ACCURACY OF THE TURBULENT ENERGY DISSIPATION RATE ESTIMATION FROM THE TEMPORAL SPECTRUM OF WIND VELOCITY FLUCTUATIONS

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The paper describes the errors in the estimates of the turbulent energy dissipation rate based on the likelihood maximum from time spectra of wind velocity measured with an acoustic anemometer and a Doppler lidar.

1. INTRODUCTION

At present some methods are available for determining the turbulent energy dissipation rate in the atmosphere. As a rule, the regularities of inertial interval of turbulent variations of the air flow rate are used.[1–3] The assessment of the dissipation rate from the time spectrum of the measured wind velocity is most generally employed. In this case the hypothesis of frozen turbulence is used.[1]

The accuracy of the dissipation rate estimates from the wind velocity time spectrum is governed by various factors, namely, the state of the atmosphere (intensity and typical scales of turbulence), duration and geometry of sounding, method of data processing, noise, and so on. Therefore when planning the measurements of the dissipation rate it is important to have the results of preliminary calculations of the error of the parameter being considered depending on the above-mentioned factors. This paper presents the analysis of such an error when estimating the turbulent energy dissipation rate from time spectrum of wind velocity based on the likelihood maximum.

2. ESTIMATION OF THE DISSIPATION RATE

Let we have a the succession of the values of wind velocity $V(\Delta tm)$, measured during $T = \Delta tM$, where $m = 0, 1,\ldots, M - 1$; Δt is the discreteness interval. Using the fast Fourier transform we can obtain from this succession a one-sided function of spectral density

$$\hat{S}\left(\frac{k}{T}\right) = \frac{2\Delta t}{M}\left|Z\left(\frac{k}{T}\right)\right|^2, \tag{1}$$

where

$$Z\left(\frac{k}{T}\right) = \sum_{m=0}^{M-1} V(\Delta tm)\exp\left[-j\,2\pi\frac{mk}{M}\right],$$

$k = 0, 1, \ldots, M/2$.

Let us assume that for analyzing the time statistical characteristics of the measured wind velocity, $V(t)$, the hypothesis of frozen turbulence is used.[1] Then at large T when $T \gg \tau_k$ (here τ_k is the time of wind velocity correlation) the frequency interval $[f_1, f_2]$ can be chosen within the limits of which the estimate of $\hat{S}(f)$ is unbiased (i.e., $\langle\hat{S}(f)\rangle = \hat{S}(f)$, where $\langle\ldots\rangle$ is the averaging over the ensemble) and its average value is described by the formula

$$S(f) = \varepsilon^{2/3}\,Q(f), \tag{2}$$

where ε is the mean rate of the turbulent energy dissipation; $Q(f)$ is the function whose parameters are the mean wind velocity U and the size of the volume sounded. In particular, when measuring the longitudinal component of wind velocity at a fixed point $Q(f) = 0.15\,U^{2/3}/f^{5/3}$, Refs. 1–3. The frequency $f_1 = k_1/T$ is the lower limit of the inertial interval, and $f_2 = k_2/T$ corresponds to the highest frequency at which the noise contribution to the measured spectrum can be neglected.

Having used Eq. (2) we obtain the estimate of the turbulent energy dissipation rate from n estimates of the spectral density $\hat{S}_i = \hat{S}[(k_0 + i)/T]$ at the frequencies falling within an interval being studied, where $k_1 \le k_0$ and $k_0 + n \le k_2$, $i = 1, 2,\ldots,n$. The complex random value $Z(k/T)$ has due to the condition $T >> \tau_k$ the normal probability density distribution with zero mean value and equal variances for the real and imaginary part Z, proportional to the mean spectrum $S(k/T)$. Then the probability density of the $\hat{S}_i \sim |Z|^2$ spectrum estimate will be distributed exponentially. In this case, the $\hat{S}_i$ and $\hat{S}_l$ estimates (at $i \ne 1$) are independent. Hence, the probability density of the vector $\hat{\mathbf{S}} = \{\hat{S}_1, \hat{S}_2,\ldots,\hat{S}_n\}$ has the form

