Linear-system approach and the theory of optical transfer operator

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We consider the problem of remote sensing of the Earth's surface through the atmosphere with the account of contribution from the reflecting underlying surface using models of the Earth's atmospheric radiation. The mathematical model of radiative transfer in the atmosphere–surface system is formulated as the optical transfer operator within the frames of the classical linear-system approach by use of the method of influence function and spatial-frequency characteristics.

Introduction

Numerous experimental and theoretical investigations into the solar radiation transfer in the atmosphere–surface system (ASS) and self-radiation of the Earth made it possible to achieve a reliable knowledge about the radiation field of our planet. Moreover, this also enabled establishing explicit and quantitative connections between radiation characteristics and optico-physical parameters of the atmosphere and Earth's surface that are responsible for the radiation regime of the Earth and transfer characteristics of the atmosphere in the systems of vision and remote sensing.[1–25]

First spectrographic experiments on observing various types of natural formations on the Earth's surface in the visible spectral range (manned spacecraft Soyuz–7, 9, and 13, orbital stations Salyut, Salyut–3 and 5) and combined synchronous sub-satellite geophysical experiments (Soyuz–7 and 9) were prepared and conducted under the supervision of K.Ya. Kondrat'ev. To reduce spectroscopic data collected from space to the level of the underlying surface, the transfer function of ASS was introduced and estimations of its components were obtained using data of integrated ground support experiments over the key areas of the territories surveyed using spectrophotometry.[26–30]

At present, one can state that theoretical and methodical basis for the correct account for the effects of atmosphere as a scattering, absorbing, polarizing, and refracting medium in remote sensing of the atmosphere and Earth's surface have been created. At the same time, it is clear that, to increase the efficiency of space technologies, it is necessary to continue theoretical studies on interpretation, analysis, and generalization of results of photography, spectrophotometry, spectrography, tomography, videopolarimetry, radiometry, telemetry, spectropolarimetry, refractometry, visual observations, and, certainly, optimization of measurements and experiments themselves.

One can distinguish between the following types of radiation problems that make it necessary to take into account the Earth's surface. The first type includes problems of energy and radiation balance of the Earth when solar radiation serves as a source. These problems are solved mostly in the approximation of a plane model of the Earth's envelope with an implicit allowance for the contribution of a homogeneous Lambertian or non-orthotropic underlying surface. The second type includes problems of remote sensing of the atmosphere and clouds with the Earth's surface being the source of interference. The third type involves problems of remote sensing of the Earth's surface when it is necessary to eliminate (to perform atmospheric correction) or reliably take into account the influence of the atmosphere.

A significant contribution to solution of the problems on the account for atmospheric distortions in processing the images of the Earth's surface taken from satellites was made by investigations by V.V. Kozoderov.[31–34] The problem on the account for horizontal inhomogeneities of a Lambertian surface in the method of spherical harmonics for solving boundary problems of solar radiation transfer in a scattering atmosphere was stated in Ref. 34. Knowledge of the spatial frequency characteristic (SFC) of ASS was used as one of the tools to solve the boundary problem of the theory of radiative transfer for the fluctuating component of the solar radiation field in addition to the boundary problem for the mean component of the radiation intensity.[22] Similar approach was formulated in Ref. 35 with introduction of the SFC model invariant with respect to variations of albedo and illumination conditions. It uses numerical solution of the complex equation of transfer by the iteration method of characteristics.[5,6,15,36] The author of this paper was first to calculate full SFC of the ASS for slant paths of observation from the space as the input-output and phase-frequency characteristic for realistic models of a scattering and absorbing atmosphere.[5,6,15] Phase distortions that lead to image shift comparable with the refraction shift, which was estimated by

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M.V. Kabanov,[37] were studied theoretically and numerically for the first time. Theoretical estimates of amplitude-phase distortions of image transfer through a turbid medium qualitatively coincided with the results of the first one-of-a-kind experiment.[38] The model for transfer properties of the atmosphere was completely formed as the optical transfer operator (OTO) for the Lambertian surface.[5,6,15,39–43]

The first practical results of radiation correction (i.e., correction of radiometric distortions by instruments and influence of scattering and absorbing properties of the atmosphere) for digitized images of scanning satellite-based radiometers are presented in Ref. 44. These results were obtained several years earlier than analogous results in processing of the Earth's images taken from spaceborne platforms by foreign authors. One more efficient algorithm of radiation correction was developed and applied by the authors of Ref. 45. The difficulties in the allowance for non-orthotropic reflection of solar radiation by various natural formations were studied in Ref. 46. The milestones in the development of the atmospheric correction of images taken from space have been described in Ref. 47. These results and conclusions made up the basis for modern methods and tools of spaceborne physical geography.[8,13,14,18,20,22,23]

The linear-system approach

In every active or passive system of remote sensing of the Earth's surface, there are four main components: scenario, scene, i.e., distribution of brightness of observed objects or landscapes; the atmospheric channel of image transfer; a device detecting electromagnetic waves; a unit for image processing and recognition. The influence of the atmosphere manifests itself in three components: atmospheric optics mechanisms affect the formation of scenario, the transfer of its image through the medium, and are taken into account in radiation correction in analyzing the scenes.

