REPRESENTATION OF KARHUNEN-LOEVE-OBUKHOV FUNCTIONS IN THE WALSH AND HAAR BASES

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We have derived analytical relationships that enable one to make a transition from the Haar and Walsh bases in the expansions of the optical phase to a statistically optimal Karhunen−Loeve−Obukhov basis. The former bases are most convenient for use in compensating devices of the adaptive optics systems while the latter one is most correct and comprehensive when formulating criteria of an adaptive optics system closeness to an ideal diffraction-limited system. The relationships derived, if used in parallel with the algorithms of fast Walsh and Haar transforms, enable one to approach real time operation mode when reconstructing the wave fronts. Besides, the transform matrix obtained enables one to allow for the effect of the turbulence outer scale.

For making a wave front modal compensation in adaptive optics systems (AOS) a wave phase is represented as a series expansion over a given system of basis functions.[1-3] Basic criterion for choosing wave front forming devices in the given basis is simplicity of the compensating device.[2,4] In this case it is preferable that the wave front correction is performed in the basis of discrete Walsh functions and Haar wavelets which are characterized by simplicity of engineering realization and belong to a type of fast transforms. These functions take only two values $+ 1$ and $- 1$. As a result it is possible to avoid multiplication when algorithms are realized in a computer (see Fig. 1).

In this connection the execution time of the discrete Walsh transform is less than this time for the fast Fourier transform by a factor of 10 for the same data array.[5] The Haar wavelet transform is faster than the Walsh transform. Wavelets are a local basis which is stable relative to noise in the initial data, i.e. it improves the signal-to-noise ratio and allows local singularities such as a pulse, step, and power peculiarity to be extracted.[6–8] Such properties of the basis allow one to approach the real time operation mode, to reduce the time of estimation the phase expansion coefficients limited by the speed of adaptive circuit response, and to numerically simulate distortions of the wave front having a complex topology.

To estimate the potential efficiency of modal correctors the statistical indices characterizing the degree of AOS closeness to an ideal diffraction-limited system are used. These indices are the Strehl ratio, distribution of the expansion coefficients variance over the aperture, structure function of adaptive optics systems aberrations, etc.[9,10] Estimation of these important values is

essentially simplified, if the phase $S(\rho)$ is represented as a universal expansion over the Karhunen-Loeve-Obukhov (KLO) functions which agree with the turbulent medium of wave propagation

$$S(\rho) = \sum_{k=0}^{N} c_k \, \psi_k(\rho). \tag{1}$$

In accordance with the Karhunen-Loeve-Obukhov theorem the minimum in the norm of error of representing a random function within an aperture with the pupil function $W(\rho)$ is achieved when using N eigenfunctions corresponding to N largest eigenvalues of the integral operator whose kernel is the phase correlation function.[11,12] The task of seeking such eigenfunctions ψ_k reduces to solution of the Fredholm integral equation of the second kind

$$\psi_k(\rho) \, \lambda_k = \int W(\rho) \, B_s(\rho, \rho') \, \psi_k(\rho') \, \mathrm{d}^2\rho' , \tag{2}$$

where $B_s(\rho, \rho') = \langle S(\rho), S(\rho') \rangle$ is the phase correlation function, $\psi_k(\rho)$, λ_k are the eigenfunctions and eigenvalues of the integral equation (2),

$$W(\rho) = \begin{cases} 1, & |\rho| \leq R, \\ 0, & |\rho| > R \end{cases}$$

is the pupil function.

Note, that the expansion (1) is the most informative. Coefficients of the series expansion (1) do

not correlate and that simplifies using of the expansion results and their analysis. The variance of the expansion coefficients in this series is minimal as compared to any other expansion. However, practical implementation of this expansion in