$$P(\hat{\mathbf{S}}) = \left(\prod_{i=1}^{n}\frac{1}{S_i}\right)\exp\left[-\sum_{i=1}^{n}\frac{\hat{S}_i}{S_i}\right], \tag{3}$$

where, in accordance with the Eq. (2), $S_i = \varepsilon^{2/3}Q_i$, $Q_i = Q[(k_0 + i)/T]$. When the values Q_i are considered to be known, the dissipation rate $\hat{\varepsilon}$ can be

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determined from the vector $\hat{\mathbf{S}}$ based on the likelihood maximum.[4] To do this one must take the derivative of the logarithmic likelihood function with respect to ε

$$L(\varepsilon) = \ln P(\hat{\mathbf{S}}) = -\sum_{i=1}^{n}\left[\ln(\varepsilon^{2/3} Q_i) + \frac{\hat{S}_i}{\varepsilon^{2/3} Q_i}\right] \quad (4)$$

and it is necessary to solve the equation

$$\left.\frac{\mathrm{d}L(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=\hat{\varepsilon}} = 0. \quad (5)$$

As a result we have

$$\hat{\varepsilon} = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{\hat{S}_i}{Q_i}\right)^{3/2}. \quad (6)$$

3. STATISTICAL CHARACTERISTICS OF THE DISSIPATION RATE ESTIMATE

Let us divide both parts of the expression (6) by the mean value of the dissipation rate ε. Then, taking into account Eq. (2), the normalized value of the above estimate can be written as follows:

$$\hat{\varepsilon}/\varepsilon = x_n^{3/2}, \quad (7)$$

where

$$x_n = \frac{1}{n}\sum_{i=1}^{n}\frac{\hat{S}_i}{S_i}, \quad (8)$$

in accordance with Eq. (3) we have the following gamma-distribution of the probability density:

$$P(x_n) = (n^n/[\Gamma(n)])\, x_n^{n-1}\, \mathrm{e}^{-nx_n}, \quad (9)$$

where $\Gamma(n)$ is the gamma-function; $x_n \in [0, \infty]$. For the mean value of x_n^{ν} we have

$$\langle x_n^{\nu}\rangle = \int_0^{\infty} \mathrm{d}x_n\, x_n^{\nu} P(x_n) = \frac{\Gamma(\nu + n)}{n^{\nu}\,\Gamma(n)}. \quad (10)$$

Having used this formula, for the mean normalized value of $B = \langle\hat{\varepsilon}/\varepsilon\rangle$ and relative random error $E = [\langle(\hat{\varepsilon}/\varepsilon)^2\rangle - B^2]^{1/2}$ of the dissipation rate estimate we have

$$B = \Gamma(3/2 + n)/[n^{3/2}\,\Gamma(n)]; \quad (11)$$

$$E = \left[\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right) - \left(\frac{\Gamma(3/2 + n)}{n^{3/2}\,\Gamma(n)}\right)^2\right]^{1/2}, \quad (12)$$

where the characteristics of the gamma-function $\Gamma(3 + n) = n(n + 2)\Gamma(n)$ were used.

From Eq. (11) it follows that in the general case B does not equal 1, i.e., the estimate of ε is biased. The complete magnitude of the error, taking into account the biased estimate, is determined by the formula (see Ref. 5) $A = [E^2 + (B - 1)^2]^{1/2}$. At $n = 1$ we have from Eqs. (11) and (12) that $(3/4)\sqrt{\pi} \approx 1.33$ and $E = \sqrt{6 - 9\pi/16} \approx 2.06$. Such estimates of $\hat{\varepsilon}$ are 1.33 times greater than ε; and their random variance relative to ε is significant. Consider now the case of $n >> 1$ which is of interest for a practical use. In Eqs. (11) and (12) we may take the asymptotic formula[6]

$$\Gamma(3/2 + n)/\Gamma(n) \approx n^{3/2}\left(1 + \frac{3}{8}\frac{1}{n} + \ldots\right). \quad (13)$$

