THE GEOMETRIC-OPTICS APPROXIMATION FOR THE ABSORPTION CROSS SECTION OF A TWO-LAYER SPHERE

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A simple geometric-optics formula to the absorption cross section of a two-layer sphere is derived and analyzed. The case of particles with nonabsorbing shells is studied separately. The results of calculations performed using the exact theory of scattering for two-layered particles were used to determine the limits of applicability of the approximation.

Interest in the problem of the scattering of light by nonuniform particles has been increasing in the last few years.[1–2] In a number of cases (multilayer spheres, spheres with definite types of radial nonuniformity of the refractive index, etc.) exact solutions of the corresponding scattering problems have been obtained. However the calculations based on the exact formulas are quite complicated and they are not always suitable for practical studies. This is especially true for large scatterers. It is thus especially important to develop approximate approaches to the solution of this problem.

The purpose of this work is to study the properties of the absorption cross section of a two-layer sphere by the method of ray optics. In so doing a variant of the geometric-optics approach, based on adding the intensities rather than the fields, is employed. In this procedure the interference structure is averaged. We note that this type of averaging occurs in most real dispersed media owing to their polydispersity or the nonmonochromaticity of the incident radiation.

Let a plane wave with wavelength λ be incident on a particle having a core with a radius a and a shell with a radius b ($\lambda \ll a,\ b - a$). It is assumed that the surrounding medium is transparent and the material of the core and the shell can be absorbing ($m_e = n_e - i\kappa_e$ are the relative complex refractive indices of the core ($e = 1$) and the shell ($e = 2$)). We shall represent radiation incident on the particle as collection of pencils of rays, characterized by an angle of incidence φ and the spread in the azimuthal angle dX and the angle of incidence $\partial\varphi$.[3–4] Then we obtain for the energy flux dP_0, incident on an element of area of the particle $dS = b^2 \sin\varphi\, d\varphi\, dX$

$$dP_0 = I_0 \cos\varphi\, dS, \tag{1}$$

where I_0 is the intensity of the incident radiation. Part of the energy dE_{abs} absorbed by the particle can be written in the form

$$dE_{abs} = F(\varphi) dP_0, \tag{2}$$

where the function $F(\varphi)$ is to be determined. We shall calculate it. As shown in Fig. 1, at the point A of the surface of a two-layer particle the incident ray is divided into the ray reflected outward and the ray refracted inward. The relative fraction of the energy transmitted by the boundary is equal to $1 - R$ (R is the Fresnel power reflection coefficient for the first interface). As the ray propagates along [AB] part of the energy is absorbed, and the relative fraction of relative fraction of the absorbed energy is equal to $1 - \exp(-f)$ $\left(f = k_2|AB|,\ k_2 = \dfrac{4\pi\kappa_2}{\lambda}\right)$. For the ray incident on the sphere at the point C the relative fraction of the absorbed energy is equal to $1 - \exp(-p)$ ($p = k_1 d_1 + 2k_2 d_2$, $d_1 = |DM|$, $d_2 = |CD| = |MN|$, $k_1 = \dfrac{4\pi\chi_1}{\lambda}$). Of course, reflected rays also appear at points D, M, N, and B. As special calculations show, however, to a first approximation their contribution can be neglected (particles with large values of the refractive indices m_1 and m_2 are not studied here).

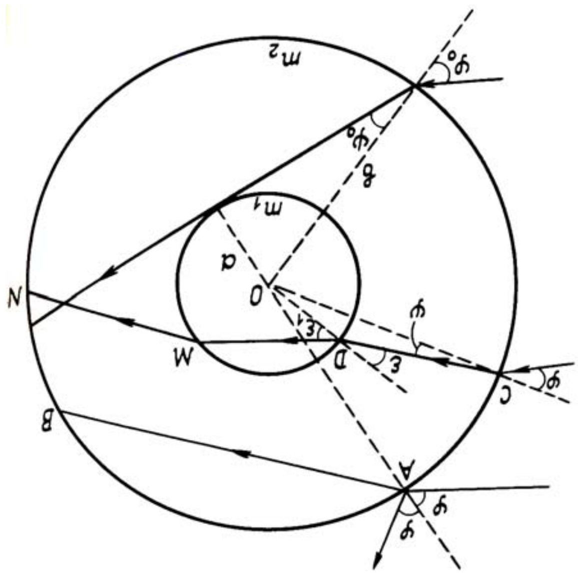

FIG. 1. The path of refracted rays in a two-layer particle.