an adaptive corrector of an AOS is rather difficult. Therefore the problem of seeking a relation between the optimal Karhunen-Loeve-Obukhov expansion and the Walsh or Haar functions is expedient.
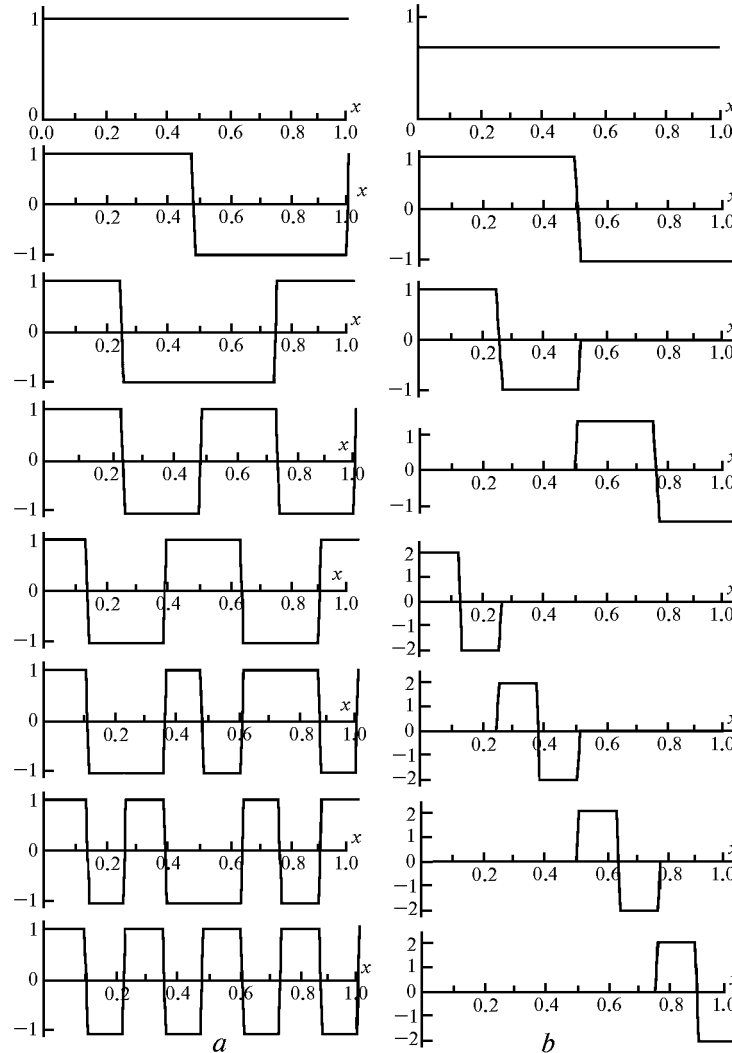

FIG. 1. *View of the first eight Walsh and Haar functions of a single coordinate: a − Walsh functions Wal(x), b − Haar functions H(x).*

Let us find the transition matrix relating the coefficients of the KLO basis expansion with the coefficients of the Walsh or Haar bases. The problem of deriving a relation between the KLO expansion and Walsh and Haar functions is identical to the problem of expanding the kernel of the integral equation (2) determining the KLO functions in the Walsh and Haar bases. The coefficients of the kernel expansion make up

the Gram matrix, its eigenvectors being the transition matrix sought and the eigenvalues are the variance of expansion coefficients.

To determine the transition matrix for a round aperture we represent the Walsh functions $Wal(\rho)$ as a product of Walsh functions of each coordinate