Having expanded Eqs. (11) and (12) into a power series over n^{-1} and having subtracted the components making the basic contribution to the corresponding characteristics, we obtain

$$B = 1, \quad E = \frac{3}{2}\frac{1}{\sqrt{n}}, \quad (14)$$

hence it follows that at $n >> 1$ the estimate of $\hat{\varepsilon}$ is unbiased and its error is proportional to $n^{-1/2}$.

4. COMPARISON WITH THE EXPERIMENTAL DATA

To check the obtained dependences of B and E on the number n of spectral components $\hat{S}_i$ chosen for estimating the dissipation rate the results of wind velocity measurements were used. The wind velocity was measured with an acoustic anemometer[7] at 6 m height above the ground.

The estimate of the mean wind velocity $\hat{U}$, obtained from these data, equals 3.45 m/s. The wind velocity spectrum is calculated by formula(1). Figure 1 shows a smoothed (averaged) over 100 statistical degrees of freedom spectrum. Dashed lines show the interval $[f_1, f_2]$. Using all the estimates of $\hat{S}_i$ in this interval, the mean dissipation rate $\bar{\varepsilon}$ is calculated by formula (6) (where the values Q_i are equal to $0.15(\hat{U}\bar{\varepsilon})^{2/3}[(k_0 + i)/T]^{-5/3}$). The mean dissipation rate equals 0.046 m^2/s^3. In calculations of the first and second moments of the value $x_n^{3/2}$ by formula (8) for S_i we use the expression $S_i = 0.15(\hat{U}\bar{\varepsilon})^{2/3}[(k_0 + i)/T]^{-5/3}$. The result of calculation of such components of S_i spectrum is given in Fig. 1 as a straight line in the frequency-range studied $[f_1, f_2]$.

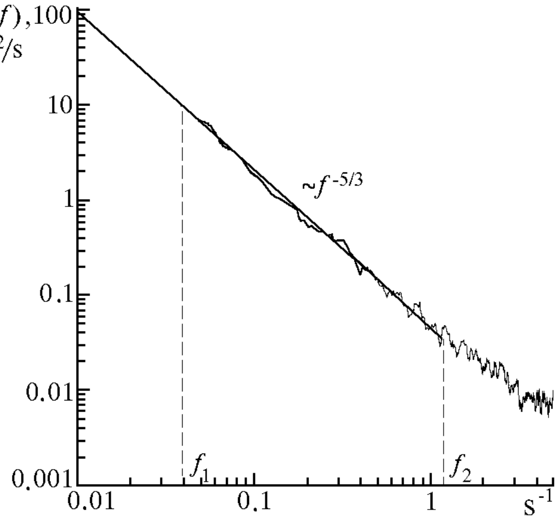

FIG. 1. Wind velocity spectrum.

The experimental dependences of B and E on n are denoted by small crosses and squares in Fig. 2.

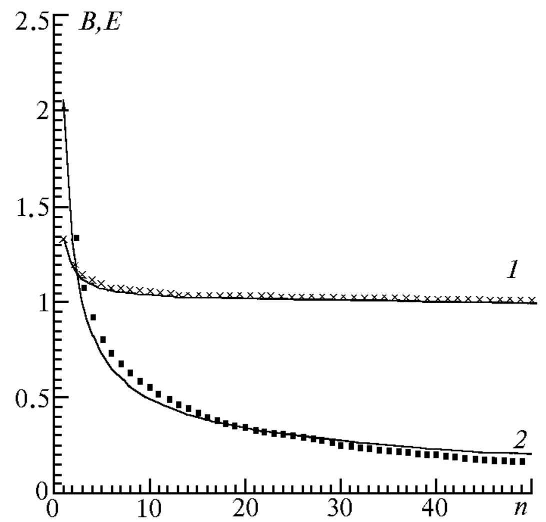

FIG. 2. The dependence of the normalized B value and of the relative random error E of the estimate of the turbulent energy dissipation rate on the number n of spectral components used in data processing. The data for B are denoted by small crosses and curve 1; the data for E are denoted by small squares and curve 2; the experimental data are denoted by small crosses and squares; the theoretical data are denoted by solid curves.