Since the variety of possible observation objects is infinite, it is expedient to use a universal approach which permits one to describe the whole observation channel by objective characteristics invariant with respect to particular structure of objects sounded, conditions of illumination, and viewing. This approach is widely used in classical optics, theories of vision, electric circuits, optical-electron schemes, photography, image processing, etc., and is known as the *linear-system approach*.[1,9,21,23]

By the system one should understand everything that realizes transformation of some input functions or actions into some output functions or reactions (responses). Systems' reactions to input actions, due to their similarity, can be described by certain generalized characteristics whose definition does not depend on a concrete form of a system (electrical, optical, radiophysical, etc.). The generality means that the functional relation connecting the input $E(x, y)$ and output $\Phi(x, y)$ two-dimensional signals of the system

$$\Phi(x,y) = (\Theta, E) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Theta(x,y,x',y')E(x',y')\,\mathrm{d}x'\mathrm{d}y' \quad (1)$$

is of fundamental character and is well known as the *superposition integral* that has meaning that the linear system is fully characterized by the sum of its responses to input actions; x, y are the horizontal coordinates. If the condition of *spatial invariance (isoplanarity)* is satisfied, the *scattering function* (SF) of the system or the *point spread function* (PSF) $\Theta(x,y,x',y')$ depends on the arguments' difference and the functional (1) takes the form of convolution

$$\Phi(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Theta(x - x', y - y')E(x', y')\,\mathrm{d}x'\mathrm{d}y'. \quad (2)$$

By use of the theorem on the convolution of a Fourier spectrum, the two-dimensional spectrum of the output signal of the system $B(p_x, p_y) = F[\Phi(x, y)]$ is obtained as the product

$$B(p_x, p_y) = \Psi(p_x, p_y)\ V(p_x, p_y), \quad (3)$$

where the spectral density of the input signal (distribution of the object's brightness) is $V(p_x, p_y) = F[E(x, y)]$. The spectral density of the scattering function $\Psi(p_x, p_y) = F[\Theta(x, y)]$ is called the *transfer function* of the system, or the *optical transfer function* (OTF).

By use of the inverse Fourier transform, one can obtain from Eq. (3) the value of the output signal of the system (brightness distribution at the output of the system)

$$\Phi(x, y) = F^{-1}[B(p_x, p_y)] = F^{-1}[\Psi(p_x, p_y)\ V(p_x, p_y)]. \quad (4)$$

Therefore, the (optical) system realizes two-dimensional Fourier transform of the product of the spectra of its scattering function and input signal.

According to Eq. (3), OTF $\Psi(p_x, p_y)$ permits one to establish a connection between two-dimensional spectra of the brightness distribution in the object plane and illumination in the image plane. Therefore, the *optical system* is a *linear filter of spatial frequencies* with a transmission coefficient $\Psi(p_x, p_y)$. In the general case, $\Psi(p_x, p_y)$ is a complex function

$$\Psi(p_x, p_y) = A(p_x, p_y)\ \exp\ [i\beta(p_x, p_y)].$$

The module $A(p_x, p_y)$ of a normalized OTF is called the two-dimensional *spatial frequency characteristics* (SFC), *frequency-contrast characteristic* (FCC), *amplitude-frequency response characteristic* (AFRC), *function of modulation transfer* (FMT), and dependence of the phase $\beta(p_x, p_y)$ on spatial frequency is called the *phase-frequency characteristic* (PFC). For a symmetric PSF, the normalized OTF coincides with FCC and the phase is $\beta(p_x, p_y) = 0$. Formally, OTF is defined as a two-dimensional Fourier spectrum of PSF. The FCC of a system is the ratio of a contrast observed in the image of diffusely luminous harmonic mire to the initial contrast depending on the mire's frequency. The PFC of the system defines the phase shift in the mire's image.

For systems with cylindrical symmetry, Fourier-Bessel or Hankel transform of zero order is used

$$\Phi(\nu) = 2\pi \int_0^\infty \Phi(\rho) J_0(2\pi\nu\rho)\rho \, d\rho.$$

The question on the advantages of one or another characteristic of a system of radiative transfer is in fact a question on the convenience of mathematical description and applied trends of a particular investigation.

The conception of (optical) spatial filtration, i.e., manipulation with spatial frequencies aimed at changing or transferring the image properties, is known for more than 100 years after the study by Ernst Abbe.[48] These works exerted deep influence on the branch of science that was later called Fourier optics.[49] This branch appeared at the frontier between classical optics and theory of information. Abbe's results directly led to description of imaging optical devices as filters of spatial frequencies of the object's field.

The key point in developing optical methods for image processing is connected with the publications by N. Nyuberg[50,51] in the thirties and R.M. Duffieux[52] in the forties. Nyuberg[50] proposed to use expansion of functions over the system of orthonormal functions as the best linear approximation for analysis of the problem of light and device. In Ref. 51, applying the idea of expanding analytical functions into a series over a complete system of orthogonal functions, Nyuberg introduced the general principle of constructing spectral devices. This paper exerted a significant influence on the development of Fourier spectrometry.