The only exception is the region with $n_1 < n_2$ and the region of total internal reflection (TIR) of radiation by the core. Here and below we shall assume that $\kappa_l \ll n_l$ ($l = 1, 2$). Then the attenuation of the TIR owing to absorption is negligibly small and a real critical angle of incidence ε_c can be introduced: $\sin\varepsilon_c = n_1/n_2$. The rays incident on the core at angles larger than the critical angle undergo the TIR and the relative fraction of the absorbed energy in this case is equal to $1 - \exp(-s)$, where $s = 2k_2 d_2$ (we neglect multiple rereflections).

It follows from what was said above that approximately

$$F(\varphi) = (1 - R)\,(1 - e^{-\gamma}), \tag{3}$$

where $\gamma = p$, s, and f depending on the point of entry of the ray into the particle.

Substituting Eq. (3) into Eq. (2) and summing all incident rays we obtain for the absorbed energy E_{abs}:

$$E_{abs} = \int dE_{abs} = \pi b^2 J_0 \left\{ \int_0^{\varphi_c} (1-R)(1-e^{-p})\sin 2\varphi d\varphi + \int_{\varphi_0}^{\varphi_c} (1-R)(1-e^{-s})\sin 2\varphi d\varphi + \int_{\varphi_0}^{\pi/2} (1-R)(1-e^{-f})\sin 2\varphi d\varphi \right\}, \tag{4}$$

where φ_c is the angle of incidence of the ray on the particle, corresponding to the critical angle ε_c, and φ_0 is the limiting angle of incidence, corresponding to the ray that is tangent to the core at one point (see Fig. 1). Obviously, for $\varphi > \varphi_0$ the rays incident on the particle do not interact with the core. It follows from trigonometric identities and Snell's law (see Fig. 1) that $\sin\varphi = n_2\sin\Psi$, $\sin\varepsilon = b/a\sin\Psi$, $\sin\varepsilon_1 = n_2\sin\varepsilon/n_1$, $\sin\Psi_0 = a/b = \nu$, whence it easily follows that

$$\varphi_0 = \arcsin(n_2\nu), \quad \varphi_c = \arcsin(n_1\nu). \tag{5}$$

Thus, for $n_2\nu \geq 1$ there is no limiting angle φ_0: all rays are focused on the core and the third integral in Eq. (4) vanishes. For $n_2 = 2$ and $n_2 = 4/3$ this corresponds to the parameters $\nu \geq 1/2$ and $\nu \geq 3/4$. For $n_2 \leq 1$ a limiting angle φ_0 exists for any value of ν. We note that if $n_1 < n_2$, then the critical angle φ_c, as follows from Eq. (5), does not depend on the material of the shell. If $n_1\nu \geq 1$, then the angle φ_c does not exist and the second integral in Eq. (4) vanishes. This is also true for $n_1 > n_2$; then it must be assumed in Eq. (4) that $\varphi_c = \varphi_0$.

Introducing the new variable of integration $\sigma = \sin^2\varphi$, we easily obtain from Eq. (4) the following expression for the absorption cross section of a two-layer sphere $C_{abs} = E_{abs}/J_0$:

$$C_{abs} = \pi b^2 \left[\int_0^{\beta} (1-R)(1-e^{-p})d\sigma + \int_{\beta}^{\beta_2} (1-R)(1-e^{-s})d\sigma + \int_{\beta_2}^{1} (1-R)(1-e^{-f})d\sigma \right], \tag{6}$$

where

$$\beta_1 = (n_1\nu)^2, \ \beta_2 = \min(1, (n_2\nu)^2), \ \beta = \min(\beta_1, \beta_2);$$

$$R = \frac{1}{2}\left(|r_1|^2 + |r_2|^2\right), \quad r_e = (\sqrt{1-\sigma} - N_e\eta)(\sqrt{1-\sigma} + N_e\eta), \quad \eta = \sqrt{1 - \sigma/n_2^2},$$

$N_1 = n_2$, and $N_2 = n_1^{-1}$. The functions $p(\sigma)$, $s(\sigma)$, and $f(\sigma)$ can be determined by applying Snell's law and trigonometric identities (see Fig. 1):

$$p(\sigma) = f(\sigma) + 4\rho_1(\kappa_1\eta_1 - \kappa_2\eta_2),$$

$$s(\sigma) = f(\sigma) - 4\kappa_2\eta_2\rho_1,$$

$$f(\sigma) = 4\kappa_2\rho_2\eta,$$

where

$$\eta_e = \sqrt{1 - \sigma/(n_e\nu)^2}, \quad \rho_1 = 2\pi a/\lambda, \quad \rho_2 = 2\pi b/\lambda.$$

We note that in most cases the second integral in Eq. (6) can be neglected. Then the approximation (6) differs from the anomalous diffraction approximation (ADA)[3,5,6] only in that the curvature of the rays in the particle and the reflection from the shell are taken into account. As $n_1 \to n_2$, $n_2 \to 1$, the refraction and reflection at the interfaces can be neglected and the expressions (6) transforms into the form corresponding to the ADA.

