FREDHOLM SERIES SOLUTION TO THE INTEGRAL EQUATIONS OF THERMAL BLOOMING

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From the solution for the linear theory of thermal blooming,[2] *the propagator is a* 2×2 *matrix that satisfies an integral equation of Fredholm type. We develop a generalized Fredholm series solution to this integral equation. Since the kernel is a matrix, the usual determinants in the Fredholm series contain ordering ambiguities. We resolve all ordering ambiguities using the standard diagrammatic representation of the series. The Fredholm denominator is computed for the case of uncompensated and compensated propagation in a uniform atmosphere with uniform wind. When the Fredholm denominator vanishes, the propagator contains poles. In the compensated case, the denominator does develop zeros. The single mode phase compensation instability gains computed from the zeros agrees with results obtained from other methods.*

1. INTRODUCTION AND BACKGROUND

The effects of thermal blooming in large beams can be broadly classified by the transverse size of the disturbances in the beam. Effects on the order of the size of the beam are called whole beam blooming. Structure much smaller than the beam diameter (typically on the order of the scintillation scale) is called small scale blooming.

Blooming converts intensity fluctuations into (unwanted) phase fluctuations. Assuming ideal beam optics, there are two main sources that produce intensity fluctuations that act as initial conditions for blooming. They are atmospheric turbulence and edge diffraction. The ability of atmospheric turbulence to produce intensity fluctuations is measured by the turbulence Fresnel number, $N_\mathrm{T} = r_0^2/\lambda L$. The smaller the turbulence Fresnel number, the bigger the ambient intensity fluctuations. The production of intensity fluctuations from edge diffraction is measured by the whole beam Fresnel number, $N_\mathrm{F} = D^2/\lambda L$, where D is the beam diameter. The smaller the beam, the smaller N_F is and the greater the effects of edge diffraction.

Thus, whether whole beam blooming or small scale blooming is dominant depends on the square of the relative size of the turbulence coherence diameter to the size of beam. As an explicit example, consider a 10–meter system with a blooming layer height of 5 km.

TABLE I. Hufnagel–Valley 5/7 turbulence profile. D = 10 m, L = 5 km, $N_\mathrm{F} = D^2/\lambda L$, *and* $N_\mathrm{T} = r_0^2/\lambda L$.

Wavelength λ	0.41 μ	0.8 μ	1.7 μ	3.8 μ	10.0 μ
Whole beam Flesnel number N_F	48780	25000	11764	5263	2000
Turbulence Flesnel number N_T	0.780	2.06	5.90	18.2	70.7

Without a doubt, large aperture high intensity HPL systems fall into the small–scale regime.

There are two physical processes involved in blooming: stimulated thermal Rayleigh scattering (STRS) and the phase compensation instability (PCI). A qualitative description of these processes is given in Ref. 1. The STRS is a process that occurs at all transverse spatial scales. The STRS occurring at spatial scales outside of the compensation band cannot be affected by the adaptive optics and therefore is completely uncorrectable. The PCI is an adaptive optics feedback instability and therefore occurs inside the compensation band. Even though PCI occurs at compensated spatial scales, it too is not completely correctable due to diffraction even if the adaptive optics system has infinite dynamic range at each compensated scale.

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The dominance of small–scale blooming has several important consequences. We demonstrated that small–scale physics is linear even though the equations of motion are nonlinear.[2] We also used our analytic computation of the structure functions and Strehl ratio to develop analytic functional scaling based on the absorption profiles.[3] In addition, the beam edges are dynamically unimportant, thus whole beam Strehl predictions can be made from infinite beam results or patches using functional reconstruction.[4] We implemented the functional absorption profile scaling and reconstruction as a systems model to make accurate Strehl predictions for large beams in real–time.[4,5] Unlike brute force 4–d wave–optics simulations, reconstruction becomes more accurate as the beam gets larger and is presently the only way to predict large beams results accurately.

In Ref. 2 we converted the linearized matrix differential equations of motion into an integral equation of motion for the Green's functions or propagators. We solved the integral equation in the time domain perturbatively as a Neumann series. The integral equation is of the Fredholm type (except that the kernel is a matrix), and therefore can be expressed as a Fredholm solution. The resolvent kernel of the Fredholm solution is written as the ratio of the Fredholm first minor over the Fredholm determinant. If the Fredholm solution is expanded in a power series (the Fredholm series solution), then the Neumann series is recovered after the division is completed. However, unlike the Neumann solution, the Fredholm determinant can be examined to determine if the propagator contains poles since the determinant will vanish at a pole. If the Fredholm solution is not expanded in a power series, then it is a nonperturbative representation of the propagator. The determinant and minor can be expressed as path integrals and are computed using functional calculus. The path integral representation of the uncompensated propagator is given in Ref. 6. The Fredholm solution is summarized in the figure below.