Considering a generalized imaging system as a linear filter, Duffieux established that energy distribution in the plane of an object or of an image and in the plane of the optical system's pupil are connected through a Fourier transform, and described distribution of light intensity in the image plane as a result of light intensity distribution in the plane of the object and *point spread function* (i.e., *pulse response*). Duffieux's ideas were very fruitful[49]: already in the 50–60s, due to general mathematical technique, it was possible to formulate the main postulates of the theory of systems (linear and nonlinear, invariant and spatially not invariant, feedback systems, etc.); to establish analogies between optics and the science about transfer of information, between optical and electrical filters, between optics and electronics, between increase of image sharpness and equalizing the transfer function; to develop synthesis of optical systems, coherent-optical and holographic methods of information processing (rapid growth with appearance of lasers); to try to control phase transmission of spatial filters by use of polarization methods and to control amplitude and phase transmission by use of the holographic method; to turn to the important problem of detecting signals by optical tools on the background with use of non-coherent, partially coherent, and coherent light; rapid growth was observed in adaptive optics (V.P. Lukin, 1986), optoelectronic systems (M.M. Miroshnikov, 1977), image processing theory (Yu.N. Pyt'ev, 1979, 1983, 1989; G.I. Vasilenko, 1985, 1986), information theory,[22,23] etc. During a decade, a large number of papers devoted to Fourier analysis of optical imaging systems was published. Thus, the basis for (mathematical) technique of the *theory of linear systems* was established (A. Marechal, M. Franson, 1960; E. O'Neill, 1963; J. Goodman, 1968; A. Papoulis, 1968, et al.).

The basis for the theory of linear (two-dimensional) systems is expressed in Eqs. (1)–(4).[21] Spatial filtration is performed by use of spatial and frequency spatial characteristics. This technique of linear transformations in spatial and spatial-frequency ranges use such concepts as pulse action (instead of a pointlike source), pulse response (instead of an image of a pointlike source),[21] and can be generalized for systems with narrow and broad monodirectional beams. In particular, such beams arise in the problems with the influence functions at anisotropically reflecting surfaces.

We consider the *atmospheric channel* as an element of an *optical system of radiative transfer* and formulate the theory of *optical transfer operator* by use of the mathematical apparatus of the linear-system approach. The objective characteristics (*point spread function*, *optical transfer function*, *modulation transfer function*, *spatial frequency characteristic*, *pulse transient function* (PTF), *scattering function of a system*, and other image characteristics of reconstructing and transmitting optical, optoelectronic, photographic, cinematographic, television, radio-engineering, controlling, and other systems) can be naturally extended to the theory of radiative transfer in optically active media.

Mathematical formulation of the problem

Atmospheric radiation of the Earth is formed under the influence of two ASS components. Connections between radiation characteristics and parameters of the atmosphere and the Earth's surface are described by solutions of the boundary problem of the theory of radiative transfer in ASS when it is important to use the theory of multiple scattering. The difficulty of the problem is that the model of the medium depends on many parameters; processes of solar energy transformation and ways of viewing and measurements are various. One has to deal with boundary problems for an integro-differential kinetic equation describing radiative transfer in absorbing, scattering, refracting, emitting, and polarizing media with one-dimensional, two-dimensional, or three-dimensional plane or spherical geometry. The theory of transfer permits one to study the influence of different factors on radiation propagated in ASS and to obtain relations between concrete parameters of the medium and characteristics of the radiation field. In this way one can determine sensitivity of the spectral brightness, angular and spatial structure of a radiation field, spatial distribution of density and fluxes of radiation under given conditions of illumination and observation to variations in the parameters.

We consider the problem of radiative transfer in a scattering, absorbing, and emitting horizontally-homogeneous plane layer which is not bounded in the horizontal direction ($-\infty < x, y < \infty$, $r_\perp = (x, y)$) and finite with respect to height ($0 \le z \le h$) in the three-dimensional Euclidean space: the radius vector is $r = (x, y, z)$. The system "atmosphere − underlying surface at the level $z = h$" is considered to be non-multiplying (without duplication). The set of all directions of radiation propagation $s = (\mu, \varphi)$, where $\mu = \cos\vartheta$, $\vartheta \in [0, \pi]$ is the zenith angle counted from the direction of the internal normal to the upper boundary of the layer $z = 0$ (the normal coincides with the z axis) and $\varphi \in [0, 2\pi]$ is azimuth counted from the positive direction of the x axis forms a unit sphere $\Omega = \Omega^+ \cup \Omega^-$; Ω^+ and Ω^- are hemispheres for the propagation directions of downward going, transmitted radiation with $\mu \in [0, 1]$ and upward going, reflected radiation with $\mu \in [-1, 0]$, respectively. The value $\varphi = 0$ is taken in the solar vertical's plane coinciding with the plane passing through the x and z axes. The solar flux is incident onto the upper boundary of the layer $z = 0$ along the direction $s_0 = (\mu_0, \varphi_0)$ with the zenith angle $\vartheta_0 \in [0, \pi/2]$, $\mu_0 = \cos\vartheta_0$, and azimuth $\varphi_0 = 0$. To write the boundary conditions in a more convenient form, let us introduce the sets $t = \{z, r_\perp, s : z = 0, s \in \Omega^+\}$, $b = \{z, r_\perp, s : z = h, s \in \Omega^-\}$, whose labels are taken from the words "top" and "bottom".