We shall study the asymptotic behavior of the expression (6) in the region of strong and weak absorption. In the limit $\kappa_2\rho_2 \to \infty$ (ρ_1 = const) it follows from Eq. (6) that

$$C_{abs} = \left[1 - r(n_2)\right] S, \quad S = \pi b^2, \tag{7}$$

and the integral $r(n_2) = \int_0^1 R d\sigma$ is calculated analytically in Ref. 7. The asymptotic behavior of Eq. (7) has a clear physical meaning: all radiation penetrating into the particle is absorbed. In the region of weak absorption we easily obtain

$$C_{abs} = \frac{3V}{2}\,(k_1 z_1 \nu + k_2 z_2) \tag{8}$$

where

$$z_1 = \int_0^{\beta} (1 - R)\, \eta d\sigma,$$

$$z_2 = \int_0^{1} (1 - R)\, \eta d\sigma - \nu \int_0^{\rho_2} (1 - R)\, \eta_2 d\sigma,$$

and V is the volume of the particle.

It follows from Eq. (8) that the volume absorption coefficient $K_{abs} = C_{abs}/V$ does not depend on the size of the particle. In the region of weak absorption this law also holds for nonspherical particles.[6] At the same time in the region of strong absorption, as follows from Eq. (7), the quantity K_{abs} decreases as the size of the scatterer increases:

$$K_{abs} = (1 - r)\, S/V, \tag{9}$$

where $r = \int_0^1 R d\sigma$. We note that the formula (9) is also valid for nonspherical particles, if r is interpreted as the relative fraction of the energy reflected by the particle into the surrounding medium. In this case r will depend on the orientation of the particle in the field of the incident wave, the shape of the particle, etc.

We shall discuss in greater detail the important case of two-layer particles with nonabsorbing shells $\kappa_2 = 0$. If the nonabsorbing shell is quite soft ($|m_2 - 1| \ll 1$), then in calculating the absorption cross section the shell can be neglected altogether: everything is determined by the absorption cross section of the core.[6] If the shell is hard, then, as analysis of the results of numerical calculations performed based on the exact theory shows, as the size of the shell increases (a = const) C_{abs} approaches some asymptotic value.[8,9] The value of the parameter $\nu_{as} = a/b$ for which this asymptotic behavior is reached can be determined from simple physical arguments. Rays with angles of incidence $\varphi \le \varphi_0$, where the angle φ_0 is defined in Eq. (5), are focused on the core. If $n_2\nu > 1$, then all incident rays are focused on the core. As ν decreases (a = const) the parameter $n_2\nu$ decreases and rays which are not focused by the shell on the core. For the boundary conditions we must obviously use $n_2\nu_{as} = 1$, whence $\nu_{as} = n_2^{-1}$; this is also confirmed by the exact calculations.

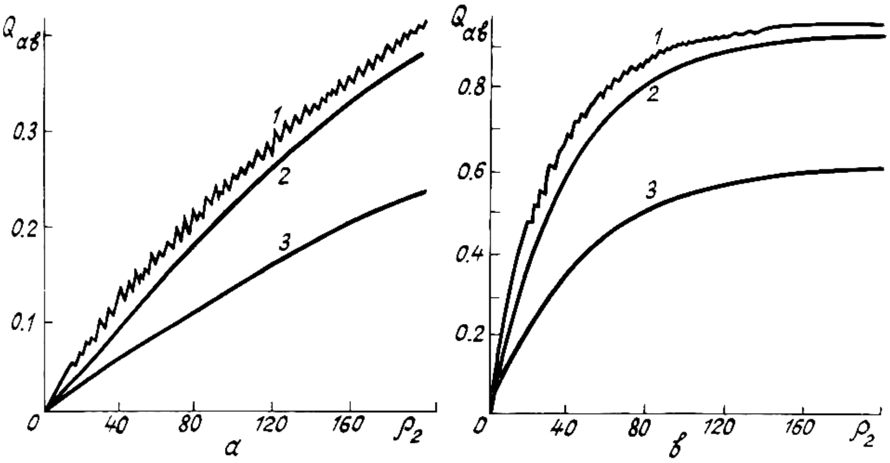

FIG. 2. The absorption efficiency factor Q_{abs} of a two-layer-particle as a function of the diffraction parameter ρ_2 (1 calculation according to the exact theory, 2 calculation using the formula (6), 3 the ADA)[5–6] with $n_1 = 1.5$, $n_2 = 1.34$, $\nu = 0.8$: a $\chi_1 = 10^{-3}$, $\kappa_2 = 10^{-4}$, b $\chi_1 = 10^{-2}$, $\kappa_2 = 0$.

