Some problems in smoothing lidar return signals

S.N. Volkov, B.V. Kaul, V.A. Shapranov, and D.I. Shelefontuk

Institute of Atmospheric Optics, Siberian Branch of the Russian Academy of Sciences, Tomsk

Received April 26, 2000

An optimum approach to smoothing the profiles of atmospheric parameters measured in the photon counting regime is proposed in the paper. The application of the generalized method of least squares is substantiated as well as of the choice of the order of a linear model and of the validity criterion for *a priori* setting the variance profile. A possibility purposefully setting the window of moving average is discussed.

Introduction

Investigations into the atmospheric processes related, for example, to changes in radiation, cyclonic activity, etc., by means of lidar sounding have become possible up to the height of the tropopause. In so doing, vertical profiles of a number of parameters (signals) to be studied are experimentally measured. The difficulties that may arise in processing the measured signals are the following: statistical properties of the physical processes under investigation are not known exactly and are different at different altitudes. Besides, the contribution of noise to useful signal is non-uniform and significantly increases with the increasing sounding distance. The subsequent nonlinear transformations make the problem of smoothing the measured signals more difficult.

The problems of the type, when the noise properties and its contribution to the measured signal are known, but the statistical properties of the physical process itself are studied insufficiently, are mainly reduced to the use of classical method of least squares or, in less frequent cases, to Backus-Hilbert smoothing method, Tikhonov method, etc.[1–3] The method of least squares (MLS) attracts attention, first of all, by simplicity of its realization,[4] for which it is enough to have the medium-power calculation facilities. As a rule, smoothing is performed by means of the moving average, i.e., a smoothing interval slides along the signal. An optimum approach to the problem of smoothing of the profiles of measured atmospheric parameters is considered in this paper as applied to processing the return signals of the Raman scattering channel obtained in the photon counting mode at the Siberian Lidar Station.

1. Statement of the problem

Most commonly sounding of the atmospheric parameters is as follows: a short laser pulse is transmitted into the atmosphere, the radiation backscattered from the atmosphere is collected by a receiving mirror to the photoelectric converter and then enters the recording device in the form

$$Y(x) = A\,\frac{c\tau}{2}\,I_0\,KG(x)\,x^{-2}\,\beta_\pi(x)\,T^2(x) + q(x), \quad (1)$$

where

$$T^2(x) = \exp\left\{-2\int\limits_0^x \varepsilon(x')\,\mathrm{d}x'\right\}; \quad (2)$$

$Y(x)$ is the recorded signal, A is the area of the receiving mirror, c is the speed of light, τ is the laser pulse duration, I_0 is the laser radiation power, K is the total lidar transmission coefficient; $G(x)$ is the geometrical function of the lidar, x is the distance, $\beta_\pi(x)$ is the volume coefficient of backscattering, $\varepsilon(x)$ is the extinction coefficient, and $q(x)$ is the noise.

The lidar equation (1) was obtained taking into account the following approximations:

(1) atmospheric processes are considered in the single scattering approximation;

(2) the state of the atmosphere does not change during the measurement time;

(3) the sounding pulse duration does not exceed 10–20 ns;

(4) parameters of the sounding device are supposed to be constant during the whole measurement cycle.

In the photon counting mode, the signal is accumulated as a response from a certain number of sounding pulses. The path of sounding is divided into the fixed number of range intervals of the length Δx. The histogram-like profile of the measured characteristic, as function of distance from the source can be written in the form

$$Y(x_i) = AN_0\,K \int\limits_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} G(x)\,x^{-2}\,\beta_\pi(x)\,T^2(x)\mathrm{d}x + q(x_i), \quad (3)$$

0235-6880/00/08 702-05 $02.00

where i is the strobe number, and

$$x_i = \Delta x \left(i + \frac{1}{2}\right), \quad (i = 1, \ldots, z); \tag{4}$$

N_0 is the total number of the emitted quanta.

The probability of recording the photoelectron within an interval for the Raman scattering signals is significantly less than 1 per 1 laser shot, then the statistics of photocounts, that in the general case is described by the binomial distribution, is considered in the approximation of a Poisson process, for which, as is known, the mean value is equal to the variance.

