LIDAR GEOMETRICAL FACTOR IN A SMALL-ANGLE APPROXIMATION

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An exact analytical description of the geometrical factor of a lidar is obtained as an integral of a product of the Bessel functions within the framework of a small-angle approximation. A way to represent it in terms of elementary functions is proposed. The behavior of the geometrical factor along a path is analyzed for the case of sounding with separated source and receiver.

Solution of inverse problems in dense media sounding requires a correct determination of the relation between singly and multiply scattered components in a lidar signal.[1] Among the factors determining the behavior of this relation, the geometrical factor stands out, which carries information about the effect of geometrical parameters of the lidar sounding scheme upon the single scattering signal. In this paper, we obtain the exact analytical description of the geometrical factor within the framework of the small-angle approximation, propose a way to represent it in terms of elementary functions, and analyze the behavior along a sounding path with separated source and receiver.

1. INITIAL EQUATIONS. FORMULATION OF THE PROBLEM

Suppose that the scattering medium occupies the domain $z > 0$, the source and the receiver of optical radiation are placed in the plane $z = 0$, and their axes are oriented in parallel to the OZ axis and separated by distance d (Fig. 1).

To describe the spatial-angular structure of the light field at the output of the radiation source with the center in the origin of coordinates, we use a model having the circular symmetry of the form

$$I_0(\mathbf{r}, \mathbf{n}) = A\ \Sigma(r, R_s)\ \Omega(\gamma, \gamma_s) \qquad (1)$$

with stepwise intensity distribution

$$\Sigma(s, t) = \Omega(s, t) = U(t - s)\ , \qquad (2)$$

where $U(t)$ is the unit step function (Heaviside function); the factor $A = P_0/(\pi R_s \gamma_s)^2$, P_0 is power, R_s and γ_s are radius of the output aperture and source divergence angle, respectively; $r = |\mathbf{r}|$, $\mathbf{r} = (x, y)$ are transverse coordinates; $\gamma = (\mathbf{n}\hat{}\mathbf{z}_0)$ is the angle between the direction $\mathbf{n}$ and OZ axis.

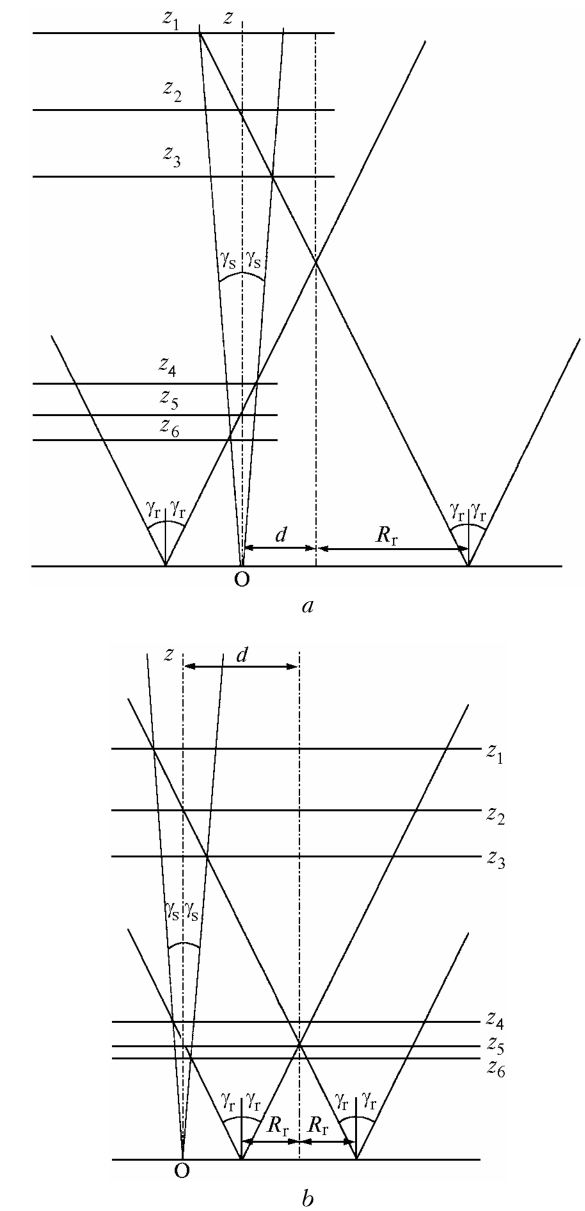

FIG. 1. Lidar geometry: $d < R_r$ (a); $d > R_r$ (b).

