Method of flexible adaptive piezoceramic mirror approximation by a limited number of Zernike polynomials

D.A. Bezuglov and R.A. Zabrodin

Rostov Service Institute,
Branch of the South-Russian State University of Economics and Service

Received February 27, 2006

A minimization method for Zernike polynomials is developed and investigated by the example of a piezoceramic mirror. The method takes into account the geometrical characteristics of response functions and the contribution of each polynomial to the final error of the phase front approximation according to statistics of phase fluctuations. To approximate aberrations of the turbulent atmosphere and response functions, the basis of Zernike polynomials close to the optimum is used. The approach suggested allows one to simplify essentially the construction of the phase front sensor: to reduce the number of its channels in low-order adaptive optical systems.

Introduction

Flexible mirrors with a totally deformable surface are used in adaptive optical systems of aperture sensing and phase conjugation as an executing device for compensation of non-stationary phase distortions appeared in the process of the wave front propagation through an optically inhomogeneous medium.[1–3] These mirrors are key elements of many modern systems for the radiation control and correction. Therefore, their capabilities determine to a large extent the parameters and the spectrum of solvable problems of the system.[4,5]

One of the basic requirements to the phase front corrector is a possibility of introducing necessary distortions into the incident radiation phase. Application of phase front correctors based on piezoelectric ceramic plates provides a better phase profile approximation as compared to the segmented corrector, because the phase distortions are usually smooth.[3–5] The adaptive mirrors based on piezoelectric plates, the mathematical apparatus for their description, as well as empirical profiles of responses of the flexible adaptive mirrors have been developed and investigated.[3–6]

When producing adaptive optical systems of phase conjugation, the phase distribution on the adaptive optical system aperture is, as a rule, measured.[7,8] Then these measurements are recalculated into the basis of response functions of a flexible mirror by one of the numerical methods. Note that the universal expansion conforming to a number of optimality conditions is the Karhunen–Loeve expansion.[2] It is characterized by the following properties meeting its optimality: the minimal root-mean-square error at confining the given number of terms in the infinite expansion series; deriving of the most amount of information (in comparison with any other expansion) about the function presented by the truncated series, no matter what number of terms is retained; and noncorrelatedness of expansion coefficients, which simplifies the further application of expansion results and their analysis.

However, the Karhunen–Loeve expansion has essential disadvantages: it requires a great body of *a priori* information (for example, knowledge of correlation functions of the measured characteristic), which is often absent or insufficient. The expansion eigenfunctions of the distorted field characteristics have a rather complex structure, and their practical realization as correcting devices with a variable function basis turns to be difficult.[2]

To approximate turbulent atmospheric aberrations, the system of Zernike polynomials, orthogonal (orthonormalized) inside a unit circle (or a circle with the radius R) is close to optimum.[1,2,9] In this case, it seems convenient to measure the phase distribution with the phase front sensor[7,8] immediately in the form of a limited number of Zernike polynomials and then, with the expansion of response functions of a flexible adaptive mirror as a linear combination of the same polynomials in hand, to recalculate the corresponding control signals. It often turns out that the response functions of individual electrodes of phase front correctors are described precisely enough by a limited number of Zernike polynomials, and the contribution of these polynomials themselves into the final error of phase distribution approximation depends nonlinearly on the polynomial number. Thus, there appears a problem to choose properly a set of polynomials in order to organize the control in a particular adaptive optical system.

This article presents the minimization technique for Zernike polynomials used in description of the flexible mirror profile, which takes into account the geometry of response functions and the contribution of each polynomial into the final error of the phase front approximation according to statistics of the phase fluctuations.

