# Second harmonic generation by focusing a beam into a uniaxial crystal with crossed cylindrical lenses. Approximation of a preset field

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A method is proposed, which allows the solution of the second-harmonic generation (SHG) problem in a uniaxial nonlinear crystal to be presented as a double integral in the approximation of a preset field at the fundamental frequency. This fact makes the final analytical equation simpler and more convenient for practical use than the known Boyd–Kleinman approximation. Within the frameworks of the method proposed, the problem of SHG by use of a laser beam focused with crossed cylindrical lenses into a crystal is solved. It is shown that, in a certain case of practical interest (ray optics approximation), the equation for the second-harmonic field can be written through elementary functions and is free of quadratures.

## Introduction

Owing to the number of unique properties of the output radiation, metal-vapor lasers are now finding a wide-ranging application in research and technology. The range of applications can be extended significantly by using not only fundamental frequencies of these lasers, but also their harmonics lying usually in the UV spectral region. The most promising from this viewpoint (being more powerful) are copper vapor lasers,[1] emitting simultaneously at two wavelengths, green (510.6 nm) and yellow (578.2 nm). With these lasers, using a nonlinear $\beta$-$BaB_2O_4$ (BBO) crystal, one can obtain the discretely tunable radiation at three wavelengths: 255.3 nm (second harmonic (SH) of the green line), 289.1 nm (SH of the yellow line), and 272.2 nm (sum frequency of the two fundamental lines). It is clear that the practical significance of this approach depends directly on the efficiency of nonlinear conversion, but certain problems arise here.

The point is that the output pulse power of metal-vapor lasers, operating in the kilohertz region of the pulse repetition frequencies, is not high enough to ensure the considerable manifestation of nonlinear effects in the crystals known by now. Nevertheless, it is possible to improve the situation with these lasers and to obtain practically significant efficiencies of the nonlinear conversion (even with low-power radiation[2,3]), if the laser beam is focused into a crystal in an optimal way. The aim of this paper is just the theoretical study of this issue.

It is known (see, for example, Ref. 4) that the maximum efficiencies of harmonic generation processes should be expected in the case of the cylindrical focusing of a laser beam into a nonlinear crystal. The term "cylindrical focusing" is understood in the most general sense as creation of such conditions, at which the divergence of the laser beam appears to be significantly different in two mutually orthogonal planes. The principal optical plane of the anisotropic nonlinear crystal is taken as one of these planes. It should be noted in this connection that we restrict this consideration to only uniaxial media possessing quadratic nonlinearity. The cylindrical focusing can be realized in different ways, using different number of optical elements, but we consider only one, the most general case (including all others as particular cases). In this case, it is assumed that the laser beam is focused using two crossed (at the angle of 90°) cylindrical lenses with different focal lengths.

All the calculations were conducted assuming a preset field at the fundamental frequency.[4,5] The Boyd–Kleinman method,[6] so modified somewhat for a simpler numerical simulation, was used as a basis. To achieve the best agreement between the model representations used and the actual experimental conditions, the problem was solved taking into account the radiation refraction at the entrance and exit facets of the crystal. For this purpose we made use of the results from Ref. 7. The laser radiation was assumed fully spatially coherent. As applied to metal-vapor lasers, this means that the calculated results reconstruct most accurately the actual situation in the cases when special measures are undertaken in the experiment to improve the degree of the spatial coherence of the laser radiation. We mean unstable resonators with a large confocal parameter, systems of spatial filtering of a beam, etc.