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Further, let us suppose that the sensitivity function of the receiving system with respect to spatial and angular coordinates $D(\mathbf{r}, \mathbf{n})$ is also of a stepwise form and, for a receiver with circular symmetry and the optical axis coinciding with OZ can be represented in the form similar to Eq. (1)

$$D(\mathbf{r}, \mathbf{n}) = \Sigma(r, R_r)\, \Omega(\gamma, \gamma_r)\ , \tag{3}$$

where R_r and γ_r are radius of the input aperture and the receiver field-of-view angle.

Let the position of the receiving aperture center be determined by the radius-vector $\mathbf{d}$ on the plane $z = 0$. Then, if the medium is irradiated by a δ-pulse source with unit energy, the expression for power of a lidar signal coming to the input of the receiving system at time t can be written as a two-dimensional convolution on the plane $z = ct/2$ (Refs. 2 and 3):

$$P(t) = \frac{c}{2}\, \mu_\pi \left(\frac{ct}{2}\right) \iint\limits_S E_s(\mathbf{r}, z)\, E_r(\mathbf{d} - \mathbf{r}, z)\, d\mathbf{r}\ , \tag{4}$$

where c is light speed; $\mu_\pi(z)$ is backscattering coefficient; $E_s(\mathbf{r}, z)$ and $E_r(\mathbf{r}, z)$ are spatial irradiances created in the medium by a stationary source with radiance distribution at the input to the medium $I_0(\mathbf{r}, \mathbf{n})$ given by Eq. (1) and a fictitious source with radiance distribution $D(\mathbf{r}, \mathbf{n})$ given by Eq. (3), respectively. The function $E(\mathbf{r}, z)$ can be calculated in the small-angle approximation of the radiation transfer theory by equations presented in Ref. 4. Using the properties of the two-dimensional Fourier transform for a function with circular symmetry and the convolution theorem, we can write equation (4) in the following form in the single-scattering approximation[4]:

$$P(z, d, \gamma_s, R_r, \gamma_r) = F \left(\frac{c}{2}\right) \mu_\pi(z)\, z^{-2}\, e^{-2\tau(z)}\ , \tag{5}$$

where the factor $F = 4\pi \dfrac{R_r \gamma_r}{\gamma_s}\, G(d, z\gamma_s, R_r, z\gamma_r)$ describes the lidar signal as a function of geometrical parameters of the sounding scheme and it can be defined as a lidar geometrical factor. For stepwise apertures, the function $G(d, z\gamma_s, R_r, z\gamma_r)$ can be represented as an integral (A10) (see Appendix) with the following substitution of parameters: $r = d$, $a = z\gamma_s$, $b = R_r$, $c = z\gamma_r$.

It is evident that direct calculation of the function $G(d, z\gamma_s, R_r, z\gamma_r)$ by Eq. (A10) is a very tedious problem. A way to transform the integral (A10) to the form (A13) more convenient for analysis and calculations is described in the Appendix.