1. Approximation of the corrector response functions

To approximate the response function of a flexible adaptive mirror, we use a system of Zernike polynomials,

0235-6880/06/09 729-05 $02.00

orthogonal (orthonormalized) inside a unit circle presented in polar coordinates r, θ (Refs. 1, 2, 9)

$$Z_j(r,\theta) = \begin{cases} \sqrt{n+1}R_n^m(r)\sqrt{2}\cos m\theta \\ \text{for even polynomials and } m \neq 0, \\ \sqrt{n+1}R_n^m(r)\sqrt{2}\sin m\theta \\ \text{for odd polynomials and } m \neq 0, \\ \sqrt{n+1}R_n^0(r), \text{ for } m = 0, \end{cases} \quad (1)$$

where

$$R_n^m(r) = \sum_{s=0}^{(n-m)/2} \frac{(-1)^s (n-s)!\, r^{n-2s}}{s!\left[(n+m)/2-s\right]!\left[(n-m)/2-s\right]!}.$$

Quantities n and m are always integer and satisfy the condition $n \le m$, and the difference $n - |m|$ is even. Index j is the ordinal number of the mode, depending on n and m. The orthogonality condition in a circle of the unit radius has the form

$$\int_0^{2\pi}\int_0^1 Z_j(r,\theta) Z_j(r,\theta)\, \mathrm{d}r\mathrm{d}\theta = \delta_{j'}, \quad (2)$$

where

$$W(r) = \begin{cases} 1/\pi & \text{at } |r| \le 1, \\ 0 & \text{at } |r| > 1; \end{cases}$$

$\delta_{j'}$ is the Kronecker symbol.

The response functions of the phase corrector can be presented in the Zernike basis as

$$S_i(r,\theta) = \sum_{j=1}^{N} a_{ij} Z_j(r,\theta), \quad i = \overline{1,M}, \quad (3)$$

where a_{ij} is the coefficient of Zernike expansion at jth basis function for ith response function of the corrector $S_i(r,\theta)$; N is the number of polynomials; M is the number of the corrector response functions.

The phase front described by the corrector is

$$\varphi_k(r,\theta) = \sum_{i=1}^{M} S_i(r,\theta) b_i, \quad (4)$$

where b_i are the coefficients of the phase front expansion in terms of the corrector response functions, i.e., the piezoelectric mirror.

In fact, b_i are the controlling signals in operation of the adaptive optical system. The phase front measured by the sensor[7,8] in the Zernike basis has the following form:

$$\varphi_{\text{meas}}(r,\theta) = \sum_{j=1}^{N} Z_j(r,\theta) c_j, \quad (5)$$

where c_j are coefficients of the phase front expansion in terms of Zernike polynomials (signals from the phase front sensor output).

Having substituting Eq. (4) into Eq. (3), we obtain

$$\varphi_k(r,\theta) = \sum_{i=1}^{M}\sum_{j=1}^{N} a_{ij} Z_j(r,\theta) b_i. \quad (6)$$

It should be noted that the number of response functions M is a constructive parameter, and the number of polynomials N can be determined on the basis of well-known relations[3]:

$$N = \left[\left(\frac{-0.2944(D/r_0)^{\frac{5}{3}}}{\ln R_{\text{Sht}}}\right)^{\frac{2}{\sqrt{3}}}\right], \quad (7)$$

where D is the aperture diameter; R_{Sht} is the Strehl number, determining the quality of the adaptive correction and chosen usually between 0.7 and 0.8; [*] is the operator of calculation of the number's integer part; r_0 is the Fried radius determined as[2,3]:

$$r_0 = \left(0.423 k_1^2 \int_0^L C_n^2(l)\mathrm{d}l\right)^{-\frac{3}{5}}, \quad (8)$$

where L is the turbulent layer thickness; $C_n^2(l)$ is the structural constant of the atmosphere; $k_1 = 2\pi/\lambda$ is the radiation wave number.

To determine the expansion coefficients of the phase front in terms of the corrector b_i response functions, equate the expression (5) to (6):

$$\varphi_k(r,\theta) = \varphi_{\text{meas}}(r,\theta). \quad (9)$$

or, taking into account Eqs. (4) and (5):

$$\sum_{i=1}^{M}\sum_{j=1}^{N} a_{ij} Z_j(r,\theta) b_i = \sum_{j=1}^{N} Z_j(r,\theta) c_j. \quad (10)$$

The solution of the latter equation can be obtained by minimizing the following square form:

$$\sum_{j=1}^{N}\left(\sum_{i=1}^{M} a_{ij} b_i - c_j\right)^2 \to \min. \quad (11)$$