## 1. Linear approximation

Let us suppose that we deal with a laser beam, propagating in a vacuum ($n = n_0 = 1$) along the axis $Z$ and having a plane phase front along a rather long path. The latter means that the transverse radius of the beam $a_0$ (the consideration is restricted to the

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case of centrally symmetric fields) is rather large as compared to the wavelength. Accurate to the phase factor $\exp(ikz)$, the equation for such a beam in the most general case among all cases of interest can be written in the following form

$$A(\mathbf{r}) = A_0 \exp[-(x/a_0)^{2m_x}]\exp[-(y/a_0)^{2m_y}], \quad (1)$$

where $k = 2\pi/\lambda$; $m_x$, $m_y$ = 1, 2, 3, …;

$$A_0 = \sqrt{8\pi P/cI},$$

$$I = \left[\int_{-\infty}^{+\infty}\exp[-2(x/a_0)^{2m_x}]\mathrm{d}x\right]\left[\int_{-\infty}^{+\infty}\exp[-2(y/a_0)^{2m_y}]\mathrm{d}y\right];$$

$P$ is either the mean power ($P_{\text{mean}}$) or pulse power ($P_{\text{p}}$), and in the approximation of rectangular pulses $P_{\text{p}} = P_{\text{mean}}/\tau F$, $\tau$ is the pulse duration; $F$ is the pulse repetition frequency; $c$ is the speed of light.

We suppose that the beam under consideration is focused using a system, consisting of two thin crossed cylindrical lenses $L_x$ and $L_y$, spaced by the distance $L_{xy}$. The lens $L_x$ has the focal length $f_x$ and focuses the beam in the plane $XZ$. The lens $L_y$ with the focal length $f_y$ contracts the beam in the plane $YZ$. For simplicity, the distance $L_{xy}$ between the lenses is chosen so that the waist planes of both lenses coincide.

Then we suppose that the infinite layer, between the planes $z = z_c$ and $z = z_c + L$, is an isotropic medium with the refractive index equal to $n$. The absorption is neglected, that is, we believe that $n$ is a real constant. The lenses $L_x$ and $L_y$ are assumed to be spaced by $z_{cx}$ and $z_{cy}$ from the beginning of this layer (this layer will be referred to as a crystal and thought to be isotropic). Our task is to calculate the field at an arbitrary distance $z_0$ from the exit from the crystal.

We restrict our consideration to the quasioptical approximation, that is, we assume that the following condition is fulfilled

$$\alpha_x = \frac{a_0}{f_x} \sim \alpha_y = \frac{a_0}{f_y} << 1. \quad (2)$$

According to results from Ref. 7, we obtain for the complex amplitude of the solution of interest

$$A(x_0, y_0, z_0) = A_x(x_0, z_0)\, A_y(y_0, z_0), \quad (3)$$

where for $j = x, y$ $(x_{0j} = x_0, y_0)$

$$A_j(x_{0j}, z_0) = \sqrt{-\frac{ik}{2\pi z_j}}\sqrt{A_0 T_1 T_2} \times$$

$$\times \int_{-\infty}^{+\infty} \mathrm{e}^{-\left(\frac{x}{a_0}\right)^{2m_j}} \mathrm{e}^{-ik\frac{x^2}{2f_j}} \mathrm{e}^{ik\frac{(x-x_{0j})^2}{2z_j}} \mathrm{d}x;$$

$$z_j = z_{cj} + L/n + z_0,$$

and the following is obviously fulfilled:

$$|z_x - z_y| = L_{xy};$$

$T_1 = 2/(n + 1)$, $T_2 = 2n/(n + 1)$ are the Fresnel coefficients for the refraction at the entrance and exit facets of the crystal, written in the approximation of the normal incidence [this takes place in view of Eq. (2)].

Let $\Delta_f$ determine the distance from the common waist plane of the lenses to the center of the crystal (for definiteness, we assume that $\Delta_f > 0$, if the laser beam is focused before the center of the crystal). We believe that $\Delta_f$ is the initial, i.e., known parameter, that is, the waist position relative to the crystal is preset. In this case, to make use of Eq. (3), it is necessary to find the distances $z_{cx}$ and $z_{cy}$. Using the results from Ref. 7, we find

$$z_{cj} = \begin{cases} z_{pj} + \Delta_f + L\left(\frac{1}{2} - \frac{1}{n}\right), & \text{if } \Delta_f < 0, |\Delta_f| \ge L/2, \\ z_{pj} + \Delta_f - L/2, & \text{if } \Delta_f \ge L/2, \\ z_{pj} + \frac{1}{n}(\Delta_f - L/2), & \text{if } |\Delta_f| < L/2, \end{cases} \quad (4)$$

where $z_{pj}$ is the position of the waist plane of the $j$th lens (measured from the lens) in a vacuum.