Substituting the values $r = d$, $a = z\gamma_s$, $b = R_r$, $c = z\gamma_r$ into the transformed expression (A13) and taking into account that the function $B(\nu, z\gamma_s, z\gamma_r)$ entering into it equals zero for $\nu \geq z\gamma_s + z\gamma_r$, let us write the following expression for the function $G(d, z\gamma_s, R_r, z\gamma_r)$, which determines behavior of the lidar geometrical factor:

$$G(d, z\gamma_s, R_r, z\gamma_r) = \int\limits_0^{z\gamma_s + z\gamma_r} A(\nu, d, R_r)\, B(\nu, z\gamma_s, z\gamma_r)\, \nu\, d\nu\ . \tag{6}$$

Since the functions $A(\nu, d, R_r)$ (A1) and $B(\nu, z\gamma_s, z\gamma_r)$ (A6) forming the integrand in Eq. (6) can be expressed in terms of elementary functions and integration limits are finite, calculation by the Eq. (6) is a simpler problem.

2. ANALYSIS OF THE GEOMETRICAL FACTOR

Equation (6) can be further simplified under certain relations between the parameters in the integrand. For definiteness, we assume that $\gamma_r > \gamma_s$. This makes it possible to divide the integration domain in Eq. (6) into two parts and present the integral as a sum

$$G = \int\limits_0^{z\gamma_r - z\gamma_s} A(\nu)\, B(\nu)\, \nu\, d\nu + \int\limits_{z\gamma_r - z\gamma_s}^{z\gamma_r + z\gamma_s} A(\nu)\, B(\nu)\, \nu\, d\nu = G_1 + G_2\ . \tag{7}$$

Decomposition of the function G given by Eq. (7) into summands implies the corresponding decomposition of a lidar signal $P(z) = P_1(z) + P_2(z)$. According to Eqs. (A8) and (A9), the function $B(\nu, z\gamma_s, z\gamma_r)$ in the first integral of Eq. (7) is constant what permits us, taking into account Eq. (A7), to write

$$G_1 = \frac{\gamma_s}{2\gamma_r} \int\limits_0^{z\gamma_r - z\gamma_s} A(\nu, d, R_r)\, \nu\, d\nu = \frac{z\gamma_s}{2\gamma_r}\, (\gamma_r - \gamma_s)\, B(d, z\gamma_r - z\gamma_s, R_r)\ . \tag{8}$$

So, on the base of Eq. (A8), for the component $P_1(z)$ of a lidar signal we have

$$P_1(z, d, \gamma_s, R_r, \gamma_r) = \left(\frac{c}{2}\right) \mu_\pi(z) \times e^{-2\tau(z)}\, z^{-2} \left[U_{R_r}(d) ** U_{z\tilde{\gamma}_r}(d) \right]\ , \tag{9}$$

where $\tilde{\gamma}_r = \gamma_r - \gamma_s$. With substitution of $\tilde{\gamma}_r$ for γ_r, Eq. (9) describes a lidar signal for a simplified model with a point mono-directed source.[5]

Let us dwell on the structure of the lidar signal component $P_1(z)$ given by Eq. (9). In the far zone of reception (see Fig. 1) which is of the most practical interest, as a rule, for

$$z > z_1 = (R_r + d)/\tilde{\gamma}_r \qquad (10)$$

the lidar signal $P_1(z)$ is defined by the equation

$$P_1(z) = (c/2)\ \mu_\pi(z)\ z^{-2}\ S_r\ e^{-2\tau(z)}\ , \qquad (11)$$

where $S_r = \pi R_r^2$ is the area of the receiving aperture. As follows from Eq. (10), if $\gamma_s \to \gamma_r$, this case can be realized only at infinity.

In the near zone of reception which is defined by the condition

$$z < z_4 = |R_r - d|/\tilde{\gamma}_r\ , \qquad (12)$$

for $d > R_r$ we obtain $P_1(z) = 0$, and for $d < R_r$

$$P_1(z) = (c/2)\ \mu_\pi(z)\ \tilde{\Omega}_r\ e^{-2\tau(z)}\ , \qquad (13)$$

where $\tilde{\Omega}_r = \pi\ \tilde{\gamma}_r^2$ can be considered as effective solid angle of reception.