Intensity (energy brightness) of radiation $\Phi(r, s)$ in ASS is sought as a solution of the general boundary problem (GBP for $R \not\equiv 0$) of the theory of transfer

$$K\,\Phi = F^{in}, \quad \Phi\big|_t = F^0, \quad \Phi\big|_b = \varepsilon R\Phi + F^h \qquad (5)$$

with the following linear operators:
transfer operator

$$D \equiv (s, \mathrm{grad}) + \sigma(z) = D_z + \left(s_\perp, \frac{\partial}{\partial r_\perp}\right),$$

$$D_z \equiv \mu \frac{\partial}{\partial z} + \sigma(z);$$

integral of collisions

$$S\,\Phi \equiv \sigma_s(z) \int_\Omega \gamma(z, s, s')\, \Phi(z, r_\perp, s')\, \mathrm{d}s',$$

$$\mathrm{d}s' = \mathrm{d}\mu'\mathrm{d}\varphi', \quad S(1) \le 1;$$

operator of reflection

$$[R\,\Phi](h, r_\perp, s) \equiv \int_{\Omega^+} q(r_\perp, s, s^+)\, \Phi(h, r_\perp, s^+)\, \mathrm{d}s^+, \qquad (6)$$

which is uniformly bounded: $R(1) = q^*(r_\perp, s) \le 1$; the integro-differential operator $K \equiv D - S$; the one-dimensional operator $K_z \equiv D_z - S$; $\gamma(z, s, s')$ is the phase scattering function; $\sigma(z)$ and $\sigma_s(z)$ are vertical profiles of the coefficients of attenuation (extinction) and scattering; $q(r_\perp, s, s^+)$ is the kernel of the reflection operator; the parameter $0 \le \varepsilon \le 1$ represents the interaction between radiation and the underlying surface; $F^{in}(z, s)$, $F^0(r_\perp, s^+)$, $F^h(r_\perp, s^-)$ are sources of insolation (the external solar flux, self-radiation of the medium).

The boundary problem (5) is linear and its solution can be sought as a superposition $\Phi = \Phi_a + \Phi_q$. The background radiation of the atmosphere Φ_a is determined as the solution of the first boundary problem of the theory of transfer (FBP) with vacuum boundary conditions

$$K\Phi_a = F^{in}, \quad \Phi_a\big|_t = F^0, \quad \Phi_a\big|_b = F^h \qquad (7)$$

for a layer with transparent or absolutely black (non-reflecting) boundaries ($R \equiv 0$). It can contain three background components: $\Phi_a = \Phi_a^{in} + \Phi_a^0 + \Phi_a^h$. Each of the components can be calculated separately as a solution to the FBP (7) with the sources F^{in}, F^0, F^h, respectively.

The problem for illumination Φ_q caused by the effect of reflecting underlying surface is the GBP

$$K\Phi_q = 0, \quad \Phi_q\big|_t = 0, \quad \Phi_q\big|_b = \varepsilon R\Phi_q + \varepsilon E, \qquad (8)$$

where the source $E(r_\perp, s) \equiv R\Phi_a$ is the brightness (illumination, irradiance) caused by the background radiation.

The general boundary problem (5) for a plane layer is a mathematical idealization of radiative transfer in scattering, absorbing, and emitting media. It describes real radiative processes in ASS sufficiently adequately. The variety of underlying surfaces (without regard to elevations and orography) which is described by the operator (6) and boundary sources can be grouped into four main types (or their combinations): horizontally-homogeneous isotropic; horizontally-homogeneous anisotropic; horizontally-inhomogeneous isotropic; horizontally-inhomogeneous anisotropic. If at least one of the functions F^0, F^h, q depends on $r_\perp$, the solution of GBP (5) is determined in a five-dimensional phase volume $(x, y, z, \vartheta, \varphi)$ and GBP is not solvable by numerical methods without restrictions on the horizontal dimensions of the layer. Solutions of a three-dimensional GBP refer to the class of generalized solutions.

There exists whole branch of mathematics dealing with calculations of fundamental solutions of partial differential equations by Fourier method. An important part played the suggestion referred to the equation of transfer in the textbook on mathematical physics written by Acad. V.S. Vladimirov for Moscow Physico-Technical Institute.[53] Theoretical constructions and calculation algorithms for the optical transfer operator are based on the theory of generalized solutions, theory of integral transforms of distributions and series of the general theory of regular perturbations (asymptotic method). The approach developed on the basis of rigorous mathematical foundations is called the *method* *of influence functions and spatial frequency characteristics* (IF and SFC method).[5,6,15,54,55] In the theory of generalized solutions, IF is the fundamental solution of FBP and GBP. It is a universal characteristic of a radiative transfer system, invariant with respect to concrete values and structure of radiation sources and parameters of reflection of the boundary. The term includes the whole variety of well known terms: SF, PSF, PTF, Green's function, etc. and methodologically unites one-, two-, and three-dimensional boundary problems. The term SFC is introduced as a two-dimensional Fourier spectrum of the SF by the horizontal coordinates. Identity of the concepts of SFC, OTF, MTF, FCC, etc. is evident. Note that IF and SFC can depend on several variables considered as parameters. A.S. Monin[56] and B.B. Kadomtsev[57] were the first to apply the IF apparatus in the theory of transfer. The series of the general theory of (regular) perturbations (asymptotic theory) are being employed for rather a long time.[15,25,54,55,58–61]