Having calculated the corresponding partial derivatives and equating them to zero:

$$\frac{\partial}{\partial b_i}\sum_{j=1}^{N}\left(\sum_{i=1}^{M} a_{ij} b_i - c_j\right)^2 = 0, \quad (12)$$

we obtain the system of N linear equations:

$$\sum_{i=1}^{M} a_{ik} a_{ji} b_i = \sum_{i=1}^{M} c_j a_{ji}, \quad i,k = \overline{1,N}; \quad (13)$$

$$\mathbf{A}_1 \cdot \mathbf{B} = \mathbf{D}, \quad (14)$$

where $\mathbf{A}_1$ is the matrix $N \times N$; $\mathbf{B}$, $\mathbf{D}$ are vector-columns of N size, and the elements A_1 and $\mathbf{D}$ are determined as

$$A_{1kj} = \sum_{i=1}^{\mathrm{M}} a_{ik} a_{ji}, \quad D_j = \sum_{i=1}^{M} c_j a_{ji}. \quad (15)$$

Thus, the found coefficients of phase front expansion in terms of the corrector b_i response functions can be used for organization of control in some adaptive optical system, and $\mathbf{A}_1$ is, in fact, the matrix of transition between the Zernike basis and the basis of response functions.

Note that the number of Zernike polynomials used in description of the corresponding response functions (5), can be decreased, taking into account their contribution into approximation of response function $S_i(r,\theta)$ and the contribution of the corresponding polynomial into description of statistics of the phase fluctuations. In this case, the expenses for the building of adaptive optical system can be lowered, because at insignificant deterioration of the functioning quality within acceptable limits (Strehl number) the number of channels in the phase front sensor can be decreased. This is due to the fact that it is not necessary to calculate the expansion coefficients of the phase front c_j, which further, owing to the construction of the used corrector, cannot be used in the building of the adaptive optical system.

To do this, let us calculate by the least square method the coefficients of the response function $S_i(r,\theta)$ expansion into the Zernike series in polar coordinates accurate to the Nth term, where N is predetermined by the expression (7). The Zernike basis in the given case is convenient due to the fact that both the response functions $S_i(r,\theta)$ and statistics of phase atmospheric fluctuations can be presented in it.[1,2] Let us minimize the quadratic form for the ith response function:

$$\sum_{j=1}^{N}\left(\sum_{k=1}^{K} a_{ij}Z_j(r_k,\theta_k)-w_k\right)^2 \to \min, \qquad (16)$$

where w_k are the experimentally measured values of the response functions in points of the mirror aperture; $Z_j(r_k,\theta_k)$ are the polynomial values in these points, respectively; $k=\overline{1,K}$; K is the number of points, in which the response functions values are measured

$$\frac{\partial}{\partial a_{ij}}\sum_{j=1}^{N}\left(\sum_{k=1}^{K} a_{ij}Z_j(r_k,\theta_k)-w_k\right)^2 = 0. \qquad (17)$$

As a result, a system of linear algebraic equations is obtained:

$$\sum_{j=1}^{N} a_{ij}\left[\sum_{k=1}^{K} Z_l(r_k,\theta_k)Z_j(r_k,\theta_k)\right] = \sum_{j=1}^{N} w_k z_l(r_k,\theta_k),$$

$$l=\overline{1,N}, \qquad (18)$$

where matrix coefficients have the form

$$M_{lj} = \sum_{k=1}^{K} Z_l(r_k,\theta_k)Z_j(r_k,\theta_k),$$

and the vector elements of free terms are determined by the relation

$$B_{1l} = \sum_{k=1}^{K} w_k z_l(r_k,\theta_k).$$

Due to orthogonality of the Zernike polynomials, the matrix M should have a diagonal form in the general case, when the number of measuring points K tends to infinity, i.e., at transition to the integral of the form (2). However, since the number of measurement points is finite, this condition is not fulfilled, and the matrix M, strictly speaking, is not diagonal. Thus, the conducted calculation experiment has shown that even at increase of the number of measurement points up to 10^5, the non-diagonal terms are presented in M: a_{38}, a_{27}, and so on., although their number is small.

