Differential and statistical invariants of a wavy surface. Part 2. Properties of a Gaussian surface

A.N. Dubovik

P.P. Shirshov Institute of Oceanology, Russian Academy of Sciences, Moscow

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It is shown that the spectrum of a Gaussian surface has only the second-order rotational axis, and its moments of higher than the second order are degenerated in such a way that only three of them are independent and only two invariants are nonzero. The conditions for decomposition of the spectrum into one-dimensional systems are revealed, and the joint statistical distribution of the mean and differential curvatures at horizontal surface points is found. Within the framework of the Gaussian model, a simple optical method is suggested for simultaneous remote measurement of the second- and fourth-order invariants of spectral moments.

Introduction

In Ref. 1, we have considered the general theory of invariants of a wavy surface, which is interesting from the viewpoint of surface development. At the same time, it forms the theoretical prerequisites needed for rigorous justification and development of remote optical methods for sea investigations. This paper is a continuation of Ref. 1. Here we consider the statistical properties of the Gaussian model of a surface, which presents the greatest practical interest in the theory of sea roughness.

As is well-known, in a wide range of conditions, the wavy sea surface can be thought roughly Gaussian providing the statistical distributions of its derivatives are described by the normal law. Significant deviations from the Gaussian statistics are observed only in the near-shore zone (in shallow water) and under the conditions of developing roughness in deep water, when the surface is significantly inhomogeneous and a wave collapse occurs. For the Gaussian surface, the requirement of the external symmetry is connected with invariance of the spur and determinant of correlation matrices. It will be shown that this causes specific degeneration of invariants (even moments) and imposes rigid requirements on the symmetry of the spectrum. In this paper, we consider also the characteristics of the angular structure of the two-dimensional spectrum at its decomposition into some simple (one-dimensional) systems and derive a joint statistical distribution of the mean and differential curvatures at horizontal surface points.

To apply the theoretical results to practical investigation of the wavy surface, the paper suggests a simple optical method for simultaneous remote measurement of invariants of the second- and fourth-order moments of the spectrum.

1. Degeneration of invariants and spectrum properties

For the Gaussian surface, the even moments of the $p + q = 2n$ order form a symmetric correlation matrix[2]

$$M_{2n} = \begin{pmatrix} m_{2n,0} & m_{2n-1,1} & \cdots & m_{n,n} \\ m_{2n-1,1} & m_{2n-2,2} & \cdots & m_{n-1,n+1} \\ \vdots & \vdots & & \vdots \\ m_{n,n} & m_{n-1,n+1} & & m_{0,2n} \end{pmatrix} \quad (1)$$

in the statistical distribution of derivatives of the nth order. However, for any matrix, its spur and determinant

$$\mu_{2n} = \mathrm{Sp}\, M_{2n} = \sum_{r=1}^{n+1} (M_{2n})_{rr'} \quad \Delta_{2n} = \det M_{2n} \quad (2)$$

are rotation invariants.[3] The presence of additional determinants expressing the requirement of the external symmetry indicates that degeneration (linear dependence) of moments is possible. In such a case, the components $H_{2n}^{(r)}$ and $h_{2n}^{(r)}$ of some invariants $V_{2n}^{(r)}$ vanish (hereinafter U_{2n} and $V_{2n}^{(r)}$ denote invariants, as accepted in Ref. 1, where the equations for them are given up to the sixth order).

Since Δ_{2n} is a nonlinear function of moments, degeneration is determined by only the matrix spur. For the second-order moments we have $\mu_2 \equiv U_2$. Consequently, in this case all three moments are independent, and the direction moment is described by the well-known equation[2]:

$$2m_2(\varphi) = \mu_2 + (\mu_2^2 - 4\Delta_2)^{1/2} \cos\, [2\varphi - \varphi_2^{(1)}],$$

$$\varphi_2^{(1)} = \arctan\, [2m_{11}/(m_{20} - m_{02})]. \quad (3)$$

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However, for moments of higher orders $\mu_{2n} \neq U_{2n}$. From comparison of U_4 and μ_4, it follows that $(m_{40} + m_{04})$ and m_{22} are invariants. Examining transformation of the moments at rotation through the angle φ, we can show that