The analytical representations of $z_p$ are known only for the Gaussian beams (see, for example, Ref. 5), that is, if $m_x = m_y = 1$ in Eq. (3). On the other hand, if we require the fulfillment of the following condition

$$D_j = (2f_j / ka_0^2) << 1, \quad (5)$$

then for practical estimates in Eq. (4) we can take with a good accuracy that

$$z_{pj} \approx f_j. \quad (6)$$

From here on, we will use just this approximation (the case of a rather sharp focusing), because it is just this situation, which is most interesting to us in solving nonlinear problems.

It is likely possible to eliminate integrals from Eq. (3) only in two cases, namely when Eq. (1) defines the Gaussian beam[5] or if the solution of the problem is sought in the ray optics approximation. Let us consider the latter possibility in a more detail.

It can easily be seen that, for the case when condition (5) holds, one can choose the distance $z_j$, by increasing $z_0$, so large that the below condition is certainly met

$$\left|\frac{1}{z_j} - \frac{1}{f_j}\right| >> \frac{2m_j}{ka_0^2}. \quad (7)$$

This means that we are interested in finding the solution to the problem in an region, which sufficiently far from the lens waist plane, that is, where the ray optics approximation yields the result close to the exact solution of the problem. The validity of the above-said can be easily checked, by using the condition (7) for the known solution of the problem on propagation of the Gaussian beam.[5]

The fulfillment of Eq. (7) allows the integrals over $x$ and $y$, in Eq. (3), to be calculated asymptotically, using the method of stationary phase.[8] As a result, for the most interesting case of $|\Delta_f| < L/2$ we can write (the constant phase change equal to $\pi$ is omitted):

$$A(x_0, y_0, z_0) = A_x(x_0, z_0)\, A_y(y_0, z_0), \tag{8}$$

where

$$A_j(x_{0j}, z_0) = \sqrt{A_0 T_1 T_2 / q_j}\; \mathrm{e}^{-\left(\frac{x_{0j}}{a_j}\right)^{2m_j}}\, \mathrm{e}^{ik\frac{x_{0j}^2}{2R_j}};$$

$$q_j = \left|1 - \frac{z_j}{f_j}\right|,\ a_j = a_0 q_j;$$

$$R_x \approx R_y = z_j - f_j = \frac{1}{n}(\Delta_f + L/2) + z_0 \equiv R,$$

and the last equation is the direct consequence of Eq. (6).

Equation (8), determining the properties of paraxial beams in the ray optics approximation, is used in what follows as one of the boundary conditions of the nonlinear problem. It should be noted that this representation of the field could be obtained from the solution of the corresponding wave equations modified for the case of the infinitely short wavelengths. This possibility is considered, for example, in Ref. 5.

## 2. Approximation of a preset field for the second harmonic generation. General solution

Within the frameworks of this study, we are interested in the steady-state second harmonic generation (SHG) mode in a homogeneous uniaxial crystal with a quadratic optical nonlinearity. The radiation of a laser beam, propagating along the $Z$ axis of a Cartesian coordinate system, is believed to be monochromatic and spatially coherent. In addition, we restrict the consideration to the approximation of a preset field and to the scalar *ooe*-interaction. The first of these two restrictions is of principle, as the generalization to other types of interaction can easily be obtained by analogy. For the case under study, the complex slowly varying amplitudes (below, we shall use the term “field” for brevity) at the fundamental frequency $A_1(x, y, z)$ and at its second harmonic $A_2(x, y, z)$ are, as known, the solutions of the following equations[4,5]:

$$\frac{\partial A_1}{\partial z} + \frac{1}{2ik_{1o}}\left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2}\right) = 0, \tag{9.1}$$

$$\frac{\partial A_2}{\partial z} + \rho\frac{\partial A_2}{\partial x} + \frac{1}{2ik_2^e}\left(\frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_2}{\partial y^2}\right) = i\sigma A_1^2 \mathrm{e}^{-i\Delta_k z}, \tag{9.2}$$

where

$$k_{1o} = kn_o(\omega),\quad k_2^e = 2kn^e(2\omega, \theta),\quad k = \omega/c;$$