Fredholm solution:

– propagator = free propagator + resolvent

– resolvent is the ratio: $\dfrac{\text{Fredholm first minor}}{\text{Fredholm determinant}}$

perturbative	nonperturbative
Express Fredholm first minor and Fredholm determinant as a power series in λ – Fredholm series solution	Compute Fredholm first minor and Fredholm determinant using functional calculus – Fredholm determinant is a sourceless path integral – Fredholm first minor is the second moment of a path integral with sources

In this paper we will use the Fredholm series solution to compute the Fredholm determinant for case of uncompensated propagation and for compensated propagation. We will find for uncompensated propagation that the determinant does not vanish so that the STRS propagator does not contain poles. When phase–only compensation is introduced, the determinant does vanish at isolated points so that the compensated propagator does contain poles. The physical process associated with the existence of these poles is PCI.

2. INTEGRAL EQUATIONS OF MOTION

For phase and intensity fluctuations, $\hat{S}_1$ and $\hat{I}_1$, about a plane wave, the combined heating and propagation equations of linearized isobaric thermal blooming are

$$\partial_\tau = \begin{pmatrix}\partial_\xi & -1\\ 1 & \partial_\xi\end{pmatrix}\begin{pmatrix}\hat{I}_1\\ \hat{S}_1\end{pmatrix} + \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\hat{I}_1\\ \hat{S}_1\end{pmatrix} =$$

$$= \begin{pmatrix}0\\ \beta\,\delta(\tau)\,\delta\,\hat{n}(\xi)\end{pmatrix} \tag{1}$$

or

$$\left(D^{(0)} + V\right)\begin{pmatrix}\hat{I}_1\\ \hat{S}_1\end{pmatrix} = \begin{pmatrix}0\\ \beta\,\delta(\tau)\,\delta\,\hat{n}(\xi)\end{pmatrix}, \tag{2}$$

where $\xi = \boldsymbol{\kappa}^2 z/2\kappa$ is nondimensional altitude z, τ is nondimensional time, and $\beta = 4\kappa^2/\boldsymbol{\kappa}^2$ where $\kappa = 2\pi/\lambda$ and $\boldsymbol{\kappa}$ is the transverse spatial frequency. The blooming vertex operator V converts intensity into phase at one point

$$\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}I\\ S\end{pmatrix} = \begin{pmatrix}0\\ S_{\text{bloom}}\end{pmatrix}. \tag{3}$$

The Green's functions or propagator $G(\xi, \xi_0; \tau - \tau_0)$ for the phase and intensity fluctuations satisfies

$$\left(D^{(0)} + V\right) G(\xi, \xi_0; \tau - \tau_0) =$$

$$= \begin{pmatrix}\delta(\xi - \xi_0)\,\delta(\tau - \tau_0) & 0\\ 0 & \delta(\xi - \xi_0)\,\delta(\tau - \tau_0)\end{pmatrix}. \tag{4}$$

The propagator also satisfies the integral equation[2]

$$G(\xi, \xi_0; \tau - \tau_0) = G^{(0)}(\xi, \xi_0; \tau - \tau_0) -$$

$$-\int d\tau_1 \int d\xi_1\, G^{(0)}(\xi, \xi_1; \tau - \tau_1)\, V(\xi_1)\, G(\xi_1, \xi_0; \tau_1 - \tau_0), \tag{5}$$

where $G^{(0)}(\xi, \xi_0; \tau - \tau_0)$ is the free propagator (no blooming, $V = 0$, just turbulence). If there are no adaptive optics, or $\boldsymbol{\kappa}$ lies outside of the compensation band, the uncompensated free propagator is[2]

$$G^{(0)}(\xi, \xi_0; \tau - \tau_0) = H(\xi - \xi_0)\, H(\tau - \tau_0)\times$$

$$\times \begin{pmatrix}\cos(\xi - \xi_0) & \sin(\xi - \xi_0)\\ -\sin(\xi - \xi_0) & \cos(\xi - \xi_0)\end{pmatrix}, \tag{6}$$

where H is the Heaviside step function. When the transverse spatial frequency $\boldsymbol{\kappa}$ is in the compensation band, the compensated propagator, call it $G_{\text{PC}}(\xi, \xi_0; \tau - \tau_0)$, satisfies the same integral equation, Eq. (5), as the uncompensated propagator. The only difference is that the free propagator is modified to include the new boundary condition at $\xi = 0$ introduced by the adaptive optics. We will return to this in Sec. 6.