In Fig. 2 solid curves show similar theoretical dependences (curve *1* – B, curve *2* – E) calculated by formulas (11) and (12). The error of the experimental values E does not exceed 15%. This figure demonstrates quite good agreement between theory and the experiment. At $n = 10$ the estimate $\hat{\varepsilon}$ is practically unbiased ($B \approx 1$), but the relative error is great and is about 50%.

5. THE ACCOUNT FOR ERROR IN THE MEAN WIND VELOCITY ESTIMATE

According to Eq. (14) the error of estimate of the dissipation rate can be made as small as desired by increasing n. So, at a maximum value $n = 800$, realistic for the experiment, $E \approx 0.05$. However, in real situations, the error will be larger because of the error in the estimate of the mean wind velocity $\hat{U}$. The experimental data, presented in Fig. 2, may be represented by the expressions $\langle \hat{\varepsilon}\hat{U}/\varepsilon U \rangle$ and $[\langle (\hat{\varepsilon}\hat{U}/\varepsilon U)^2 \rangle - \langle \hat{\varepsilon}\hat{U}/\varepsilon U \rangle^2]^{1/2}$, where U is the true mean wind velocity. Therefore, taking into account the difference between $\hat{U}$ and U, for the normalized estimate of the dissipation rate, obtained from the results of the wind velocity measurements at a point, we must use the following expression instead of Eq. (7):

$$\hat{\varepsilon}/\varepsilon = (U/\hat{U})\, x_n^{3/2}. \tag{15}$$

In the measurements with a cw Doppler lidar the time spectrum of wind velocity $S(f)$ is the product of the wind velocity time spectrum at a fixed point, being at the center of a volume sounded, by $H(f)$, the function of a low-frequency filter determined by a longitudinal size Δz of this volume. The form of this function is given, in particular, in Ref. 8. At $\Delta z \to 0$ the function $H(f) \to 1$ and then the normalized estimate of the dissipation rate will be determined by Eq. (15). In the other limiting case of large values of Δz the following formula is derived for the spectrum in Ref. 8:

$$S(f) = 0.06\, \varepsilon^{2/3}\, (U \sin\gamma)^{5/3}\, (1/\Delta z)\, f^{-8/3}, \tag{16}$$

where γ is the angle between the wind direction and the axis of the sounding beam.

The data on U and γ can be obtained, for example, from additional measurements using a scanning Doppler lidar. It is reasonable that the measured value of the velosity $\hat{U}$ is different than the true mean value of U. From Eqs. (2), (6), (16) and (17), taking into account this difference and ignoring the error in estimates of the angle γ, we obtain the expression

$$\hat{\varepsilon}/\varepsilon = (U/\hat{U})^{5/2}\, x_n^{3/2}. \tag{17}$$

It is assumed that in Eqs. (15) and (17) the error $\tilde{U} = \hat{U} - U$ is small as compared with U. Then for $\hat{\varepsilon}/\varepsilon$ one can write the approximate formula

$$\hat{\varepsilon}/\varepsilon = (1 - \alpha\, \tilde{U}/U)\, x_n^{3/2}, \tag{18}$$

where for the case of point measurements $\alpha = 1$, and when measuring with a Doppler lidar in a large volume

$\alpha = 5/2$. If the estimate of wind velocity is unbiased $<(\hat{U} = U)>$, then, as it was noted above, at large n the estimate $\hat{\varepsilon}$ is also unbiased ($B = 1$). From Eqs. (7)–(14) and (18) at $n >> 1$ for the relative error E, taking into account the fluctuations of the estimate of mean wind velocity we obtain

$$E = [9/(4n) + \alpha^2 \sigma^2_{\hat{U}}]^{1/2}, \quad (19)$$

where $\sigma^2_{\hat{U}} = <(\tilde{U}/U)^2>$ is the relative variance of the wind velocity estimate.