Influence functions of the boundary problem in the theory of transfer

Let us consider a FBP

$$K\Phi = 0, \quad \Phi\big|_t = 0, \quad \Phi\big|_b = f(s^h, r_\perp, s). \tag{9}$$

The parameter $s^h \in \Omega^-$ can be absent. The problem (9) corresponds to a linear ASS and its generalized solution is represented as a linear functional, i.e., the superposition integral

$$\Phi(s^h, z, r_\perp, s) = P(f) \equiv (\Theta, f) \equiv$$
$$\equiv \frac{1}{2\pi}\int_{\Omega^-} ds_h^- \int_{-\infty}^{\infty} \Theta(s_h^-, z, r_\perp - r'_\perp, s)\, f(s^h, r'_\perp, s_h^-)\, dr'_\perp, \tag{10}$$

whose kernel is the IF $\Theta(s_h^-, z, r_\perp, s)$, i.e., the solution of FBP

$$K\Theta = 0, \quad \Theta\big|_t = 0, \quad \Theta\big|_b = f_\delta \tag{11}$$

with the parameter $s_h^- \in \Omega^-$ and source $f_\delta(s_h^-, r_\perp, s) = \delta(r_\perp)\,\delta(s - s_h^-)$. In fact, the IF Θ describes a radiation field in a layer with non-reflecting boundaries. The field is created due to processes of multiple scattering of a stationary narrow beam propagated along the direction s_h^-. The source of the beam is at the boundary $z = h$ at the origin of the system of horizontal coordinates x, y.

If the source $f(r_\perp)$ is isotropic and horizontally inhomogeneous, the solution of FBP (9) is sought using a linear functional-convolution

$$\Phi(z, r_\perp, s) = P_r(f) \equiv (\Theta_r, f) \equiv$$
$$\equiv \int_{-\infty}^{\infty} \Theta_r(z, r_\perp - r'_\perp, s) f(r'_\perp)\, dr'_\perp \tag{12}$$

with the kernel

$$\Theta_r(z, r_\perp, s) = \frac{1}{2\pi}\int_{\Omega^-} \Theta(s_h^-, z, r_\perp, s)\, ds_h^-. \tag{13}$$

IF Θ_r coincides with PSF and satisfies FBP

$$K\Theta_r = 0, \quad \Theta_r\big|_t = 0, \quad \Theta_r\big|_b = \delta(r_\perp). \tag{14}$$

In the case of an anisotropic and horizontally inhomogeneous source $f(s^h, s)$, the solution of FBP (9) is determined by the linear functional

$$\Phi(s^h, z, s) = P_z(f) \equiv (\Theta_z, f) \equiv$$
$$\equiv \frac{1}{2\pi}\int_{\Omega^-} \Theta_z(s_h^-, z, s)\, f(s^h, s_h^-)\, ds_h^- \tag{15}$$

with the kernel

$$\Theta_z(s_h^-, z, s) = \int_{-\infty}^{\infty} \Theta(s_h^-, z, r_\perp, s)\, dr_\perp. \tag{16}$$

IF Θ_z is the solution of the one-dimensional FBP

$$K_z\Theta_z = 0, \quad \Theta_z\big|_t = 0, \quad \Theta_z\big|_b = \delta(s - s_h^-) \tag{17}$$

and describes the radiation field formed in the layer onto the boundary $z = h$ of which an outer parallel wide flux is incident along the direction $s_h^- \in \Omega^-$. FBP (17) is similar to the usual problem for a one-dimensional plane layer illuminated by the solar flux.

For an isotropic and horizontally homogeneous source, the solution of FPB (9)

$$\Phi(z, s) = fW(z, s), \quad f = \text{const}, \tag{18}$$

is calculated by KF

$$W(z, s) = \frac{1}{2\pi}\int_{\Omega^-} ds_h^- \int_{-\infty}^{\infty} \Theta(s_h^-, z, r_\perp, s)\, dr_\perp =$$
$$= \int_{-\infty}^{\infty} \Theta_r(z, r_\perp, s)\, dr_\perp = \frac{1}{2\pi}\int_{\Omega^-} \Theta_z(s_h^-, z, s)\, ds_h^-, \tag{19}$$

which is also called the transmission function complicated by the contribution of multiple scattering and is determined as a solution of the one-dimensional FBP

$$K_z W = 0, \quad W\big|_t = 0, \quad W\big|_b = 1. \tag{20}$$

The relations (13), (16), (19) can be used as accuracy criteria in calculation of KF Θ, Θ_r, Θ_z by solutions of simpler FBP (14), (17), (20). Functionals (12), (15), (18) are particular cases of the functional (10). The influence functions Θ, Θ_r, Θ_z, W form a complete set of basic models of fundamental solutions of the first and general boundary problems of the theory of radiative transfer in a plane layer and objective invariant characteristics of a linear ASS.