3. FREDHOLM SERIES

Since we are interested in computing the location of the poles, if any, in the uncompensated and compensated propagators, let us apply the Laplace transform to the integral equation of motion, Eq. (5). Let s be the Laplace transform variable dual to time τ. After multiplying by s, Eq. (5) becomes

$$sG(\xi, \xi_0; s) = G^{(0)}(\xi, \xi_0) -$$

$$- \frac{1}{s} \int \mathrm{d}\, \xi_1\, G^{(0)}(\xi, \xi_1)\, V(\xi) \Big(s\, G(\xi_1, \xi_0; s) \Big) , \qquad (7)$$

where in the uncompensated case,

$$G^{(0)}(\xi, \xi_0) = H(\xi - \xi_0) \begin{pmatrix} \cos(\xi - \xi_0) & \sin(\xi - \xi_0) \\ -\sin(\xi - \xi_0) & \cos(\xi - \xi_0) \end{pmatrix} . \qquad (8)$$

A Fredholm equation of the second kind is of the form[7]:

$$G(x, y) = G_0(x, y) + \lambda \int \mathrm{d}\, z\, K(x, z)\, G(z, y) ,$$

where $K(x, y)$ is the kernel. Comparing with Eq. (7) we see that $\lambda = - 1/s$, and that $K = G^{(0)} V$. It is more convenient to associate the blooming vertex operator V with $\int \mathrm{d}z$ than with the kernel K, hence we will apply the Fredholm series solution to an integral equation of the form:

$$G(x, y) = G_0(x, y) + \lambda \int \mathrm{d}\, z\, K\, (x, z)\, V(z)\, G(z, y) , \qquad (9)$$

The solution to Eq. (9) is written in terms of a resolvent kernel, $R(x, y)$.

$$G(x, y) = G_0(x, y) + \lambda \int \mathrm{d}\, z\, R(x, z)\, V(z)\, G_0(z, y). \qquad (10)$$

The Fredholm solution expresses the resolvent kernel as the ratio,

$$R(x, y) = D_1(x, y, \lambda)/D(\lambda), \qquad (11)$$

where $D_1(x, y, \lambda)$ is called the Fredholm first minor, and $D(\lambda)$ is called the Fredholm determinant.

The Fredholm series solution expands D_1 and D as power series in λ.

$$D_1(x, y, \lambda) = K(x, y) - \lambda \int \mathrm{d}\, z\, V(z) \begin{vmatrix} K(x, y) & K(x, z) \\ K(z, y) & K(z, z) \end{vmatrix} +$$

$$+ \frac{\lambda^z}{2!} \int \int \mathrm{d}z\, \mathrm{d}z'\, V(z) V(z') \begin{vmatrix} K(x, y) & K(x, y) & K(x, z') \\ K(z, y) & K(z, z) & K(z, z') \\ K(z', y) & K(z', z) & K(z', z') \end{vmatrix} + \ldots ,$$

$$D(\lambda) = 1 - \lambda \int \mathrm{d}z\, V(z)\, K(z, z) +$$

$$+ \frac{\lambda^z}{2!} \int \int \mathrm{d}z\, \mathrm{d}z'\, V(z)\, V(z') \begin{vmatrix} K(x, y) & K(x, z) \\ K(z, y) & K(z, z) \end{vmatrix} + \ldots . \qquad (12)$$

4. DIAGRAMMATIC REPRESENTATION

The series solution can be transformed into pictures or Feynman diagrams using the following legend[7]:

$$K(x, y) \leftrightarrow \Big|{}^{x}_{y} \quad ,$$

$$\int \mathrm{d}\, z\, K(x, y)\, V(z)\, K(z, y) \leftrightarrow \big|\, V(z) \quad ,$$

$$\int \mathrm{d}z\, K(z, z)\, V(z) \leftrightarrow \bigcirc ,$$

$$\int \mathrm{d}\, z\, K(x, y)\, K(z, z)\, V(z) \leftrightarrow \big| \bigcirc \quad .$$

The first Fredholm minor, $D_1(x, y, \lambda)$, in Eq. (12) becomes

$$D_1(x, y, \lambda) =$$

$$= \Big| - \lambda \Big(|\bigcirc - | \Big) + \frac{\lambda^z}{2!} \Big(|\bigcirc \bigcirc + 2| - |\bigcirc - 2|\bigcirc \Big) + \ldots ,$$

while the Fredholm determinant, $D(\lambda)$, in Eq. (12) is

$$D(\lambda) = 1 - \lambda \bigcirc + \frac{\lambda^2}{2} \Big(\bigcirc\ \bigcirc - \bigcirc \Big) + \ldots .$$

The Fredholm determinant can also be written as

$$D(\lambda) = \exp\Big(- \lambda \bigcirc - \frac{\lambda^z}{2} \bigcirc - \frac{\lambda^3}{3} \bigcirc - \ldots \Big) ,$$

which can be checked by expanding the exponential in a power series and comparing with the diagrams or equations above.