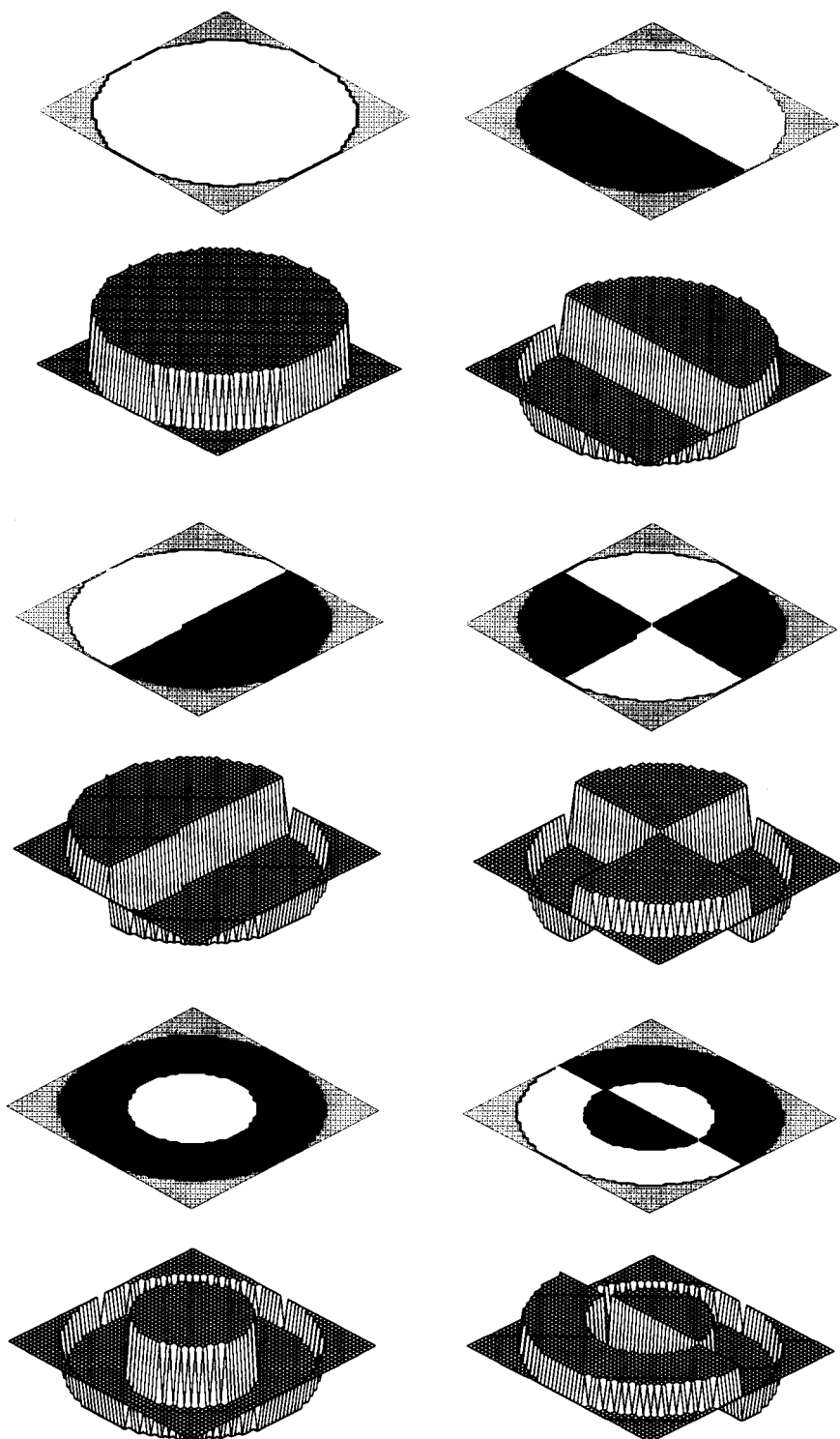$$Wal_{nm}(\rho) = Wal_n(\rho)\ Wal_m(\theta),$$

FIG. 2. *Spatial form of the Walsh functions* $Wal_{nm}(\rho)$, $n = 0,1$, $m = \overline{0,3}$.

where $\rho = \{x, y\} = (\rho, \theta)$. The spatial view of these functions is presented in Fig. 2. The Haar functions $H(\rho)$ can be represented in the same way

$$H_{nm}(\rho) = H_n(\rho) \, H_m(\theta).$$

Normally the mean wave phase is inessential in the majority of adaptive optics problems, therefore we omit the expansion term characterizing the phase averaged over the aperture.[1-3] Taking this into account the wave front distortion within the aperture

may be presented in the form

$$S(\rho) = \varphi(\rho) - \varphi_{av} , \qquad (3)$$

where

$$\varphi_{av} = \int W(\rho)\,\varphi(\rho)\,d^2\rho \qquad (4)$$

is the phase averaged over the aperture.

By taking the relation (4) into account the correlation function

$$B_s(\rho, \rho') = <S(\rho)\,S(\rho')> = <(\varphi(\rho) - \varphi_{av})\,(\varphi(\rho') - \varphi_{av})>$$

can be presented as the phase structure function $D(\rho - \rho')$, Ref. 9

$$B_s(|\rho - \rho'|) = -\frac{1}{2}\,D(\rho - \rho') +$$

$$+ \frac{1}{2}\int W(\rho)\,D(\rho - \rho')\,d^2\rho +$$

$$+ \frac{1}{2}\int W(\rho')\,D(\rho - \rho')\,d^2\rho' -$$

$$- \frac{1}{2}\int d^2\rho \int d^2\rho'\,W(\rho)\,W(\rho')\,D(\rho - \rho') . \qquad (5)$$