Spatial-frequency characteristics

By use of Fourier transform with respect to the horizontal coordinate $r_\perp$ we obtain that

$$g(p) = F[f(r_\perp)] \equiv \int_{-\infty}^{\infty} f(r_\perp) \exp[i(p, r_\perp)] \, dr_\perp,$$

$$B \equiv F[\Phi], \qquad (21)$$

where the spatial frequency $p = (p_x, p_y)$ takes only real values $(-\infty < p_x, p_y < \infty)$, in the class of tempered distributions,[53] FBP (9) can be reduced to FBP for the parametric one-dimensional complex equation of transfer (CFBP):

$$L(p)B = 0, \quad B\big|_t = 0, \quad B\big|_b = g(s^h, p, s) \qquad (22)$$

with the linear operator

$$L(p) \equiv D_z - i(p, s_\perp) - S;$$

$$(p, s_\perp) = p_x \sin\vartheta \cos\varphi + p_y \sin\vartheta \sin\varphi.$$

The solution of CFBP (22) is represented as a linear functional

$$B(s^h, z, p, s) = \Pi(g) \equiv (\Psi, g) \equiv$$

$$\equiv \frac{1}{2\pi} \int_{\Omega^-} \Psi(s_h^-, z, p, s) \, g(s^h, p, s_h^-) \, ds_h^-. \qquad (23)$$

The kernel of Eq. (23) is SFC $\Psi(s_h^-, z, p, s) = F[\Theta(s_h^-, z, r_\perp, s)]$ with the parameters $s_h^- \in \Omega^-$ and p is the solution of CFBP

$$L(p)\Psi = 0, \quad \Psi\big|_t = 0, \quad \Psi\big|_b = g_\delta. \qquad (24)$$

This CFBP is obtained as a result of applying the Fourier transform (21) to FBP (11)

$$g_\delta(s_h^-, s) \equiv F[f_\delta(s_h^-, r_\perp, s)] = \delta(s - s_h^-).$$

Together with the SFC model Ψ (24) for the case of a horizontally inhomogeneous and anisotropic source in FBP (9), the set of basic models includes SFC

$$\Psi_r(z, p, s) \equiv F[\Theta_r(z, r_\perp, s)] = \frac{1}{2\pi} \int_{\Omega^-} \Psi(s_h^-, z, p, s) \, ds_h^-,$$

which satisfies CFBP

$$L(p)\Psi_r = 0, \quad \Psi_r\big|_t = 0, \quad \Psi_r\big|_b = 1, \qquad (25)$$

when the source in FBP (9) is isotropic and horizontally inhomogeneous.

The following relations are valid: if $\Psi = F[\Theta]$, then $\Theta = F^{-1}[\Psi]$; if $g = F[f]$, then $f = F^{-1}[g]$. We obtain the following connections for the functionals:

$$\Pi(g) = F[P(f)]; \quad P(f) = F^{-1}[\Pi(g)].$$

The optical transfer operator

Based on the general theory of regular perturbations, one can use the series

$$\Phi_q(s^h, z, r_\perp, s) = \sum_{k=1}^{\infty} \varepsilon^k \Phi_k$$

to reduce GBP (8) to a system of recurrent FBP of the type (9)

$$K\Phi_k = 0, \quad \Phi_k\big|_t = 0, \quad \Phi_k\big|_b = E_k \qquad (26)$$

with the sources $E_k = R\,\Phi_{k-1}$ for $k \geq 2$, $E_1 = E$. The following operation describes interaction of radiation with the boundary by the IF Θ:

$$[Gf]\,(s^h, h, r_\perp, s) \equiv R(\Theta, f) = \int_{\Omega^+} q(r_\perp, s, s^+)\,(\Theta, f)\, ds^+.$$

Solutions of the system of FBP (26) are obtained as linear functionals (10)

$$\Phi_1 = (\Theta, E),$$

$$\Phi_k = (\Theta, R\,\Phi_{k-1}) = (\Theta, G^{k-1}E).$$

An asymptotically exact solution of GBP (8) is obtained in the form of a linear functional (10), i.e., the optical transfer operator

$$\Phi_q = (\Theta, Y), \qquad (27)$$

where the scenario of the optical image or brightness of the underlying surface

$$Y \equiv \sum_{k=0}^{\infty} G^k E = \sum_{k=0}^{\infty} R\,\Phi_k, \quad R\,\Phi_0 = E, \qquad (28)$$

is a sum of the Neumann series over orders of radiation reflection for the base with the allowance for multiple scattering in the medium. The following majorant estimate is valid:

$$\|Y\| \leq \sum_{k=0}^{\infty} \|R\,\Phi_k\| \leq \|E\| \sum_{k=0}^{\infty} (q_* c_*)^k = \frac{\|E\|}{1 - q_* c_*} \leq \frac{q_* \|\Phi_d\|}{1 - q_* c_*},$$

$$\|\Phi_k\| = \operatorname{vrai} \sup_{z, r_\perp, s} \quad |\Phi_k| \leq q_*^{k-1} c_*^k \|E\|,$$

$$\|R(1)\| \leq \operatorname{vrai} \sup_{r_\perp, s^-} \int_{\Omega^+} |q(r_\perp, s^-, s^+)| \, ds^+ = q_* \leq 1,$$

$$P(1) = W(z, s), \quad \|P(1)\| \leq \sup_{z, s} W = c_* \leq 1.$$

The scenario satisfies the Fredholm-type equation of the second kind

$$Y = R(\Theta, Y) + E, \qquad (29)$$

which is called the equation of “near-land photography“.[45] In the general case $R(\Theta, Y) \neq (R\,\Theta, Y)$. The total radiation of ASS and space photography are described by the functional

$$\Phi = \Phi_a + (\Theta, Y). \qquad (30)$$

In terminology of Fourier images (21), the components of the perturbation series

$$B_q(s^h, z, p, s) \equiv F[\Phi_q(s^h, z, r_\perp, s)] = \sum_{k=1}^{\infty} \varepsilon^k B_k,$$

$$B_k \equiv F[\Phi_k], \qquad (31)$$

satisfy a system of recurrent CFBP of the type (22). The Fourier image of the reflection operator (6) is determined by the following formula ($\nu \equiv F[q]$):

$$[TB](h, p, s) \equiv F[R\Phi] =$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dp' \int_{\Omega^+} \nu(p - p', s, s^+)\, B(h, p', s^+)\, ds^+.$$