As a result, solution of the system (18) can be obtained by any known method, for example, by the Gaussian method. However, under certain conditions (wrong choice of approximation points), the obtained system can be ill-posed ($|M|\approx 0$). In this case, the problem of matrix inversion is unstable in terms of Adamar, and the Gaussian method is not applicable because of fast accumulation of the calculation error. Hence, when choosing the approximation grid, it is necessary to minimize the number of points, where the condition $Z_j(r_k,\theta_k)=0$ holds. This can be done through introduction of a small angular shift $\Delta\theta$ to the chosen coordinate system, when constructing the computational algorithm.

To estimate the necessary number of polynomials after restoration of the mirror profile, normalize the obtained coefficients:

$$a_{ij}^{*} = a_{ij}\Big/\sum_{j=1}^{N} a_{ij}. \qquad (19)$$

Calculate the contribution of each term from the Zernike expansion into the quality of the Kolmogorov turbulence correction, using the expressions[3]

$$\Delta J^{j,j+1}\left(\frac{D}{r_0}\right) = \Delta_{j+1}\left(\frac{D}{r_0}\right)-\Delta_j\left(\frac{D}{r_0}\right),\quad j=\overline{1,N-1}, \qquad (20)$$

where $\Delta_j\left(\frac{D}{r_0}\right) = 0.2944\, j^{-\frac{\sqrt{3}}{2}}\left(\frac{D}{r_0}\right)^{\frac{5}{3}}$ is the root-mean-square residual error. Normalize the obtained differences:

$$\Delta J^{*j,j+1} = \Delta J^{j,j+1}\Big/\sum_{j=1}^{N}\Delta J^{j,j+1}. \qquad (21)$$

The final expression for the normalized expansion coefficients with accounting for the weight coefficients of each polynomial contribution to the residual error of the phase front approximation has the form

$$a_{ij}^{\text{n}} = \Delta J^{*j,j+1}a_{ij}^{*}. \qquad (22)$$

Having chosen the threshold Δa_{ij}^{n}, it is possible to sequentially eliminate from the expression (3) the Zernike coefficients with minimal contribution. The decline in the phase front correction quality can be estimated by the following expression:

$$\sigma^2 = \frac{1}{K}\sum_{k=1}^{K}(a_{ij}^{*} - a_{ij}^{\text{n}})Z_j(r_k,\theta_k)^2, \qquad (23)$$

where the corresponding jth values of a_{ij}^{n} are replaced with 0; σ is the root-mean-square deviation.

2. Numerical experiment

To study the possibility of the proposed technique realization, we have used the experimental results for a flexible adaptive mirror based on the TsTBS-3 piezoelectric ceramic plates.[3,5] The mirror reflecting surface of 50 mm in diameter is formed by silver electrodes, spray-coated on the plate surface of 1 mm thickness. Figure 1 presents a sample with five controlling electrodes. Measurements[3,5] were carried out in planes a, b, c, and d at a step h of 5 mm. Consider the application of the proposed technique by an example of only four lateral controlling electrodes.