$\theta$ is the angle between the optical axis of the crystal, lying in the plane $XZ$, and the axis $Z$ of the coordinate system; $\Delta_k = k_2^e - 2k_{1o}$ is the wave detuning; $\rho$ is the birefringence angle; $\sigma$ is the coefficient of nonlinear coupling.

Equations (9) are written accurate to terms of the order of $\mu^2$, where the value of the small parameter $\mu \ll 1$ is determined by the divergence (2) of the fundamental radiation. The anisotropy angle is also assumed small $(|\rho| \sim \mu)$. If one restricts the consideration to situations, in which

$$\frac{|\Delta_k|}{k} \sim \mu, \tag{10}$$

then in Eq. (9) outside the exponent we can use

$$n_o(\omega) \approx n^e(2\omega, \theta) \equiv n\,. \tag{11}$$

Or for wave numbers

$$2k_{1o} \approx k_2^e \equiv 2k. \tag{12}$$

Let the plane $z = 0$ be the entrance facet of the nonlinear crystal and in this plane

$$A_2(x, y, 0) = 0\,. \tag{13}$$

In this case, the solution of Eq. (9.2), satisfying the boundary condition (13), is the function

$$A_2(x_0, y_0, z) = i\sigma \int_0^z \mathrm{e}^{-i\Delta_k t}\left[-\frac{ikn}{\pi(z-t)}\right] \times$$
$$\times \left[\iint_{-\infty}^{+\infty} A_1^2(x, y, t) \exp\left[ikn\frac{(x - x_0 + \rho z - \rho t)^2}{(z-t)}\right] \times\right.$$
$$\left.\times \exp\left[ikn\frac{(y - y_0)^2}{(z-t)}\right]\mathrm{d}x\mathrm{d}y\right]\mathrm{d}t, \tag{14}$$

where $z$ is an arbitrary distance passed by the laser beam inside the crystal.

The validity of the above-said can be easily checked by the direct substitution of Eq. (14) into Eq. (9.2). In writing Eq. (14), we supposed that the transverse dimensions of the laser beam inside the nonlinear crystal are much smaller than the cross size of the crystal itself, which has allowed the infinite limits to be used for the integrals over $x$ and $y$.

Denote the length of the nonlinear crystal as $L$. Then for all $z \le L$ Eq. (14) determines the part of the SH field, which was formed at the distance $z$. This component can be considered now as an independent extraordinary wave, propagating linearly along the $Z$ axis absolutely independent of the processes, which will occur at a distance between $z$ and the chosen observation plane (inside or behind crystal). Let the space $z > L$ be vacuum ($n = n_0 = 1$), and we are interested in the form of the field (14) in the plane $L_0 = L + z_0$, that is, at the distance $z_0$ from the crystal exit. The result of such a "linear" propagation of the SH wave from an arbitrary plane $z$ to $L_0$ will be denoted as a function $A_2(x, y, z; L_0)$. After quite standard calculations, which are omitted here, we find that

$$A_2(x_0, y_0, z, L_0) = iT_2\sigma \int_0^z \mathrm{e}^{-i\Delta_k t}\left[-\frac{ik}{\pi t_L}\right] \times \left[\int_{-\infty}^{+\infty}\!\!\int A_1^2(x, y, t)\exp\left[ik\frac{(x - x_0 + \rho L - \rho t)^2}{t_L}\right] \times \left[ik\frac{(y - y_0)^2}{t_L}\right]\mathrm{d}x\mathrm{d}y\right]\mathrm{d}t, \qquad (15)$$

where $t_L = z_0 + (L - t)/n$. It is taken into account here that the field (15) passes the path $L - z$ in the anisotropic medium. The losses for the reflection from the exit facet of the crystal are taken into account as well.