The operation of interaction between radiation and the boundary is introduced by the SFC:

$$[Qg]\,(s^h, h, p, s) \equiv F[Gf] = T(\Psi, g) =$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dp' \int_{\Omega^+} \nu(p - p', s, s^+)\, (\Psi, g)\, ds^+.$$

The terms of the series (31) are obtained as linear functionals (23)

$$B_1 = (\Psi, V), \quad B_k = (\Psi, TB_{k-1}) = (\Psi, Q^{k-1}V);$$

$$V \equiv F[E].$$

The sum of the series (31) (the Fourier image of the asymptotically exact solution of GBP (8) in the class of tempered functions) is the linear functional (23)

$$B_q = (\Psi, Z), \quad \Phi_q = F^{-1}(\Psi, Z). \qquad (32)$$

The Fourier image of the scenario is the sum of the Neumann series over the orders of radiation reflection from the underlying surface (in the terminology of Fourier images)

$$F[Y] = Z = \sum_{k=0}^{\infty} Q^k V = \sum_{k=0}^{\infty} TB_k. \qquad (33)$$

IF $\Theta(s_h^-, z, r_\perp, s)$ and SFC $\Psi(s_h^-, z, p, s)$ are used to solve GBP (8) with the following set of pairs of functions of the source and characteristic of reflection:

$$E(r_\perp, s),\ q(r_\perp, s, s');\ E(r_\perp, s),\ q(s, s');$$

$$E(s),\ q(r_\perp, s, s');\ E(r_\perp),\ q(r_\perp, s, s');$$

$$E(r_\perp),\ q(s, s');\ E,\ q(r_\perp, s, s').$$

IF $\Theta_r(z, r_\perp, s)$ and SFC $\Psi_r(z, p, s)$ are kernels of the functionals when the source and reflection parameter form the following pairs:

$$E(r_\perp, s),\ q(r_\perp, s');\ E(r_\perp, s),\ q(s');\ E(s),\ q(r_\perp, s');$$

$$E(r_\perp),\ q(r_\perp, s');\ E(r_\perp),\ q(s');\ E,\ q(r_\perp, s').$$

By use of IF $\Theta_z(s_h^-, z, s)$ one can determine the functionals in the case of the following sources and reflection parameters: $E(s)$, $q(s, s')$; E, $q(s, s')$. By IF $W(z, s)$, one can find the solution for the pair E, $q(s')$.

The functional (27) is a mathematical model of radiative transfer in ASS. It is adequate to the initial GBP (8) for different structures of the source E and types of the underlying surface regardless of the ASS dimensionality (one, two, or three-dimensional). Instead of calculating the series over the reflection orders in the full phase volume of the GBP problem (8), it is sufficient to calculate a finite Neumann series only for the scenario at the boundary $z = h$ (28). Spatial and angular distributions of the contribution of illumination, i.e., solutions of GBP (8), can be sought by use of a linear functional, OTF (27). In the presence of horizontal inhomogeneities on the Earth's surface, one can use OTF in the form of the functional (27) with the kernel being IF or (32) with the SFC as the kernel. Here IFs are calculated either directly (for instance, by the Monte-Carlo method), or using the small-angle approximation,[62–65] or by SFCs considered as solutions of CFBP (24) or (25). Different schemes of OTF and structuring of the total radiation field of ASS (30) (see Refs. 5, 6, 8, 11, 18, 22, 43, 45, 47, 66, 67) differ either in the representation of the scenario (28) (or (33)) or in methods of solving the equation (29). I.V. Mishin[66] could not solve the problem for an anisotropically reflecting underlying surface, and A.A. Ioltukhovski[68] drew on the results by T.A. Sushkevich and, trying to re-expose them, committed gross mathematical errors.

Within the frames of the strict theory of OTF and linear-system approach, the method of IF and SFC is generalized for problems that allow for polarization[69–73] and for two-medium systems of transfer (atmosphere–ocean, atmosphere–cloudiness, atmosphere–hydrometeors, atmosphere–plant cover) with an internal separation boundary,[74–79] and horizontally inhomogeneous atmosphere.[5,15,41,42] Note that the transfer characteristics of ASS with the allowance for polarization were first studied by K.Ya. Kondrat'ev and O.I. Smoktii.[80] An outstanding contribution to the solution of the problem of polarization contrast was brought by G.V. Rozenberg,[81] T.A. Sushkevich and S.A. Strelkov,[15] researchers from the Institute of Atmospheric Optics, Siberian Branch of the Russian Academy of Sciences.[21]

Finally, the initial GBP (8) is reduced to a linear functional, and the linear-system approach to solving problems of remote sensing of the Earth's surface is formulated. The approach precisely defines the manifestation of nonlinear effects caused by multiple reflections from a surface in the "scenario" formation. These effects are described by linear transfer characteristics of an isolated layer of the atmosphere.