Thus, one can determine the noise variance profile $q(x)$ from the measured signal. The statistics $q(x)$ for the mean value greater than 9 is satisfactorily described by the Gaussian probability distribution.[5]

Assuming that physical processes in the atmosphere are described by smooth functions, let us pass, for the convenience of subsequent consideration of the smoothing problem, from the histogram representation of signal to the following one:

$$Y(x_i) = f(x_i) + q(x_i), \quad (i = 1, \ldots, z), \tag{5}$$

where $f(x_i)$ is the signal without the noise described by the smooth function; $q(x_i)$ is the random noise described by the normal distribution with the zero mean value and the variance $\sigma^2(x_i)$; x_i is the distance to the middle of the ith strobe.

If the signal has undergone a nonlinear transformation, the transformed profile of the noise variance $q(x)$ is determined by first terms of the Taylor series:

$$\sigma^2(x_i) \approx \sum_j \left[\frac{\partial Y(x_i)}{\partial Y'_j}\right]^2 \sigma_j^2(x_i), \; (i = 1, \ldots, z), \tag{6}$$

where the transformed profile $Y(x_i) \equiv Y(Y'_1(x_i), \ldots, Y'_k(x_i))$; $Y'_j(x_i)$ are the measured signals, $\sigma_j^2(x_i)$ are the corresponding profiles of the noise variance.

Thus, the problem of smoothing noisy signals is stated as follows: assuming that the contribution of noise to the measured or transformed signal $Y(x)$ of the form (5) is known, it is necessary to reconstruct the signal $f(x)$ without noise by means of the optimal algorithm and to estimate accuracy of the reconstruction.

2. Synthesis of the algorithm

As a rule, $f(x)$ is determined as a linear combination of basic functions. The basis for selecting the basic functions is the circumstance that the closer is the properties of the basic function to the sought one, the more stable and quick is the solution.

Let us represent $f(x)$ in the form of a linear combination of the known basic functions $\varphi_1, \ldots, \varphi_m$:

$$V = \sum_{j=1}^{m} c_j \, \varphi_j. \tag{7}$$

Then

$$Y(x_i) = \sum_{j=1}^{m} c_j \, \varphi_j(x_i) + q(x_i), \quad (i = 1, \ldots, z). \tag{8}$$

Formula (8) is called in statistics the linear model, the number of basic functions m is called the order of the linear model.

The method of least squares (MLS) has the optimal properties among the algorithms for smoothing used for finding the order of the linear model and coefficients c_j.

The MLS-estimates $\hat{c}_j$ of the parameters c_j are determined from the condition

$$\min Q = \sum_{i=1}^{z} \omega_i \left\{Y(x_i) - \sum_{j=1}^{m} c_j \, \varphi_j(x_i)\right\}^2, \tag{9}$$

where $\omega_i = 1/\sigma^2(x_i)$, for which, according to the condition of minimum, one has that

$$\frac{\partial Q}{\partial c_j} = 0, \quad (i = 1, \ldots, m), \tag{10}$$

which, in its turn, leads to the linear system of equations[6] relative to c_j:

$$\sum_{j=0}^{m-1} c_j \, b_{lj} = g_l, \quad (l = 0, \ldots, m-1), \tag{11}$$

where

$$b_{lj} = \sum_{i=1}^{z} \omega_i \, P_l(x_i) \, P_j(x_i); \quad g_l = \sum_{i=1}^{z} \omega_i \, Y(x_i) \, P_l(x_i).$$

In the case when the errors are normal, the MLS-estimates $\hat{c}_j$ of the parameters c_j have the following properties:

(1) the estimates $\hat{c}_j$ are not biased, consistent, and effective;

(2) the estimates $\hat{c}_j$ are the estimates of the method of maximum likelihood;

(3) the random value or the residue

$$Q = \sum_{i=1}^{z} \omega_i \left\{Y(x_i) - \sum_{j=1}^{m} \hat{c}_j \, \varphi_j(x_i)\right\}^2 \tag{12}$$

has χ^2-distribution with $n - m$ degrees of freedom;

(4) the estimates $\hat{c}_j$ and

$$s^2 = \frac{1}{n-m} \sum_{i=1}^{z} \omega_i \left\{Y(x_i) - \sum_{j=1}^{m} \hat{c}_j \, \varphi_j(x_i)\right\}^2 \tag{13}$$

are sufficient for c_j and σ^2.