Equation (15) is, generally speaking, the sought solution of the SHG problem in the approximation of a preset field, but it has one significant disadvantage. This solution appears to be very complicated and, in the general case, assumes the calculation of fivefold integrals. An exception, apparently the only one, is the SHG problem in the case of Gaussian beams. In this case the form of the function $A_1(x, y, t)$ is known and the integrals from Eq. (15) over the transversal coordinates can be calculated exactly. This result is known as Boyd–Kleinman formula. Since this particular case does not comprehend all other situations, it seems worth demonstrating the method, which yields the result analogous to Eq. (15), but in a markedly simpler form. The reasoning is as follows.

To speak generally, the boundary condition for the field at the fundamental frequency [that is, for Eq. (9.1)] can be set wherever possible, because in this case we deal with the purely linear problem. Therefore, we assume that the function $A_1$ is known at the plane $L_0$ [for which the solution (15) is written]:

$$A_1(x,\ y,\ z = L_0) \equiv A_1(x, y, L_0). \qquad (16)$$

Then, obviously, inside the crystal we have that

$$A_1(x, y, t) = \frac{ik}{2\pi T_2 t_L}\int_{-\infty}^{+\infty}\!\!\int A_1(\xi, \eta, L_0)\mathrm{e}^{-ik\frac{(x-\xi)^2+(y-\eta)^2}{2t_L}}\mathrm{d}\xi\mathrm{d}\eta, \qquad (17)$$

where $t_L$ is determined in Eq. (15).

After two times substitution of Eq. (17) into Eq. (15) and making rather simple calculations, we obtain the following result:

$$A_2(x_0, y_0, z; L_0) = (i\sigma/T_2)\int_0^z \mathrm{e}^{-i\Delta_k t}\left[\frac{ik}{\pi t_L}\right] \times \left[\int_{-\infty}^{+\infty}\!\!\int \mathrm{e}^{-ik\frac{x^2+y^2}{t_L}} A_1(x_0 - \gamma_t - x, y_0 - y, L_0) \times A_1(x_0 - \gamma_t + x, y_0 + y, L_0)\mathrm{d}x\mathrm{d}y\right]\mathrm{d}t, \qquad (18)$$

where $\gamma_t = \rho(L - t)$.

Equation (18) is just the sought general solution of the nonlinear problem. This result is fully equivalent to that presented by Eq. (15), but much simpler, because now it is necessary to know the form of the field at the fundamental frequency in only one plane, namely, at $L_0$.

The solution (18) can be reduced to even simpler form, if we assume that the observation plane is infinitely far (or, at least, very far as compared to the crystal length) from the waist plane. In this case, [see the comments to Eq. (7)] for the boundary condition (16) we can use Eq. (8). Moreover, it seems to be quite justified to seek the solution of the SH field in a similar form as well, that is, to suppose that

$$A_2(x, y, z; L_0) = U_2(x, y, z; L_0)\,\mathrm{e}^{i2k\frac{(x-\rho L)^2+y^2}{2R}}, \qquad (19)$$

where the radius of curvature of the wave front is determined by Eq. (8).

Taking the above-said into account, we substitute Eqs. (8) and (19) into Eq. (18), set $z_0$ tending to infinity, and after elementary transformations obtain

$$U_2(x_0, y_0, z; L_0) = i\sigma\left(\frac{ik}{T_2\pi z_0}\right)\int_0^z \mathrm{e}^{-iQ_0 t} \times \left[\int_{-\infty}^{+\infty}\!\!\int \mathrm{e}^{-ikV_t(x^2+y^2)}U_1(x_0 - x, y_0 - y, L_0) \times U_1(x_0 + x, y_0 + y, L_0)\mathrm{d}x\mathrm{d}y\right]\mathrm{d}t, \qquad (20)$$

where

$$Q_0 = \Delta_k - 2k\frac{x_0\rho}{z_0};\ \ V_t = \left(\Delta_f - L/2 + t\right)/nz_0^2,$$

and

$$U_1(x, y, L_0) = \frac{A_0T_1T_2}{\sqrt{q_xq_y}}\exp\left[-\left(\frac{x}{a_x}\right)^{2m_x}\right]\exp\left[-\left(\frac{y}{a_y}\right)^{2m_y}\right].$$