$$(m'_{40} + m'_{04}) = (m_{40} + m_{04}) + h_4^{(2)} \sin 4\varphi + + (1/2) H_4^{(2)} \sin^2 2\varphi, \qquad (4)$$

where the prime stands for parameter values in the turned coordinate system. It follows from here and from equations for the components $H_4^{(2)}$ and $h_4^{(2)}$ (Ref. 1) that

$$H_4^{(2)} = h_4^{(2)} = 0; \quad m_{13} = m_{31}, \quad m_{40} + m_{04} = 6m_{22}, \qquad (5)$$

that is, only three of five moments are independent. Taking m_{40}, m_{04}, and m_{31} as independent moments, we can replace U_4 and $V_4^{(1)}$ with simpler independent invariants

$$m_{22} = (m_{40} + m_{04})/6, \quad \delta = m_{40}\, m_{04} - 4m_{31}^2, \qquad (6)$$

which allow any others connected with the fourth-order moments to be written. In particular, using the results of Refs. 1 and 4, we find

$$U_4 = 8m_{22}, \quad \mu_4 = 7m_{22},$$

$$V_4^{(1)} = 2\,(9m_{22}^2 - \delta), \quad \Delta_4 = m_{22}\,(\delta - m_{22}^2),$$

$$\langle u_0^2 \rangle = \langle v_0^2 \rangle = P_0 = 2m_{22}, \quad \langle u_0^4 \rangle = 3P_0^2 = 12\, m_{22}^2,$$

$$\langle K_0^2 \rangle = S = \delta + 3m_{22}^2,$$

$$\langle u_0^4 \rangle - \langle v_0^4 \rangle = 4Q_0 = 4m_{22}^2 = P_0^2, \qquad (7)$$

where P_0, Q_0, and S are the designations of invariants accepted in Refs. 9 and 10 at analysis of the statistics of radiation reflected from the surface; u_0, v_0, and K_0 are the mean, differential, and total (Gaussian) curvatures at horizontal points. With allowance for Eq. (6), the fourth moment of the one-dimensional spectrum can be represented as

$$m_4(\varphi) = 3m_{22} + (9m_{22}^2 - \delta)^{1/2} \cos\,[2\varphi - \varphi_4^{(1)}],$$

$$\varphi_4^{(1)} = \arctan\,[4m_{31}/(m_{40} - m_{04})]. \qquad (8)$$

Degeneration of higher-order moments can be easily found using the condition $H_4^{(2)} = h_4^{(2)} = 0$ and the recurrence formulas for derivatives of the correlation function (see Eqs. (31) in Ref. 1). As a result, for moments of the sixth order we have

$$H_6^{(r)} = h_6^{(r)} = 0 \quad (r = 2, 3); \quad m_{15} = m_{51} = (5/3)m_{33},$$

$$15m_{42} = 2m_{60} + m_{06},\ 15m_{24} = m_{60} + 2m_{06}, \qquad (9)$$

that is, here, as above, only three moments are independent (m_{60}, m_{06}, m_{51}). In this case, $(m_{60} + m_{06})$ and $(m_{42} + m_{24})$ are invariants, and the direction moment is determined by the equation

$$2m_6(\varphi) = (m_{60} + m_{06}) + + [(m_{60} - m_{06})^2 + 36m_{51}^2]^{1/2} \cos\,[2\varphi - \varphi_6^{(1)}],$$

$$\varphi_6^{(1)} = \arctan\,[6m_{51}/(m_{60} - m_{06})]. \qquad (10)$$

Then, continuing this consideration, we can make certain that in the general case

$$H_{2n}^{(r)} = 0, \quad h_{2n}^{(r)} = 0 \quad (2 \le r \le n). \qquad (11)$$

Thus, we have shown that moments of the higher than the second order are degenerated in such a way that only three of them are linearly independent and only two invariants U_{2n} and $V_{2n}^{(1)}$ are nonzero. In this case the degeneration (linear relation between the moments) is described by Eqs. (11). The similar conclusions are apparently valid for even derivatives of the correlation function as well.