The order of the linear model (8) is estimated in MLS by testing the hypothesis

$$Q < \chi_\alpha^2(n - m), \tag{14}$$

where Q is the residue of the estimates by MLS with $n - m$ degrees of freedom; $\chi^2_\alpha(n - m)$ is the quantile of the χ^2-distribution with $n - m$ degrees of freedom and significance level α. If the hypothesis has been rejected, the linear combination $V = \sum_{j=1}^{m} c_j \varphi_j$, is increased by 1 starting from $m = 1$, then the residue Q is calculated again, and testing of the hypothesis is repeated until accepted.

For practical applications, the values $\chi^2_{0.05}(n - m)$ with the significance level 0.05 at the number of degrees of freedom equal to 3 is found with the accuracy of 2% by the approximate formula from Ref. 6:

$$\chi^2_{0.05}(n - m) \cong \frac{1}{2}\left(\sqrt{2(n - m) - 1} + 1.65\right)^2. \quad (15)$$

In the case when the *a priori* data on the profile of the variance of the error are not accurate, the iteration procedure is to be applied lying in testing the hypothesis on the equality of the variance of the errors before smoothing to the MLS–estimate of the variance after smoothing by use of the dispersion relation using the F–criterion:

$$\frac{\theta'}{\theta}\frac{Q}{Q'} < F_\alpha(\theta, \theta'), \quad (16)$$

where Q is the residue of the previous estimate by MLS with θ degrees of freedom; Q' is the residue of the subsequent estimate by MLS with θ' degrees of freedom; $F_\alpha(\theta,\theta')$ is the quantile of the Fisher–Snedecore distribution with the level of significance α and the degrees of freedom θ and θ'.

If the hypothesis has been rejected, the value

$$\eta^2(x) = \sigma^2(x)\, Q'/\theta' \quad (17)$$

is accepted for the next iteration as an estimate of the variance, where $\sigma^2(x)$ is the initial *a priori* data on the profile of the error variance.

If the basic functions are orthogonal, i.e., $\sum_{i=1}^{z} \omega_i\, \varphi_l(x_i)\, \varphi_j(x_i) = 0$ at $l \neq j$, one can diminish the bulk of calculations. One can construct the basic functions based on the power-law basis, that realizes the orthogonal transformation of the system of linear equations (11). Then, when determining the order of the linear model, the coefficients at the orthogonal basic functions in expression (14) are not recalculated, but remain the same and are calculated only for the added orthogonal basic function.

The number of basic functions, as well as the accuracy of reconstruction is determined by the value of the smoothing interval. By the interval of smoothing we mean the number of signal readouts selected, or the range interval, in which the signal behavior is considered. The greater the smoothing interval, the higher the degree of polynomial can be needed for describing $f(x)$ in approximating over the power-law basis.

Optimum approach in this case is in using the moving average, when a small interval of smoothing moves along the measured signal. The values of thus smoothed function are taken as the values at the center of the interval. As a rule, when determining the order of the linear model on the selected interval, when the order of the linear model m has exceeded the value of 10–12 or has approached the value of the interval of smoothing, one decreases the trial interval of smoothing n by 1 and repeats the calculations.

Let us consider the selected interval of smoothing on the sounding path. For a convenience of consideration, let us move the origin of the coordinate system to the center of the trial interval of smoothing $[-\nu, \nu]$ of the size n, consisting of integers, where $\nu = (n - 1)/2$, and n is an odd number.

If the functions $\varphi_1, \ldots, \varphi_m$ are the Chebyshev orthogonal polynomials $P_0, \ldots, P_{m-1}$, the moving average can be presented in the following form:

$$Y(x_{s+i}) = \sum_{j=1}^{m} {}^s c_j\, P_j(x_{s+i}) + q(x_{s+i}),$$
$$(s = 1 + \nu, \ldots, z - \nu,\ i = -\nu, \ldots, \nu), \quad (18)$$

where the index s determines the current position of the center of the trial interval of smoothing in moving average; i is the inner index of the interval of smoothing.