In the intermediate zone

$$z_4 < z < z_1 \qquad (14)$$

the geometrical factor $F = U_{R_r}(d) ** U_{z\tilde{\gamma}_r}(d)$ for the signal $P_1(z)$ is calculated with use of the last formula of Eqs. (A9). The family of generalized dependences $F/(\pi R_r^2)$ versus the dimensionless variable $\rho = z\gamma_r/R_r$ is presented in Fig. 2 for different values of the parameter $\eta = d/R_r$. Curve *1* in Fig. 2 corresponds to the conjoint sounding scheme ($d = 0$). The part of curve *1* to the left of the point $M_1(\rho < 1)$ refers to the near zone, while points placed on the straight line $F/(\pi R_r^2) = 1$ to the right of the point M_1 refer to the far zone. The length of the intermediate zone vanishes in the conjoint sounding scheme. With increase of η from 0 to 1, the boundary of the near zone displaces along the curve *1* towards the point $\rho = 0$ (points M_2, M_3 ...) and the near zone disappears at $d = R_r$ (curve *5*). With further increase of $\eta > 1$, the parts of the abscissa, where the function F vanishes, correspond to the near zone.

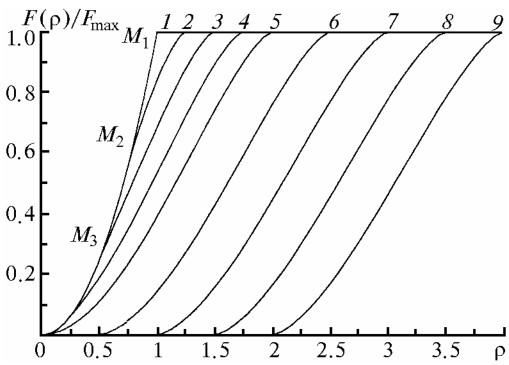

FIG. 2. Parametric family of the geometric factor $F(\rho)/F_{max}$, $\rho = z\gamma_r/R_r$ *at different values of the ratio* $\eta = d/R_r$; η = 0; 0.25; 0.5; 0.75; 1.0; 1.5; 2.0; 2.5; 3.0 *(curves 1–9).*

Let us briefly dwell on the second summand in Eq. (7)

$$G_2 = \int\limits_{z\gamma_r - z\gamma_s}^{z\gamma_r + z\gamma_s} A(\nu, d, R_r)\ B(\nu, z\gamma_s, z\gamma_r)\ \nu\ d\nu. \qquad (15)$$

Its importance increases with increase of γ_s because $G_1 \to 0$ when $\gamma_s \to \gamma_r$. According to Eqs. (A2) and (A5), the function $A(\nu, d, R_r) = 0$ if $\nu \ge d + R_r$ The last condition is always true, if the lower integration limit in Eq. (15)

$$z\gamma_r - z\gamma_s \ge d + R_r\ . \qquad (16)$$

However, the restriction (16) is equivalent to assignment of a boundary of the far zone for the integral G_1. Therefore, $G_2 = 0$ in the far reception zone and the lidar signal $P(z) = P_1(z)$ given by Eq. (11). Another situation for which $A(\nu, d, R_r) = 0$ is realized in the case $\nu \le |d - R_r|$, $d > R_r$. This is possible when

$$z \le z_6 = |d - R_r|\ /\ (\gamma_r + \gamma_s)\ . \qquad (17)$$

The last condition defines the near zone for the integral G_2 given by Eq. (15) (see Fig. 1). In this zone $G_2 = 0$ and $P(z) = 0$ for $d > R_r$. Note that the boundaries of the near zones for G_1 and G_2 (points z_4 and z_6 in Fig. 1) do not coincide. For $d < R_r$, $A(\nu, d, R_r) = R_r^{-1}$ in the near zone (17) and, based on Eq. (A16), we can write the expression

$$G = z^2\ \gamma_r\ \gamma_s/(4\ R_r) \qquad (18)$$

for the integral G given by Eq. (6) and

$$P(z) = (c/2)\ \mu_\pi(z)\ \Omega_r\ e^{-2\tau(z)} \qquad (19)$$

for the lidar signal.