We will find a solution of the equation (2) in a factorized form

$$\psi(\rho) = R(\rho)\,\Theta(\theta) . \qquad (6)$$

First of all we determine a form of the azimuthal function $\Theta(\theta)$. Note, that the function $\Theta(\theta)$ must be continuous and periodical function of the angle θ with the period 2π. By substituting the solution (6) into equation (2) we obtain

$$\int_0^{2\pi}\int_0^R W(\rho)\,B_s(\rho, \theta, \rho', \theta')\,R(\rho)\,\Theta(\theta)\,\rho\,d\rho\,d\theta =$$

$$= \lambda\,R(\rho)\,\Theta(\theta) . \qquad (7)$$

Let us change the variables $\zeta = \theta' - \theta$, $d\zeta = d\theta$, $\theta' = \zeta + \theta$

$$\int_0^{2\pi}\int_0^R W(\rho)\,B_s(\rho, \rho', \zeta)\,R(\rho')\,\Theta(\zeta + \theta)\,\rho\,d\rho\,d\zeta =$$

$$= \lambda R(\rho)\,\Theta(\theta) . \qquad (8)$$

From equation (8) it follows that $\Theta(\zeta + \theta) = \Theta(\zeta)\,\Theta(\theta)$ due to continuity, periodicity, and uniqueness of the solution. The general solution of this equation with the period 2π is well known and has the form of $\exp(il\theta)$. Hence, the function $\Theta(\theta)$ is

$$\Theta(\theta) = \exp(il\theta) , \qquad (9)$$

where $l \in \mathbf{Z}$, i.e. $l = 0, \pm 1, \pm 2, \pm 3,...$.

By expanding $\Theta(\theta)$ into a series over the functions $Wal(\theta)$ and $H(\theta)$ or, what is the same, applying the fast Walsh and Haar transformations we determine the azimuthal transition matrix.

To determine the radial transition matrix we have to substitute the expression (9) into (8) and after certain operations obtain the homogeneous integral Fredholm equation of the second kind

$$\int_0^R \rho\,d\rho W(\rho)\,R(\rho) \int_0^{2\pi} d\zeta\,\exp(il\zeta)\,B_s(\rho, \rho', \zeta) =$$

$$= \lambda R(\rho) . \qquad (10)$$

The task of seeking a relation between the basis $R(\rho)$ with the Walsh and Haar bases is equivalent to a problem of expanding the kernel of the equation (10) into a series over the Walsh and Haar functions. In this case it is necessary to determine an explicit form of the kernel in equation (10). Let us introduce the designation

$$M_l(\rho, \rho') = \int_0^{2\pi} d\zeta\,\exp(il\zeta)\,B_s(\rho, \rho', \zeta) . \qquad (11)$$

By substituting the explicit form of $B_s(\rho, \rho', \zeta)$ we have (11)

$$M_l(\rho, \rho') = -\frac{1}{2}\int_0^{2\pi} d\zeta\,\exp(il\zeta)\,D(\rho - \rho') +$$

$$+ \frac{1}{2\pi}\int d^2\rho \int_0^{2\pi} d\zeta\,\exp(il\zeta)\,W(\rho)\,D(\rho - \rho') +$$

$$+ \frac{1}{2\pi}\int d^2\rho' \int_0^{2\pi} d\zeta\,\exp(il\zeta)\,W(\rho')\,D(\rho - \rho') -$$

$$- \frac{1}{4\pi^2}\int_0^{2\pi} d\zeta\,\exp(il\zeta)\int d^2\rho\int d^2\rho'\,W(\rho)\,W(\rho')D(\rho - \rho') .$$

The view of the kernel for $l = \pm 1, \pm 2, \pm 3,...$, i.e. for $l \neq 0$, is simplified because three last terms turn out to be zero due to the orthogonality of the functions $\exp(il\theta)$. Hence, the kernel takes the form

$$M_l(\rho, \rho') = -\frac{1}{2}\int_0^{2\pi} d\zeta\,\exp(il\zeta)\,D(\rho - \rho') .$$

For Kolmogorov spectrum of the turbulence[1,2] the structure function has the form

$$D(\rho) = \frac{6.88}{r_0^{5/3}} \rho^{5/3} ,$$

where r_0 is the Fried radius. Let us now apply the Fourier-Bessel transform H_0 to the function $D(\rho)$

$$D(\rho) = \int\limits_0^\infty D(\varkappa) J_0(\varkappa\rho)\varkappa \, d\varkappa, \qquad D(\varkappa) = \frac{6.88}{r_0^{5/3}} \frac{1}{\varkappa^{11/3}} ,$$

where $J_0(x)$ is the zeroth order Bessel function of the first kind. Taking into account the shift theorem, convolution theorem, and the fact that the Fourier-Bessel transform for the pupil function has the form

$$W(\rho) \xrightarrow{H_0} \frac{J_1(\varkappa\rho)}{\varkappa} R ,$$

for $l \neq 0$ we obtain

$$M_l(\rho, \rho') = - \frac{6.88\pi}{r_0^{5/3}} \int\limits_0^\infty \frac{J_l(\varkappa\rho')J_l(\varkappa\rho)\varkappa \, d\varkappa}{\varkappa^{11/3}} .$$