From Eq. (19) it follows that in the case of small errors of wind velocity estimates satisfying the condition

$$\alpha^2 \sigma^2_{\hat{U}} \ll 9/(4n), \quad (20)$$

the values of E for a point meter and a Doppler lidar will not differ at one and the same number n. If we have the inverse condition

$$\alpha^2 \sigma^2_{\hat{U}} \gg 9/(4n), \quad (21)$$

then at the same accuracy of the mean wind velocity measurements the error of the lidar dissipation rate is 2.5 times as large as that obtained with a point meter, in particular, the acoustic anemometer that used in the experiment. Since at large n the condition (21) is quite realistic, the requirements for the accuracy of estimating the mean wind velocity in the Doppler lidar measurements of the dissipation rate should be more rigid than in the case of the acoustic anemometer.

Assume that in the point measurement the velocity $\hat{U}$ is determined by the time averaging over the period $T \gg \tau_k$. Then for $\sigma^2_{\hat{U}}$ under stationary conditions the following formula[2] is valid:

$$\sigma^2_{\hat{U}} = 2\sigma^2_U \tau_k/T, \quad (22)$$

where $\sigma^2_U = \langle V^2\rangle/U^2 - 1$ is the relative variance of instantaneous values of wind velocity.

Thus for E we have

$$E = [9/(4n) + 2\sigma^2_U \tau_k/T]^{1/2}. \quad (23)$$

In the experiment, whose results are shown in Figs. 1 and 2, $\sigma^2_U = 0.12$, $\tau_k = 15$ s, and $T = 720$ s. For such parameters the second component in Eq. (23) equals 0.005. The maximum n in this experiment was $n = 800$ and the first component in Eq. (23) was 0.0028. As a result the relative error of the dissipation rate measurement is about 10% that is twice as much as the corresponding value obtained above without the account for fluctuations of the estimates of the mean wind velocity. Assume that the estimate of $\hat{\varepsilon}$ was obtained from the Doppler lidar data at the same n and $\sigma^2_{\hat{U}}$ ($n = 800$, $\sigma^2_{\hat{U}} = 0.005$). Then according to Eq. (19), where $\alpha = 5/2$, the relative error of such an estimate equals approximately 24%.

6. CONCLUSION

This paper describes the analysis of the influence of the number of unsmoothed estimates of the spectrum and measurement error of the mean value of wind velocity on the accuracy of the estimate of the turbulent energy dissipation rate based on the likelihood maximum. The paper presents the measurements of wind velocity using an acoustic anemometer (the “point” meter) and a cw Doppler lidar at a large size of the volume sounded. It is shown that the influence of the measurement error of mean wind velocity on the accuracy of estimates of the dissipation rate is more essential in the case of a Doppler lidar measurements.

It should be noted that in the case of wind velocity measurement at a fixed point the use of the hypothesis of turbulence “freezing”, aimed at setting in Eq. (2) the function $Q(f_i)$ in an explicit form is always practically justified. The same can be stated for the wind velocity measured with a Doppler lidar in a large volume sounded provided that the side wind was strong when the fluctuations of the averaged over the sounding volume wind velocity are mainly defined by the transfer of turbulent eddies by the mean flow without evolution of eddies during the lifetime in the range of the laser beam localization. It has been found experimentally that at small angles γ and low wind[9,10] for the analysis of the wind velocity determined from a large volume of lidar sounding the hypothesis of the “frozen” turbulence is unacceptable. Therefore, if Eq. (16) is used for such conditions, a considerable bias of the estimate of the turbulent energy dissipation rate can occur.

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