Degeneration of the moments significantly simplifies their angular dependence, which at $V_{2n}^{(1)} \neq 0$ is determined only by the functions cos2φ and sin2φ. This means that the spectrum of the Gaussian surface has only the second-order rotation axis, that is, it cannot have incidental peaks, besides peaks in the given direction and the direction opposite to it. As is well-known, the developed spectrum of wind-induced waves has the same symmetry, and this is one of the arguments in favor of the Gaussian model of a wavy surface. Equation (11) determines the following properties of the spectrum:

1) moments with transposed odd indices p and q are identical:

$$m_{qp} = m_{pq} \quad (p \text{ and } q \text{ are odd}); \qquad (12)$$

2) moments $m_{2s,\,2s}$ and sums of moments with transposed even indices are invariants, and the following relations are fulfilled:

$$m_{2n-2k,\,2k} + m_{2k,\,2n-2k} = [C_n^k / C_{2n}^{2k}]\,(m_{2n,\,0} + m_{0,\,2n}) =$$

$$= 2^{k+1-2n}\, C_{2n-4k}^{n-2k}\, U_{2n}, \quad 0 \le k \le E(n/2), \qquad (13)$$

where $E(z)$ is the integer part of z. With allowance for these properties, the spectral moments with respect to the direction φ can be represented as

$$m_{2n}(\varphi) = (1/2)\, p_{2n}\,[1 + \varepsilon_n \cos\,(2\varphi - \varphi_{2n}^{(1)})],$$

$$\varepsilon_n = q_{2n}/p_{2n},$$

$$\varphi_{2n}^{(1)} = \arctan\,[(2n)\, m_{2n-1,\,1}/(m_{2n,\,0} - m_{0,\,2n})],$$

$$p_{2n} = m_{2n,\,0} + m_{0,\,2n},$$

$$q_{2n}^2 = (m_{2n,\,0} - m_{0,\,2n})^2 + 4n^2\, m_{2n-1,\,1}^2, \qquad (14)$$

where the invariants p_{2n} and q_{2n} can be expressed in terms of U_{2n} and $V_{2n}^{(1)}$. The invariants ε_n introduced here represent the anisotropy coefficients of the nth order and play an important part in the surface statistics. In particular, they characterize the angular orientation (sharpness) of the spectrum and vanish in the isotropic case.

If the spectrum is symmetric about the direction φ_0 (principal wave direction), then the conditions $\varphi_{2n}^{(1)} = 2\varphi_0$ are fulfilled, and we have the extra relations

$$\sin 2\varphi_0 = h_2^{(1)}/V_2^{(1)} = h_4^{(1)}/V_4^{(1)} = \ldots = h_{2n}^{(1)}/V_{2n}^{(1)}, \quad (15)$$

from which it follows that all the relations $h_{2r}^{(1)}/h_{2s}^{(1)}$, $H_{2r}^{(1)}/H_{2s}^{(1)}$ are invariants as well (r, s = 1, 2, 3,...).

As is well-known, the condition $\Delta_{2n} = 0$ corresponds to decomposition (degeneration) of the spectrum into n one-dimensional spectra.[2] This means that for the non-degenerated (two-dimensional) spectrum the following condition should be fulfilled: $0 \le \varepsilon_n < (\varepsilon_n)_{\max}$, where $(\varepsilon_n)_{\max} \le 1$ is the solution of the equation $\Delta_{2n} = 0$. Consider this issue in more detail and find some $(\varepsilon_n)_{\max}$. Rearranging columns and rows in the correlation matrix M_{2n}, we can represent it in the form[3]:

$$M_{2n} = \begin{pmatrix} A_{2n} & \vdots & B_{2n} \\ \cdots\cdots & \vdots & \cdots\cdots \\ B'_{2n} & \vdots & C_{2n} \end{pmatrix}, \quad (16)$$

where, with allowance for moment degeneration, the square submatrices A_{2n} and C_{2n} include only the moments $m_{2n,\,0}$ and $m_{0,\,2n}$, while the submatrices B_{2n} and B'_{2n} (not necessarily square) include only the moments $m_{2n-1,\,1}$. In the direction $\varphi = \varphi_{2n}^{(1)}/2$ (or φ_0 for the symmetric spectrum) we have