The resolvent is the ratio of the two sets of pictures above. To order λ, Eq. (11) is

$$R(x, y, \lambda) =$$

$$= \frac{\Big| - \lambda \Big(|\bigcirc - | \Big)}{1 - \lambda \bigcirc} = \frac{\Big| \Big(1 - \lambda \bigcirc \Big) + 1 \Big|}{1 - \lambda \bigcirc} = \Big| + \lambda \Big| .$$

After the division is completely carried out, the result is the Neumann series for the resolvent. The diagrammatic representation of the Neumann series is

$$R(x, y, \lambda) = \Big| + \lambda \Big| + \lambda^2 \Big| + \lambda^3 \Big| + \ldots .$$

In our case, we are considering a generalization of the Fredholm series solution in that the kernel, $K(\xi, \xi_0) = G^{(0)}(\xi, \xi_0)$, Eq. (8), is a matrix. Since matrix multiplication does not commute in general, the order in which the elements in the determinants in Eq. (12) are taken matters. The same applies to the factors of V. This means that the order in which the pictures appear also matters. For example, since the kernel is a matrix, then

$$\int dz\, K(x, y)\, K(z, z)\, V(z) \neq \int \mathrm{d}\, z\, K(z, z)\, V(z)\, K(x, y)\ ,$$

or, in pictures,

$$\big|\bigcirc \neq \bigcirc\big|\ .$$

Furthermore, the resolvent is the ratio of two matrices. What this really means is that the resolvent is the matrix product of the first minor and the inverse matrix of the determinant. Since the matrices do not commute, an ordering ambiguity arises. Namely, do we take

$$R(\xi, \xi_0; s, \lambda) = D_1(\xi, \xi_0; s, \lambda)\, D^{-1}(\lambda)$$

or

$$R(\xi, \xi_0; s, \lambda) = D^{-1}(\lambda)\, D_1(\xi, \xi_0; s, \lambda)\ .$$

The diagrams can be used to select the proper ordering that defines a consistent ordering convention. That is, the proper order of terms in the minor D_1 is set by the need to be able to symbolically factor $D(\lambda)$ out of D_1 either on the left side or the right side. For example, suppose we choose the ordering

$$\big|\bigcirc\ ,$$

instead of

$$\bigcirc\big|$$

for the $O(\lambda)$ term in D_1. In this case, $D(\lambda)$ factors out of D_1 to the right:

$$D_1(\xi, \xi_0, \lambda) =$$

$$= \Big| - \lambda\Big(\big|\bigcirc - \big|\Big) + \ldots = \Big(\Big| + \lambda\,\Big|\Big)\Big(1 - \lambda\bigcirc\Big) + \ldots\ .$$

Therefore, we must use $R(\xi, \xi_0; s, \lambda) = D_1(\xi, \xi_0; s, \lambda) D^{-1}(\lambda)$. Otherwise, the denominator will not cancel.

5. UNCOMPENSATED PROPAGATION

We are ready to compute the Fredholm determinant for STRS. The result is particularly simple.

From Eq. (8) we see that $K(\xi_1, \xi_1) = G^{(0)}(\xi_1, \xi_1)$ is just the unit matrix. Thus,

$$\bigcirc$$

is equal to

$$\overline{\lambda} \int\limits_0^{\xi} \mathrm{d}\, \xi_1)\, G^{(0)}(\xi_1, \xi_1)\, V(\xi_1) = \lambda\, \xi\, V = \lambda\, \xi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \qquad (13)$$

However, due to the presence of the step function, $H(\xi_i - \xi_j)$, in $G^{(0)}(\xi_i, \xi_j)$, all of the higher–order loops vanish. For example,

$$\xi_2 \bigcirc \xi_1 = 0\ , \quad \xi_1 \underset{\xi_3}{\bigcirc} \xi_2 = 0$$

since for the left–hand side loop we must have $\xi_1 > \xi_2$ and $\xi_2 > \xi_1$ which is not possible, and for the right–hand side loop we must have $\xi_1 > \xi_2 > \xi_3 > \xi_1$ which is also impossible. Therefore, we are left with

$$D(\lambda) = \exp\Big(-\lambda \bigcirc\Big)\ .$$

Since V is nilpotent (i.e., $V^2 = 0$), the exponential is easy to compute. The result is

$$D(\lambda) = \begin{pmatrix} 1 & 0 \\ -\lambda\xi & 1 \end{pmatrix}. \qquad (14)$$

and it is clear that the Fredholm determinant never vanishes for any λ. Therefore, there are no poles present in the STRS propagator. It is known that the asymptotic behavior in time of the STRS propagator is governed by a saddle point instead of a pole.[8,9]