Now in Eq. (20) we calculate the integral over $t$ and, as a result, obtain the simplest representation of the SH field

$$U_2(x_0, y_0, z; L_0) = i\sigma z\left(\frac{ik}{T_2\pi z_0}\right)\mathrm{e}^{-iQ_0z/2}\times$$
$$\times\iint\limits_{-\infty}^{+\infty}\mathrm{e}^{-ikV_z(x^2+y^2)}U_1(x_0 - x, y_0 - y, L_0)\times$$
$$\times U_1(x_0 + x, y_0 + y, L_0)\,\mathrm{sinc}\left(Qz/2\right)\mathrm{d}x\mathrm{d}y, \quad (21)$$

where

$$Q = Q_0 + k(x^2 + y^2)/nz_0^2;$$
$$V_z = (\Delta_f - L/2 + z/2)/nz_0^2;\ \ \mathrm{sinc}(x) \equiv \sin(x)/x.$$

Similar result, but for more general case, is presented in Ref. 9.

Consider now a particular case, in which the result can be written without quadratures. Assume that the laser beam waist lies outside the crystal (for definiteness, we believe that $\Delta_f > 0$). For this case, we again obtain the solution (20), in which now $V_t$ (with the allowance made for Eq. (4)) is as follows:

$$V_t = \left(\Delta_f - L/2 + t/n\right)/z_0^2. \quad (22)$$

Assume then that the following condition is fulfilled (the distance $\Delta_f$ is large enough)

$$\left|\Delta_f\right|\frac{2k\alpha_j^2}{a_j} >> \frac{1}{U_1^2}\left|\frac{\partial U_1^2}{\partial x_j^2}\right|, \quad (23)$$

where $x_j = x, y$; $\alpha_j$ are determined by Eq. (2).

This means that now the integrals, in Eq. (20), over the transverse coordinates can be estimated asymptotically. Thus, we obtain

$$U_2(x_0, y_0, z; L_0) = \frac{i\sigma}{T_2z_0}\,U_1^2(x_0, y_0, L_0)\int\limits_0^z\frac{\mathrm{e}^{-iQ_0t}}{V_t}\mathrm{d}t. \quad (24)$$

If, in addition to Eq. (23), the following condition is fulfilled

$$\left|\Delta_f\right| >> L, \quad (25)$$

then in Eq. (24) we have approximately that

$$V_t \approx \Delta_f/z_0^2. \quad (26)$$

Taking Eq. (26) into account, we obtain, instead of Eq. (24), that

$$U_2(x_0, y_0, z; L_0) = i\sigma\,z\left(\frac{z_0}{T_2\Delta_f}\right)U_1^2(x_0, y_0, L_0)\times$$
$$\times\ \mathrm{sinc}\left(\frac{Q_0z}{2}\right)\exp(-iQ_0z/2). \quad (27)$$

The physical meaning of the conditions used is quite clear. It is easy to see that Eq. (24) is the solution of the SHG problem for the divergent beam in the ray optics approximation. Consequently, condition (23) shows how far from the waist the nonlinear crystal should be for the effect of diffraction to be neglected. In addition, the fulfillment of Eq. (25) allows us to neglect the amplitude variation within the nonlinear crystal due to the geometric divergence of the beam at the fundamental frequency.

## Conclusions

Let us formulate some general remarks, which are not directly connected with the issue considered, but have, in our opinion, a certain methodological significance.