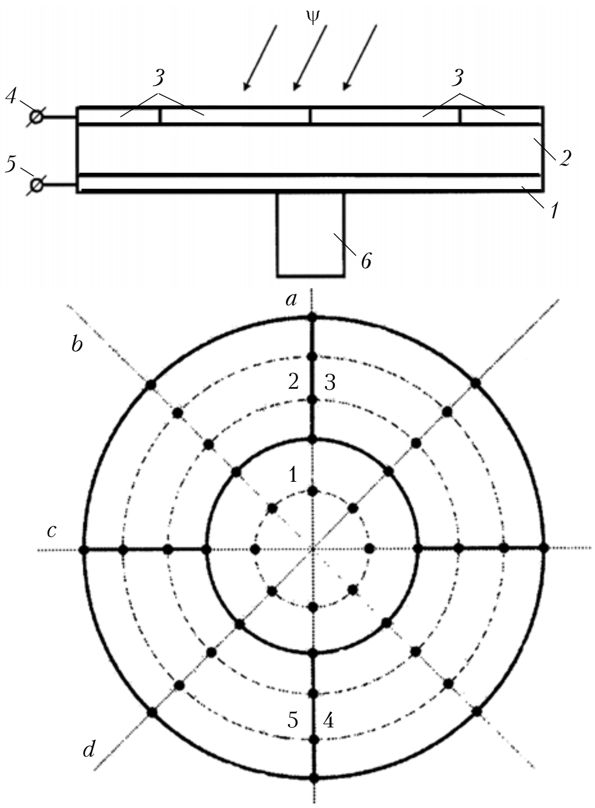

Fig.1. Flexible mirror construction: metallic base (*1*); piezoceramic plate (*2*); the controlling electrodes (*3*); terminals (*4* and *5*); support (*6*).

Figure 2 presents the response profiles of the sample at voltage application to the control electrode 2.

In our case we consider the basis of nine Zernike polynomials in the polar coordinate system, because this basis is sufficient for compensation of main aberrations.[3] The mean phase is not taken into account:

$$Z_2(r,\theta) = 2r\cos(\theta), \quad Z_3(r,\theta) = 2r\sin(\theta),$$

$$Z_4(r,\theta) = \sqrt{6}r^2\sin(2\theta), \quad Z_5(r,\theta) = \sqrt{3}(2r^2 - 1),$$

$$Z_6(r,\theta) = \sqrt{6}r^2\cos(2\theta), \quad Z_7(r,\theta) = 2\sqrt{2}r^3\sin(3\theta),$$

$$Z_8(r,\theta) = 2\sqrt{2}(3r^3 - 2r)\sin(\theta),$$

$$Z_9(r,\theta) = 2\sqrt{2}(3r^3 - 2r)\cos(\theta),$$

$$Z_{10}(r,\theta) = 2\sqrt{2}r^3\cos(3\theta). \qquad (24)$$

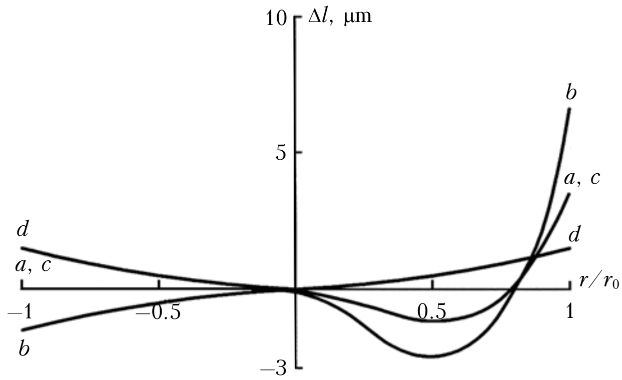

Fig. 2. Response profiles of a piezoceramic mirror.

Based on the data obtained by the mathematical modeling and using the Mathcad 11, the mirror response function was built (see Fig. 1) by applying the control voltage to the lateral electrode. By the above-described method with the use of available experimental data,[3,5] the Zernike expansion coefficients were derived for the control acts considered in the experiment. The obtained coefficients were normalized in accordance with Eqs. (19), (21), and (22). In such a way, the Zernike modes, most meaningful for compensation of atmospheric distortions used in description of response functions of the investigated piezoelectric mirror, were determined. The accepted Strehl number was equal to 0.8. The estimate has shown the modes 2, 3, 5, 8, and 9 to be most meaningful for quasi-optimum correction of the wave front. The calculation results are tabulated.