The integral G_2 is calculated by the general equation (15) in the interval $z_6 < z < z_1$.

3. CONCLUSION

This study of the analytical expression obtained for the geometrical factor of a lidar in the small-angle approximation for stepwise apertures demonstrates that the sounding path can be divided into characteristic domains or zones depending on the relation between parameters of the lidar transceiving system. They are near, intermediate, and far zones. Calculations that take into account the geometry of a lidar experiment can be considerably simplified within such zones. The expressions for transformation of integrals of products of the Bessel functions to the expressions depending on the elementary functions form the basis of this analysis. Application of these expressions facilitates analysis and physical interpretation of the results what causes their usefulness in solving other problems of optics.

APPENDIX

INTEGRALS OF PRODUCTS OF BESSEL FUNCTIONS

In this section, we present the expressions for some integrals that are necessary in analysis of the lidar geometrical factor.

1. Let us consider the following auxiliary integral of a product of the Bessel functions:

$$A(r,\, b,\, a) = \int\limits_0^{\infty} J_0(r\omega)\, J_0(b\omega)\, J_1(a\omega)\, d\omega\ . \tag{A1}$$

It can be shown that the geometrical sense of the integral (A1) is simple: its value is proportional to two-dimensional convolution of the functions

$$A(r,\, b,\, a) = (2\pi ab)^{-1}\ U_a(r)**\delta(r - b), \tag{A2}$$

where $U_a(r)$ is the unit step function in the plane XOY,

$$U_a(r) = \begin{cases} 1, & 0 \le r < a, \quad r = \sqrt{x^2 + y^2}, \\ 0, & r > a; \end{cases} \tag{A3}$$

$\delta(r)$ is the Dirac delta function. The proof of this statement is based on the convolution theorem, properties of the Fourier transforms for generalized functions, and the relation

$$\int\limits_0^{a} r\, J_0(\omega r)\, dr = \frac{a\, J_1(\omega a)}{\omega}\ , \tag{A4}$$

which is well known in the theory of Bessel functions.

In its turn, using the integration technique for expressions containing the delta function on a plane,[6] we can obtain the expression for the convolution $U_a(r)**\delta(r - b)$ in terms of the elementary functions

$$U_a(r)**\delta(r - b) = \begin{cases} 0, & r \ge a + b, \\ 0, & r \le |a - b|, \quad a < b, \\ 2b\pi, & r \le |a - b|, \quad a > b, \\ 2b\alpha, & 0 < |a - b| \le r \le a + b, \end{cases} \tag{A5}$$

where α is the angle opposite the side a in a triangle with sides a, b, and r (Fig. 3). The relations (A5) mean that the convolution $U_a(r)**\delta(r - b)$ numerically equals the length L of the arc formed by intersection of two circumferences with radii a and b, and the distance r between the centers.

Thus, the integral $A(r, b, a)$ given by Eq. (A1) can be presented in terms of the elementary functions.

2.The following step is to obtain the expression for the following integral of a product of the Bessel functions:

$$B(r,\, b,\, a) = \int\limits_0^{\infty} \omega^{-1} J_0(r\omega)\, J_1(b\omega)\, J_1(a\omega)\, d\omega\ . \tag{A6}$$

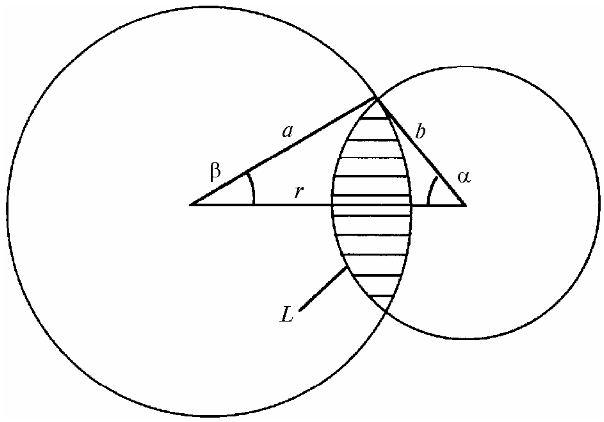

FIG. 3. Geometrical interpretation of parameters in the integrals $A(r, b, a)$ given by Eq. (A1) and $B(r, b, a)$ given by Eq. (A6)