Return to the solution (14) and note, first of all, that the structure of this equation determining the SH wave will not change, obviously, if we refuse from using the approximation of a preset field, that is, we consider the complete system of nonlinear equations instead of Eq. (9) (see, for example, Ref. 4). Of course, in this case Eq. (14) is not already a solution, but an integral equation ($A_1(x, y, t)$ now depends on $A_2(x, y, t)$) for determination of the SH amplitude. For further reasoning, the approximation of quasioptics, which we used in deriving Eq. (14) for the transformation of the Green's function, appears to be of no principle here too. Different representations of the Green's function are considered in Ref. 10, and here we shall not touch these aspects. In other words, we can conclude that Eq. (14) defines the exact structure of the representation of the SH field amplitude. Below we demonstrate that this exact representation allows a very simple physical interpretation.

As known,[4] the complex amplitude of the SH plane wave satisfies the equation

$$\frac{\mathrm{d}A_2(t)}{\mathrm{d}t} = i\sigma\mathrm{e}^{-i\Delta_kt}A_1^2(t), \quad (28)$$

which, in particular, follows from Eq. (9.2), if we neglect in it the diffraction (the amplitude of the plane wave is independent of the transverse coordinates $x$, $y$) and ignore (for the same reason) the energy walk off due to birefringence. On the other hand, Eq. (28) can be used approximately for spatially limited beams in an anisotropic medium assuming that the distance $\Delta t$, at which the nonlinear interaction occurs, is small. In this case, neither diffraction nor anisotropy can markedly change the shape of the beam. The above-said becomes mathematically rigorous, if we state that $\Delta t$ tends to zero.

Thus, the elementary increment of the SH amplitude due to the nonlinear interaction at an infinitely small distance $\mathrm{d}t$ in any crystal for any beam (with arbitrarily small, but finite dimensions in the cross section) can be written absolutely rigorously, using Eq. (28), in the following form:

$$\mathrm{d}A_2(x,y,t) = i\sigma \mathrm{e}^{-i\Delta_k t} A_1^2(x,y,t)\mathrm{d}t. \tag{29}$$

Consider now Eq. (29) as a boundary condition for the SH wave, determined in the plane $z = t$, and find the solution of the linear problem on propagation of this wave to an arbitrary plane $z \geq t$. Using the results obtained in Ref. 10, we have that, for example, in the quasioptical approximation

$$\mathrm{d}A_2(x_0, y_0, t; z) = \left[-\frac{ikn}{\pi(z-t)}\right] \times$$

$$\times \left[\int\limits_{-\infty}^{+\infty}\int \mathrm{d}A_2(x,y,t)\, \mathrm{e}^{ikn\frac{(x-x_0+\rho z-\rho t)^2}{(z-t)}}\, \mathrm{e}^{ikn\frac{(y-y_0)^2}{(z-t)}}\, \mathrm{d}x\mathrm{d}y\right] =$$

$$= i\sigma \mathrm{e}^{-i\Delta_k t}\mathrm{d}t \left[-\frac{ikn}{\pi(z-t)}\right] \times$$

$$\times \left[\int\limits_{-\infty}^{+\infty}\int A_1^2(x,y,t)\, \mathrm{e}^{ikn\frac{(x-x_0+\rho z-\rho t)^2}{(z-t)}}\, \mathrm{e}^{ikn\frac{(y-y_0)^2}{(z-t)}}\, \mathrm{d}x\mathrm{d}y\right]. \tag{30}$$

Now we can find the resultant SH field in this plane, which is obviously a sum of all elementary contributions of the form (30), formed at the distance from $t = 0$ to $t = z$. For this purpose, integrating Eq. (30) over $t$ within these limits, we obtain the exact equation (14).

The fact that our, speculative to a sufficient degree, reasoning has led to the result (14), following from the rigorous solution of the system of nonlinear wave equations, allows us to imagine the following physical pattern of the SHG process. Under the effect of laser radiation, a wave of the nonlinear (quadratic in our case) polarization is induced in the crystal. As this takes place, any medium layer $\mathrm{d}z$ appears to be a source of an elementary SH wave, whose amplitude is determined by Eq. (29). Once induced, these elementary waves then propagate along the positive direction of the $Z$ axis interacting neither with the field at the fundamental frequency nor with each other. The interaction in this case is understood as any change of the shape or the amplitude of the elementary SH wave not connected with the linear mechanisms – diffraction blooming, energy walk off due to birefringence, absorption, etc. In the arbitrarily chosen observation plane $z = \text{const}$ (inside or behind the crystal), all the elementary contributions are summed up (interfere with each other), what leads to the formation of the resultant field $A_2(x, y, z)$.