Coefficient	Mode number									RMSD
	2	3	4	5	6	7	8	9	10	
a_{ij}	0.254	0.254	−0.043	0.713	0	0.222	0.635	0.635	−0.222	–
a_{ij}^{*}	0.104	0.104	−0.018	0.291	0	0.091	0.259	0.259	−0.091	–
$\Delta J^{*j,j+1}$	0.407	0.222	0.127	0.080	0.054	0.039	0.029	0.023	0.018	–
$a_{ij}^{\text{n}} \cdot 10^{-2}$ (without 6)	4.227	2.305	−0.225	2.338	0	0.354	0.757	0.585	−0.162	0.892
$a_{ij}^{\text{n}} \cdot 10^{-2}$ (without 6, 10)	4.227	2.305	−0.225	2.338	0	0.354	0.757	0.585	0	0.921
$a_{ij}^{\text{n}} \cdot 10^{-2}$ (without 4, 6, 10)	4.227	2.305	0	2.338	0	0.354	0.757	0.585	0	0.922
$a_{ij}^{\text{n}} \cdot 10^{-2}$ (without 4, 6, 7, 10)	4.227	2.305	0	2.338	0	0	0.757	0.585	0	0.949

As follows from the table, the suggested approach allows a justified minimization of the number of Zernike polynomials in description of response functions of a piezoceramic mirror. Thus, elimination from the basis of expansion coefficients with numbers 4, 6, and 10 leads to a 3% increase of RMSD, and elimination of expansion coefficients with numbers 4, 6, 7, and 10 leads to a 6% increase of RMSD. Owing to the symmetry of control electrodes, the obtained results can be used for the rest of control electrodes as well.

Conclusions

We have developed and investigated the technique of minimization of the number of Zernike polynomials, used in description of the piezoelectric mirror profile.[3,5] The technique takes into account geometrical characteristics of the response functions and the contribution of each polynomial to the final error of phase front approximation in accordance with statistics of phase fluctuations.

To approximate aberrations of the turbulent atmosphere and response functions, the basis of Zernike polynomials close to the optimum and orthogonal (orthonormalized) inside a unit circle was used. Accounting for the fact that statistics of phase fluctuations is known for the given basis, the suggested approach allows one to essentially simplify the construction of the phase front sensor, i.e., to decrease the number of its channels. In the case, when the phase front corrector peculiarities do not allow reproducing one or other Zernike modes, to improve the correction quality, it is possible to act in the following way. First, the number N of polynomials should be chosen, which is 20–30% more than in expression (7). Then, using the suggested technique, to minimize the basis, sequentially eliminating the polynomials giving the minimal contribution into the corrector response, taking into account statistics of turbulent distortions.

Since only a limited number of polynomials is used in practice, the suggested approach can be applied to low-order adaptive optical systems.

References

1. M.A. Vorontsov and V.I. Shmal'gauzen, *Principles of Adaptive Optics* (Nauka, Moscow, 1985), 336 pp.
2. D.P. Luk'yanov, A.A. Kornienko and B.E. Rudnitsky, *Optical Adaptive Systems* (Radio i Svyaz', Moscow, 1989), 240 pp.
3. D.A. Bezuglov, Z.P. Mastropas, E.N. Mishchenko, E.N. Myasnikov, S.V. Tolstousov, and V.L. Tyurikov, Opt. Atm. **2**, No. 12, 1119–1123 (1989).
4. A.S. Sobolev, T.Yu. Cherezova, and A.V. Kudryashov, Atmos. Oceanic Opt. **18**, No. 3, 254–258 (2005).
5. D.A. Bezuglov, Z.P. Mastropas, E.N. Mishchenko, et al., "*Wave front mirror corrector,*" Inventor's Certificate No. 1695252 SSSR, MKI 5 G 02 B 5/10, Bull. No. 44 (1991).
6. Yu.N. Isaev and E.V. Zakharova, Atmos. Oceanic Opt. **12**, No. 8, 679–682 (1999).
7. D.A. Bezuglov, E.N. Mishchenko, M.I. Krymsky, and O.V. Serpeninov, "*Wave front sensor,*" Inventor's Certificate No. 1720051 SSSR, MKI 5 G 02 B 26/06, Bull. No. 10 (1992).
8. D.A. Bezuglov, E.N. Mishchenko and V.L. Tyurikov, "*Wave front sensor,*" Inventor's Certificate No. 1647496 SSSR, MKI 5 G 02 B 27/00. Bull. No. 17 (1991).
9. R.J. Noll, J. Opt. Soc. Am. **66**, No. 3, 207–211 (1976).