q_{\text{ext}} - Q_{\text{sca}})/Q_{\text{ext}}$, as a relative error of the physical optics method in the case of a plane wave. Following the approach used in Refs. 1 and 4, we obtain the inequality for estimating the value δ:

$$\delta(p) \le 1 - C(p) + D(p) . \quad (18)$$

By elementary manipulations, we may essentially approach the limits of integration of the integrals C and D and write them in the following form:

$$C(p) = \frac{p}{\pi} \int\limits_0^{\pi/2} (1 + \cos^2\varphi) \left[\frac{\sin(p \sin \varphi)}{p \sin \varphi} \right]^2 \text{d}\varphi , \quad (19)$$

$$D(p) = \frac{p}{\pi} \int\limits_0^{\pi/2} \sin^2 \varphi \left[\frac{\sin(p \sin \varphi)}{p \sin \varphi} \right]^2 \text{d}\varphi . \quad (20)$$

Thus, analytical expression (18) determines the asymptotic values of the error of the physical optics method when one of the linear dimensions of the rectangular plate tends to infinity.

The values of the integrals C and D are shown in Fig. 1 as functions of the diffraction parameter p. The curves $C(p)$ and $D(p)$ are bounded by the horizontal asymptotes $C = 1$ and $D = 0$. In this case, as the parameter p increases, the integrals C and D tend to their asymptotes faster than the corresponding intervals A and B for round[4] and square[1] plates. An error of 1% is reached in the case of a plane wave for the diffraction parameter $p = 50$, and the systematic error does not exceed 2% and 3% for $p = 26$ and 17, respectively.

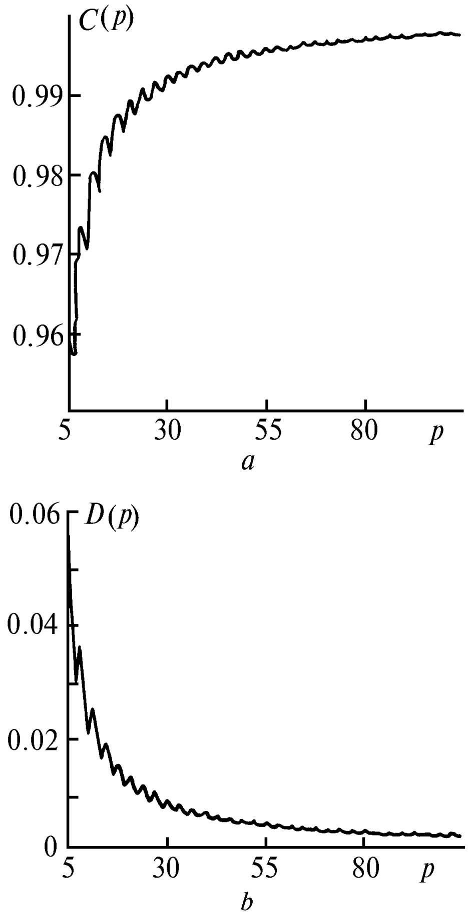

FIG. 1. Integrals C and D as functions of the diffraction parameter p.

The estimate of the relative error of the physical optics method was given in Ref. 1 for the rectangular plate in the form of inequality

$$\Delta(p, q) \le 1 - A(p, q) + B(p, q) . \quad (21)$$

The functions A and B depending on the diffraction parameters p и q are the double integrals of fast oscillating functions. The number of oscillations increases with the increase of p and q. It would entail much computation time for the numerical integration. However, in this case we can simplify the procedure of estimating the systematic error. To do this, let us replace the double integrals A and B by combinations of simpler functions C and D. Preliminary analysis of the values of A, B, C, and D for different diffraction parameters p and q makes it possible to relate them by the following expressions:

$$A(p, q) \approx C(p)\, C(q), \tag{22}$$

$$1 - B(p, q) \approx (1 - D(p))\,(1 - D(q))\ . \tag{23}$$

The larger are p and q, the better the approximate equalities hold. As a result, one can use the inequality

$$\Delta_a \le 1 - C(p)\, C(q) + D(p) + D(q) - D(p)\, D(q) \tag{24}$$

instead of estimate (21).

The values of Δ and Δ_a calculated for different diffraction parameters p and q are presented in Table I. The values of Δ and Δ_a practically coincide for large p и q. As a rule, when estimating the systematic error, it is suffice to calculate the value of Δ to two–three significant digits beyond the decimal point, which justifies replacing Δ by Δ_a. It should be noted that the integrals C and D are calculated much simpler than A and B. In addition, C and D depend only on one diffraction parameter, which makes it possible to tabulate them beforehand and give their main values in a small table. Some values of the integrals C and D are given in Table II.

TABLE I. Values of the error of the physical optics method obtained by means of calculation of double integrals A and B and definite integrals C and D for rectangular plate.

p	q	Δ	Δ_a
5	5	0.1962	0.1968
10	5	0.1479	0.1488
20	5	0.1246	0.1250
40	5	0.1128	0.1130
80	5	0.1070	0.1070
10	10	0.0985	0.0989
20	10	0.0743	0.0745
40	10	0.0622	0.0623
80	10	0.0562	0.0561
20	20	0.0497	0.0498
40	20	0.0373	0.0374
80	20	0.0313	0.0313
40	40	0.0249	0.0250
80	40	0.0186	0.0187

TABLE II. Values of the integrals C and D for different diffraction parameters p.

p	C	D
5	0.9612	0.0623
10	0.9707	0.0208
15	0.9849	0.0181
20	0.9873	0.0124
25	0.9895	0.0094
30	0.9924	0.0091
35	0.9922	0.0065
40	0.9942	0.0067
45	0.9943	0.0054
50	0.9949	0.0049
60	0.9955	0.0039
70	0.9963	0.0034
80	0.9970	0.0032
90	0.9974	0.0029
100	0.9975	0.0025
150	0.9984	0.0017
200	0.9988	0.0013
250	0.9990	0.0010
300	0.9992	0.0009
350	0.9993	0.0007
400	0.9994	0.0006
450	0.9994	0.0005
500	0.9995	0.0005
600	0.9996	0.0004
700	0.9996	0.0004
800	0.9997	0.0003
900	0.9997	0.0003
1000	0.9997	0.0002
2000	0.9999	0.0001
3000	0.9999	0.0001

Thus, the joint use of inequality (24) and data of Table II makes it possible to estimate the error of the physical optics method when it is necessary to define the cross sections of resultant beams by two linear dimensions.

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