For $l = 0$ the form of the kernel becomes more complex and for the Kolmogorov model of turbulence we obtain

$$M_0(\rho, \rho') = - \frac{6.88\pi}{r_0^{5/3}} \int\limits_0^\infty \frac{J_0(\varkappa\rho')J_0(\varkappa\rho)\varkappa \, d\varkappa}{\varkappa^{11/3}} +$$

$$+ \frac{6.88}{r_0^{5/3}} \left[\frac{1}{R} \int\limits_0^\infty \frac{J_0(\varkappa\rho)J_1(\varkappa R) \, d\varkappa}{\varkappa^{11/3}} + \frac{1}{R} \int\limits_0^\infty \frac{J_0(\varkappa\rho')J_1(\varkappa R) \, d\varkappa}{\varkappa^{11/3}} \right] -$$

$$- \frac{6.88\pi}{r_0^{5/3}} \int\limits_0^\infty \frac{J_1^2(\varkappa R) \, d\varkappa}{\varkappa^{8/3}} .$$

An advantage of the kernel representation in this form is the possibility to introduce the outer scale of the atmospheric turbulence. In this case it is sufficient to substitute $(\varkappa^2 + 1/L_0^2)^{8/6}$ instead of $\varkappa^{8/3}$ in the expressions under the integral sign, where L_0 is the outer scale of turbulence.

Let us expand $M_l(\rho, \rho')$ into a series over the Walsh functions $Wal_n(\rho)$

$$\rho \, M_l(\rho, \rho') = \sum\limits_s^N \sum\limits_p^N A_{ps}^l \, Wal_p(\rho) \, Wal_s(\rho') .$$

Note, that the Walsh functions refer to the class of multiplicative systems and form the Abelian group where the addition operation $\oplus$ is determined as the summation to module 2 performed as a bit-by-bit summation without carrying the unit into the most significant bit $Wal_n(\rho) \, Wal_m(\rho) = Wal_{n \oplus m}(\rho)$, Refs. 5, 13. Using this property the expansion coefficients can be obtained

$$\frac{1}{N^2} \int\limits_0^1 \int\limits_0^1 \rho \, M_l(\rho, \rho') \, Wal_p(\rho) \, Wal_s(\rho') \, d\rho \, d\rho' = A_{ps}^l .$$

(12)

Thus obtained matrix is the Gram matrix. By diagonalizing this Gram matrix we can obtain the expansion coefficients of the radial parts of the KLO functions in terms of Walsh functions. Performing the operations analogous to those described above we can determine coefficients of $R(\rho)$ expansion over the Haar wavelets. The expansion coefficients of the radial function $R(\rho)$ in the Haar basis can be determined in a different way by using a closed connection between the Walsh and Haar functions[13,14] expressing in the transition matrix $\mathbf{M}_{2^n}^{\text{H-W}}$

$$\mathbf{M}_{2^n}^{\text{H-W}} =$$

$$= \prod\limits_{r=1}^{n-1} \prod\limits_{p=0}^{r-1} \left(\mathbf{I}_{2^{n-1-r}} \otimes \left[\mathbf{I}_{2^r} \oplus \left[\begin{array}{c} \mathbf{I}_{2^p} \otimes \mathbf{G}_2^2 \\ \hline \mathbf{I}_{2^p} \otimes \mathbf{G}_2^3 \end{array} \right] \oplus \mathbf{I}_{2^r - 2^{p+1}} \right] \right) ,$$

where $\mathbf{I}_t$ is the unit matrix of the dimension $t \times t$; $\mathbf{G}_2^2 = [1\ 1]$, $\mathbf{G}_2^3 = [1{-}1]$; $\otimes$ is the sign of the Kronecher product, $\oplus$ is the sign of the Kronecher sum. In this case it is sufficient to multiply the matrices (12) by $\mathbf{M}_{2^n}^{\text{H-W}}$ and obtain the transition matrix for $R(\rho)$ and $H(\rho)$, sought.

Thus, in this paper we have analytically analyzed a possibility of performing a transition from the Haar and Walsh bases which allows a comprehensive and correct presentation of the potentialities of using the correcting AOS basis. The relations obtained allow one to estimate the expansion coefficients and approach the real temporal resolution of random variations of turbulence owing to the use of algorithms of fast Walsh and Haar transforms. Moreover, the Haar basis allows the distortions of the wave front with a complex topology to be simulated owing to its local properties.

In the second part of the paper we plan to discuss a numerical realization of the theoretical approach described above and present some results of calculating transition matrices.

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