In conclusion we shall compare the results of calculations of the absorption efficiency factors $Q_{abs} = C_{abs}/\pi b^2$ with the exact theory[2] and the formula (6). The calculations were performed on a BESM-6 computer using the algorithm of Ref. 2 with n_1 = 1.0, 1.34, 1.5, 1.6; $\kappa_1 = 10^{-4}, 10^{-3}, 10^{-2}$; n_2 = 1.34, 1.5, 1.6, 2.0; $\kappa_2 = 0, 10^{-4}, 10^{-3}, 10^{-2}$; ν = 0.5 (0.1) 0.9, ρ_2 = 2 (1) 200. The values of Q_{abs} in the ADA were calculated at the same time.[5,6] The following conclusions can be drawn from analysis of the computational results. The formula (6) satisfactorily describes the absorption cross section of a two-layer particle. For example, for $a \ge 12\lambda$ (Fig. 2a) and $a \ge 8\lambda$ (Fig. 2b) the error in the calculation of C_{abs} does not exceed 10%. In the other cases studied the error did not exceed 20% with $a \sim 7{\div}12\lambda$. This indicates that the approximation (6) can also be used in the case of not very large cores, where the use of the method of ray optics is problematic. As expected, the accuracy of the expression (6) decreases as the thickness of the shell decreases (primarily for nonabsorbing cores). It is sometimes stated in the literature that the anomalous diffraction approximation[10] is also applicable for quite hard particles (right up to refractive indices $n \sim 2$). For example, in Ref. 10 this conclusion is drawn based on a comparison of the extinction efficiency factors calculated with the ADA and exact formulas for two-layer particles. It is obvious from analysis of Figs. 2a and b that this rule does not hold for the absorption efficiency factors. For example, for $\rho_2 \ge 60$

(Fig. 2a) and $\rho_2 \geq 20$ (Fig. 2b) the relative error of the ADA is ~ 50% (the relative error of the formula (6) is less than 20%). As one can see from the figures presented the absorption curves $Q_{abs}(\rho_2)$ are characterized by small ripples, which cannot be described on the basis of the approximation. (6) This is connected with the fact that the interference in the drop is neglected.

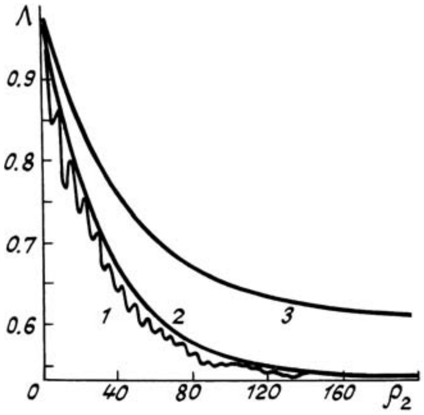

FIG. 3. The albedo Λ of a two-layer particle as a function of the diffraction parameter ρ_2 (1 – calculation using the exact theory, 2 – calculation using the formula (6); 3 – ADA)[5,6] with $n_1 = 1.5$, $n_2 = 1.34$, $\nu = 0.8$: $\chi_1 = 10^{-2}$, $\kappa_2 = 0$, $\nu = 0.9$.

Taking into account the fact that in the geometric-optics limit the extinction cross section $C_{ext} = 2\pi b^2$ is also easy to obtain based on Eq. (6) the albedo of the particle $\Lambda = 1 - C_{abs}/C_{ext}$. The values of Λ calculated in the anomalous diffraction approximation and the geometric optics approximation and by the rigorous theory for two-layer particles are compared in Fig. 3. Analysis of the figure shows that the relative error of the simple geometric-optics formula for the albedo Λ in the case studied is less than 5%, if $\rho_2 \geq 2$. In practice the quantity $1 - \Lambda$ is the more important parameter; the error in calculating this parameter in the case at hand is less that 10% for $\rho_2 > 28$.

The results of this work can be used for making different estimates and calculations in the range of diffraction parameters ρ_2 where it is difficult to perform calculations by the exact theory.

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