The authors of Ref. 6 reasoned roughly in this way and in this connection the following should be noted. It is clear that the approach to the solution of the SHG problem, which is based on the scheme (28)–(30) should cause, from the very beginning, quite justified doubts, if it is not confirmed, as in our case, by the rigorous theoretical results. In addition, in Ref. 6 the process of the SH wave propagation was considered not rigorously, but with some quite intuitive suppositions. Finally, in that paper it was assumed that the crystal is surrounded not the by air, but by some fictitious isotropic medium with the refractive index $n$, satisfying the condition (11), which allowed the refraction effects to be excluded from the consideration. Based on this, the authors of Ref. 6 have called their method "heuristic," which certainly subtracts nothing from their merits.

However, in our opinion, it is incorrect to call this approach "quasigeometric," as it was called in Ref. 4, only because it is based on Eq. (29). The validity of using Eq. (29) was already discussed above. Correspondingly, the statement that the quasigeometric Boyd–Kleinman method will coincide with the exact solution only at a long distance from a crystal[4] is not fully correct. This is not the case, because, as was shown, the result obtained by integration of Eq. (30) always coincides with the exact solution (14). It is quite different thing that in Ref. 6 the observation plane actually was remote, but only in order to reduce the integral over the longitudinal coordinate to a simpler form. Certainly, in view of this fact, the result cannot be used to describe SH at small distances from the exit facet of a crystal.

Similar possibility of simplifying the result is used in this paper as well. It is exactly because of the use of a remote observation plane, that we managed to obtain simpler equations (20)–(21) instead of Eq. (18) and used the representation (8) to set the boundary condition. In general, if we compare the solution (14), whose particular case (Gaussian beam) is the Boyd–Kleinman formula, with Eq. (18), the main result of this paper, then we can see that they are, in fact, different representations of the same rigorous solution of the problem. The only difference is that Eq. (18) is somewhat simpler and rather accurately takes into account the influence of the effects associated with the refraction of both beams at the entrance and exit facets of the crystal on the SHG. It is clear that the correctness of the above-said [that is, equivalence of the solutions (14) and (18)] should be confirmed by specific calculations within the framework of a case study by properly choosing test problems. However, these aspects call for separate investigation and are not discussed here.

## References

1. A.N. Soldatov and V.I. Solomonov, *Self-Terminating Metal-Vapor Gas-Discharge Lasers* (Nauka, Novosibirsk, 1985), 151 pp.
2. G.S. Evtushenko and V.O. Troitskii, J. Rus. Laser Res. **15**, No. 1, 28–33 (1994).
3. O.M. Prakash, S.K. Dixit, and R. Bhatnagar, IEEE J. Quant. Electron. **38**, No. 6, 603–613 (2002).
4. V.G. Dmitriev and L.V. Tarasov, *Applied Nonlinear Optics* (Fizmatlit, Moscow, 2004), 512 pp.
5. M.B. Vinogradova, O.V. Rudenko, and A.P. Sukhorukov, *Wave Theory* (Nauka, Moscow, 1990), 432 pp.
6. G.D. Boyd and D.A. Kleinman, J. Appl. Phys. **39**, No. 8, 3597–3639 (1968).
7. V.V. Kolosov and V.O. Troitskii, Atmos. Oceanic Opt. **18**, No. 9, 674–679 (2005).
8. M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, London–New York, 1964).
9. S.D. Tvorogov and V.O. Troitskii, Atm. Opt. **3**, No. 3, 235–241 (1990).
10. S.D. Tvorogov and V.O. Troitskii, Atmos. Oceanic Opt. **18**, No. 9, 664–673 (2005).