The linear system of equations corresponding to the condition (9) is the following:

$$\sum_{j=0}^{m-1} {}^s c_j\, {}^s b_{lj} = {}^s g_l, \quad (l = 0, \ldots, m - 1), \quad (19)$$

where

$${}^s b_{lj} = \sum_{i=-\nu}^{\nu} \omega_{s+i}\, P_l(x_{s+i})\, P_j(x_{s+i});$$

$${}^s g_l = \sum_{i=-\nu}^{\nu} \omega_{s+i}\, Y(x_{s+i})\, P_l(x_{s+i}); \quad (20)$$

$${}^s b_{lj} = 0 \ \text{ at } \ l \neq j.$$

The estimates ${}^s\hat{c}_j$ can be found from the expression

$${}^s\hat{c}_j = \sum_{i=-\nu}^{\nu} \omega_{s+i}\, Y(x_{s+i})\, P_j(x_{s+i}) \left(\sum_{i=-\nu}^{\nu} \omega_{s+i}\, P_j^2(x_{s+i}) \right)^{-1}. \quad (21)$$

The residue Q is calculated by the following formula:

$$Q = \sum_{i=-\nu}^{\nu} \omega_{s+i}\, Y^2(x_{s+i}) - \sum_{j=0}^{m-1} {}^s\hat{c}_j^{\,2} \sum_{i=-\nu}^{\nu} \omega_{s+i}\, P_j^2(x_{s+i}). \quad (22)$$

The orthogonal polynomials $P_k(x_{s+i})$ are calculated in the case of equidistant points x_i using recursion relationships

$$P_0(x_{s+i}) \equiv 1,$$

$$P_k(x_{s+i}) = i^k - \frac{\sum\limits_{j=-\nu}^{\nu} \omega_{s+j}\, j^k\, P_{k-1}(x_{s+j})}{\sum\limits_{j=-\nu}^{\nu} \omega_{s+j}\, P_{k-1}^2(x_{s+j})}\, P_{k-1}(x_{s+i}) - \ldots - \frac{\sum\limits_{j=-\nu}^{\nu} \omega_{s+j}\, j^k\, P_0(x_{s+j})}{\sum\limits_{j=-\nu}^{\nu} \omega_{s+j}\, P_0^2(x_{s+j})}\, P_0(x_{s+i}). \qquad (23)$$

The confidence boundaries of the estimate of the smoothed signal have the form

$$\Psi(x_{s+i}) = t_P\, \Delta(x_{s+i}), \qquad (24)$$

where t_P is the quantile of the Student distribution with $n - m$ degrees of freedom, corresponding to the confidence probability P,

$$\Delta(x_{s+i}) = \sqrt{\sum_{j=0}^{m-1} P_j^2(x_{s+i}) \left(\sum_{k=-\nu}^{\nu} \omega_{s+k}\, P_j^2(x_{s+k}) \right)^{-1}}. \qquad (25)$$

As a result, the smoothed signal takes the form

$$f(x_{s+i}) = \sum_{j=0}^{m-1} {}^{s}\hat{c}_j\, P_j(x_{s+i}) \pm \Psi(x_{s+i}), \qquad (26)$$

where i depends on the position of the interval of smoothing in the moving average on the initial signal (5).

(1) If the interval of smoothing has been placed at the beginning of the signal, $s = 1 + \nu$, then $f(x_{s+i})$ is calculated in the interval $i = -\nu, 1 - \nu, \ldots, 0$.

(2) Then, at $1 + \nu < s < z - \nu$ the values $f(x_{s+i})$ are calculated only for $i = 0$.

(3) If the interval of smoothing has been placed at the end of the signal, $s = z - \nu$, then $f(x_{s+i})$ is calculated in the interval $i = 0, 1, \ldots, \nu$, where, as before, the index s indicates, in what place of the initial signal (5) the center of the interval of smoothing is.

One can replace Eq. (25) with an approximate one, if taking into account the fact that the interval of smoothing n can be set *a priori*[8] so that the confidence boundaries of the estimate of the smoothed signal remain constant along the signal.

$$\Delta(x) \cong \sigma(x)\, \sqrt{m/n} \cong \text{const}, \qquad (27)$$

wherefrom one can derive the formula for setting the value of the trial interval of smoothing

$$n \cong \hat{m}\, \sigma^2(x) / \sigma_0^2, \qquad (28)$$

where σ_0^2 is the initial condition for the variance of the smoothed signal, and $\hat{m}$ is the *a priori* value of the order of the linear model.