The integral $b(r, b, a)$ from Eq. (A6) is connected with the integral $A(r, b, a)$ from Eq. (A1) by the expression

$$\int\limits_0^{b} A(r,\, b',\, a)\, b'\, db' = b\, B(r,\, b,\, a)\ , \tag{A7}$$

which validity is easily verified by the substitution (A4). On the other hand, taking into account the representation for the function $A(r, b, a)$ as Eq. (A2), the expression for the integral $b(r, b, a)$ from Eq. (A6) in the form[5]

$$B(r,\, b,\, a) = (2\pi ab)^{-1}\ U_b(r)**U_a(r) \tag{A8}$$

follows from Eq. (A7).

Thus, the integral $b(r, b, a)$ in Eq. (A6) is proportional to the two-dimensional convolution of the circles with radii a and b with centers spaced apart at a distance r. As follows from the geometrical considerations, the convolution is equal to the area of the circles intersection (hatched area in Fig. 3) and determined by the equations

$$U_a(r)**U_b(r) = $$
$$= \begin{cases} 0, & r \ge a + b, \\ \pi\, a^2, & r \le |\,a - b\,|\,,\ a < b, \\ \pi b^2, & r \le |\,a - b\,|\,,\ a > b, \\ a^2\beta + b^2\alpha - ab\,\sin\gamma, & 0 < |\,a - b\,| \le r \le a + b, \end{cases} \tag{A9}$$

where α, β, and γ are triangle angles opposite the sides a, b, and c, respectively.

3. In conclusion, let us consider the integral of a product of the four Bessel functions of zero and first orders

$$G(r, a, b, c) = \int_0^\infty \omega^{-2} J_0(r\omega) \times$$

$$\times J_1(a\omega) J_1(b\omega) J_1(c\omega) d\omega . \qquad (A10)$$

The function $G(r, a, b, c)$ is symmetrical with respect to rearrangement of the arguments a, b, and c. Relying on the results obtained above and using the convolution theorem for functions with circular symmetry, the integral $G(r, a, b, c)$ from Eq. (A10) can be represented in the following form:

$$G(r, a, b, c) = (2\pi a)^{-1} U_a(r)^{**}B(r, b, c) , \qquad (A11)$$

or, taking into account Eq. (A8),

$$G(r, a, b, c) = \frac{U_a(r)^{**}U_b(r)^{**}U_c(r)}{(2\pi)^2 abc} . \qquad (A12)$$

Finally, a more convenient representation for practical calculations has the form

$$G(r, a, b, c) = \int_0^\infty A(\nu, r, b) B(\nu, a, c)\nu d\nu , \qquad (A13)$$

which can be verified by substitution of the functions $A(\nu, r, b)$ from Eq. (A1) and $B(\nu, a, c)$ from (A6) with allowance for the relation

$$\int_0^\infty J_0(\nu\omega) J_0(\nu\omega') \nu d\nu = \frac{1}{\omega'} \delta(\omega - \omega'). \qquad (A14)$$

In particular, for $r = 0$, the relation

$$G(r = 0, a, b, c) = \int_0^\infty \omega^{-2} J_1(a\omega) J_1(b\omega) J_1(c\omega) d\omega =$$

$$= b^{-1} \int_0^b B(\nu, a, c) \nu d\nu \qquad (A15)$$

follows from Eq. (A13) in view of Eqs. (A2) and (A5). Based on Eq. (A4) and the convolution theorem, we can obtain the simple equation

$$G(r = 0, a, b, c) = ab/(4c), \; c \geq a + b . \qquad (A16)$$

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