6. COMPENSATED PROPAGATION

When phase–only compensation is introduced, the new propagator, $G_{\mathrm{PC}}(\xi, \xi_0; s)$, satisfies the same integral equation, Eq. (7), as the uncompensated case. The only item that changes is the propagator at $\tau = 0$ or $V = 0$, the propagator for turbulence only, $G^{(0)}_{\mathrm{PC}}(\xi, \xi_0)$. The compensated turbulence–only propagator, $G^{(0)}_{\mathrm{PC}}(\xi, \xi_0)$ is the uncompensated propagator plus an additional term to account for the new boundary conditions that arise due to the introduction of a beacon laser and the correction of phase at the ground. Let the compensation coupling for transverse spatial mode $\boldsymbol{\kappa}$ be $g(\boldsymbol{\kappa})$. Then,[2]

$$G^{(0)}_{\mathrm{PC}}(\xi, \xi_0) = G^{(0)}(\xi, \xi_0) - g(\boldsymbol{\kappa}) \begin{pmatrix} 0 \\ \sin(\xi)\cos(\xi_0) \\ 0 \\ \cos(\xi)\cos(\xi_0) \end{pmatrix}. \qquad (15)$$

Note that the additional piece does not contain the step function in altitude, $H(\xi - \xi_0)$. The new term in the turbulence–only propagator requires more pictures. We will represent the new term with double lines:

$$G^{(0)}_{\mathrm{PC}}(\xi, \xi_0) = G^{(0)}(\xi, \xi_0) - g(\boldsymbol{\kappa}) G^{(0)}_{bc}(\xi, \xi_0) \leftrightarrow \Big|\begin{matrix} x \\ y \end{matrix} - g \,\Big\|\begin{matrix} x \\ y \end{matrix}\ .$$

It the computation of the Fredholm determinant, a tremendous amount of cancellation occurs. For example, by explicit computation using Eq. (15), one can verify that

$$\begin{vmatrix} G^{(0)}_{bc}(\xi_1, \xi_1) & G^{(0)}_{bc}(\xi_1, \xi_2) \\ G^{(0)}_{bc}(\xi_2, \xi_1) & G^{(0)}_{bc}(\xi_2, \xi_2) \end{vmatrix} = 0 \Rightarrow ◎ = ◎\ ◎\ .$$

Using the pictures, one can show that the determinant reduces to

$$D(\lambda) = \exp\Big(-\lambda\bigcirc\Big)\Big(-\lambda ◎ - \lambda^2\bigcirc - \lambda^3\bigcirc - \lambda^4\bigcirc - \ldots\Big).$$

By examining the diagrams above, we note that the sum of the loops for λ^2 and greater is a single loop, where the uncompensated resolvent appears on the left–hand side of the loop and the double line $G^{(0)}_{\rm PC}$ on the right. If we represent the Neumann series for the uncompensated propagator, $G(\xi - \xi_0; s, \lambda)$, by a thick solid line:

┃ = ┃ + λ ┃ + λ^2 ┃ + λ^3 ┃ + ... ,

then the sum of the loops for λ^2 and greater that appears in the determinant can be represented as

– λ^2 ○ – λ^3 ○ – λ^4 ○ – ... = – λ^2 ○ ,

and we only have to compute a single loop. The Neumann series for the uncompensated propagator can be computed using the techniques above. The result is also given in Ref. 2. When this single loop is computed, the term

λ ◎

is cancelled and a common factor of

$$1 + g\,\lambda \int\limits_0^{\xi} d\xi_1 \frac{\sin(\alpha\xi_1)\cos(\xi_1)}{\alpha}, \quad \alpha = \sqrt{1-\lambda}\,. \qquad (16)$$

appears in front. Thus, $D(\lambda)$ will vanish if this factor vanishes. There are λ_0 that cause this factor to vanish. Therefore, the compensated propagator contains poles. This is PCI. Recall that $\lambda = -1/s$. Thus, $-1/\lambda_0$ is the single mode gain. The gains resulting from Eq. (16) above agree with those obtained by other methods,[9] and references therein.

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