3. Results of the numerical simulation

Figure 1 demonstrates the material discussed above by use of an example of smoothing a model signal (points in the figure) consisting of 400 readouts. The model signal was constructed so that one can show the characteristic peculiarities of operation of the algorithm for smoothing. The presence of an inhomogeneity with the steep fronts of the signal was simulated on the smooth signal. Then the random noise was added to the signal so that the noise variance increases by three times by the end of the sounding path. The trial interval of the moving average (curve *2*) was set equal to $n \cong \sigma^2(x)/\sigma_0^2$ in order that boundaries of the smoothed signal remain approximately constant. When the interval of smoothing has passed through the boundaries of the inhomogeneity, the order of the linear model (curve *1*) increases reaching 10, for that reason the interval of smoothing decreases. The interval decreases so that the curve adjusted on this part of the signal is described by the number of parameters no more than 10. The increase of the order of the model to more than 10–12 is not effective from the standpoint of both physical prerequisites and accumulation of the rounding-off errors in calculations. The result of smoothing the model signal is shown by curve *3*.

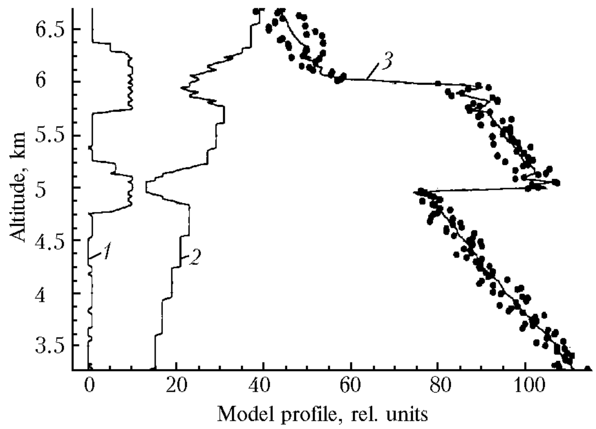

Fig. 1. The moving smoothing of the model profile by MLS: order of the linear model (*1*), width of the smoothing window (*2*), points that show the model profile (*4*), and the result of smoothing (*3*).

Conclusion

The results of numerical simulation confirm the validity of selection of the optimum algorithm for smoothing the lidar sounding signals. Representation of the measured signal in the form (5) makes it possible to apply the generalized MLS without additional suppositions about the statistical properties of the signal. The signal is considered as a consecutive series of equidistant readouts of a varying accuracy. Moving smoothing makes it possible to optimally use the calculation resources. The possibility of varying the value of the interval of smoothing allows one to keep the confidence boundaries of the smoothed signal in the prescribed limits.

The proposed optimum technique is applied to processing the results of lidar sounding in the Raman scattering channel of Siberian Lidar Station of IAO SB RAS.

References

1. A.N. Tikhonov, Dokl. Akad. Nauk SSSR **153**, 49 (1963).
2. A.N. Kolmogorov, Izv. Akad. Nauk SSSR, Ser. Matem. **5**, No. 1 (1941).
3. G. Backus and F. Gilbert, Philos. Trans. Roy. Soc. London. Ser. A **266**, 123 (1970).
4. S. Margulies, The Review of Scientific Instruments **39**, No. 4, 478–480 (1968).
5. V. Feller, *Introduction to the Probability Theory and Its Application* [Russian translation] (Izd. Inostr. Liter., Moscow, 1952).
6. V.A. Granovskii and T.N. Siraya, *Method for Experimental Data Processing in Measurements* (Energoatomizdat, Leningrad, 1990), 288 pp.
7. V.S. Korolyuk, N.I. Portenko, A.V. Skorokhod, and A.F. Turbin, *Handbook on Probability Theory and Mathematical Statistics* (Nauka, Moscow, 1985), 640 pp.
8. S.N. Volkov and A.I. Nadeev, Atmos. Oceanic Opt. **8**, No. 8, 624–626 (1995).