Main problems of three-dimensional transfer

First of all, problems of radiation correction are solvable if and only if the three-dimensional boundary problem for a plane layer infinite along the horizontal coordinates is solvable. The problem cannot be solved without measures to restrict the infiniteness. So, in first works, PSF was simulated by the Monte-Carlo method (G.A. Mikhailov, B.A. Kargin, G.M. Krekov, V.V. Belov, D.A. Usikov, et al.) which, as known, showed a good performance in local calculations; by small-angle approximations (L.S. Dolin, L.M. Romanova, E.P. Zege, V.S. Remizovich, V.V. Veretennikov, et al.); by passing from a differential problem to an integral equation (E.S. Kuznetsov, M.S. Malkevich).

The approach based on the generalized solutions and integral transforms of distributions and, as a corollary, the method of IF and SFC appeared to be a powerful mathematical apparatus. As a result, instead of the unsolvable initial boundary problem, a new mathematical model of three-dimensional radiative transfer is constructed in the form of functionals whose kernels are IF and SFC, depending on the form of the representation. These bases made it possible to develop a unified, mathematically rigorous theory for description of systems of radiative transfer in different applications and with different geometry (one-, two-, three-dimensional plane and spherical problems).

Second, to solve problems of remote sensing (vision, image transmission, etc.), it is desirable to establish an explicit connection between the solution and parameters of the transfer system (or coefficients and sources of the boundary problem). IF and SFC are invariant characteristics of the system of radiative transfer with respect to sensed (or observed, or perturbed) parameters. The constructed functionals were called OTF because they (if following the physics of phenomenon) describe radiation transfer from the objects sounded through turbid scattering and absorbing media to a receiver.

Third, in practice, all the one-, two-, and three-dimensional observation problems are realized within the frames of the linear-system approach (including the problems in holography and tomography), which lays in the base of models for inverse problems. The difference between concrete approaches is in the way by which nonlinear effects and noises are taken into account. It was possible to represent all the nonlinear approximations by linear IF and SFC, and to reduce OTF to a linear functional whose kernel is either IF or SFC, and the scenario takes into account all the linear and nonlinear effects.

By use of the unified concept of SFC as an amplitude-frequency and phase-frequency characteristic of an object, of an atmospheric channel, of a measurement device, of an image, and so on, and by the formulated set of radiation characteristics of systems of radiative transfer, a software for imitation simulation of a high-resolution spaceborne system was realized for the first time under the supervision of T.A. Sushkevich with the assistance of S.A. Strelkov.

Conclusion

Thus, the investigations carried out made it possible to obtain fundamental results in the theory of OTF. First, OTF is formulated on the unified methodological basis for the whole variety of angular and spatial structures and characteristics of reflection and radiation sources. Second, all the nonlinear approximations are represented by linear IF and SFC. Third, a full set of base models for IF and SFC is determined. It is necessary and sufficient for describing transmission characteristics of a system of radiative transfer. Fourth, the OTF is constructed in a mathematically rigorous way and physically correctly within the frames of the linear-system approach. The theory of OTF presented here describes the well known Russian and foreign theoretical results.

The method of IF and SFC is a universal mathematically strict approach to solving problems of a wide range of applications. Interpretation of methods developed by different authors as realizations of the method of IF and SFC permits one to obtain unified basic formulas for a wide class of applied problems. For these problems, the methods, tools, approaches introduced by different authors are in fact either equivalent, differing only in schemes of realization, or close to each other. So it is not expedient to personalize these methods, which became almost classical. For more details of algorithms, the reader is referred to original sources whose bibliography consists of more than 800 publications and can be found in Ref. 25. In different applications a particular, special, applied terminology was established. This complicates establishing the generality among the fields and restricts possibilities of using most advanced results from adjacent branches. At present, when theoretical and numerical investigations are performed on a mass scale due to accessibility of computers, manipulation with mathematical objects makes it necessary to use universal, generalized mathematical terms and concepts. In empirical theoretical and calculation investigations on computers, almost each investigator introduces his/her own terminology in the same field. This creates a false impression of the methods developed as of unique ones.

As seen from analysis of the state of the problem on the allowance for the Earth's surface and its remote sensing, the whole variety of approaches can be reduced to three main ones. The implicit way to take into account the underlying surface was first to appear. The second approach is the explicit way by the method of IF and SFC. The third approach is connected with functionals and conjugated equations.[61] The term IF unites all types of singularity and diffuseness of a source and all the four types of surfaces. The term SFC means two-dimensional Fourier spectra in a horizontal plane, including those of IF. In particular, when one takes a Fourier spectrum of PSF, the SFC is sometimes called the OTF or MTF.[3,11,19,21,66]

The generality of the technique described is that it can be spread for different ranges of wavelength spectrum and conditions of remote sensing. It is important that the scenario and the atmospheric channel would be considered within the frames of the theory of radiative transfer. In particular, this refers to the quasi-optic approximation for the range of millimeter waves. It is preferable to avoid frequent use of the term "optical" which narrows the applicability.

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