$$m_{2n-1,\,1} = 0, \quad m_{2n,\,0} = (m_{2n})_{\max} \equiv (1 + \varepsilon_n)\, p_{2n}/2,$$
$$m_{0,\,2n} = (m_{2n})_{\min} \equiv (1 - \varepsilon_n)\, p_{2n}/2, \quad (17)$$

where $(m_{2n})_{\max}$ and $(m_{2n})_{\min}$ are the moments in the direction specified above and in the direction opposite to it. With allowance for this, the condition of spectrum degeneration can be written as

$$\Delta_{2n} = (\det A_{2n})\,(\det C_{2n}) = 0. \quad (18)$$

Restricting our consideration to $n \le 4$, let us find determinants of the submatrices:

$$\det A_2,\ \det C_2 = (p_2/2)\,(1 \pm \varepsilon_1),$$
$$\det A_4 = (p_4/6)^2\,(8 - 9\varepsilon_2^2),\ \det C_4 = p_4/6,$$
$$\det A_6,\ \det C_6 = (p_6/15)^2\,(4\varepsilon_3^2 \pm 6\varepsilon_3 - 9),$$
$$\det A_8 = (2p_8/35)^3\,(64 - 75\varepsilon_4^2),$$
$$\det C_8 = (p_8/14)^2\,(4 - \varepsilon_4^2). \quad (19)$$

Herefrom we have that spectrum decomposition corresponds to the following values of the anisotropy coefficients:

$$(\varepsilon_1)_{\max} = 1;\ (\varepsilon_2)_{\max} = \sqrt{8}/3 = 0.9428,$$
$$(\varepsilon_3)_{\max} = 3/(1 + \sqrt{5}) = 0.9271,$$
$$(\varepsilon_4)_{\max} = 8/\sqrt{75} = 0.9238. \quad (20)$$

It should be noted that $(\varepsilon_2)_{\max}$ is equal to $\cos(\vartheta/2)$, where $\vartheta = 38°56'$ is the angle of divergence of ship waves.[5] In our opinion, such coincidence of two surface characteristics is not accidental, but discussion of this issue is beyond the scope of this paper.

2. Statistics of surface curvature

Now let us find the joint density of the distribution $W_2(u_0, v_0)$ of the mean and differential curvature at horizontal surface points (this function for the isotropic case was obtained in Ref. 4). As usually, we start from the distribution of the second derivatives[2,4]:

$$W_3(\zeta_{20}, \zeta_{11}, \zeta_{02}) = \frac{1}{(2\pi)^{3/2}\,\Delta_4^{1/2}} \times$$
$$\times \exp\left[-\frac{1}{2\Delta_4}\sum_{p,}^{3}\sum_{q=0}^{3} B_{pq}\,\xi_p\,\xi_q\right], \quad (21)$$

where $(\xi_1, \xi_2, \xi_3) \equiv (\zeta_{20}, \zeta_{11}, \zeta_{02})$ and B_{pq} is the algebraic cofactor of the matrix element $(M_4)_{pq}$ in the determinant Δ_4. In this case we have[1]

$$\zeta_{20,\,02} = u_0 \pm v_0 \cos(2\varphi - \varphi_2),$$
$$\zeta_{11} = v_0 \sin(2\varphi - \varphi_2), \quad (22)$$

where $\varphi_2 = \arctan[(2\zeta_{11}/(\zeta_{20} - \zeta_{02})]$. Assume that the axis x coincides with the direction $\varphi_4^{(1)}/2$. Then substituting Eqs. (22) into Eq. (21) and taking into account the fact that the Jacobian of transformation of variables is equal to $4\,|\,v_0\,|$, we obtain

$$W_3(u_0, v_0, t) = \frac{|\,v_0\,|}{4\pi\,(\pi m_{22}^3\,\rho)^{1/2}} \times$$
$$\times \exp\left[-\frac{(2u_0 - 3\varepsilon_2\, v_0 \cos t)^2}{16\rho m_{22}} - \frac{v_0^2}{2m_{22}}\right], \quad (23)$$

where the following designations are introduced: $\rho = 1 - (9/8)\,\varepsilon_2^2 \equiv 1 - [\varepsilon_2^2/(\varepsilon_2)_{\max}^2]$ and $t = 2\varphi - \varphi_2$ $(0 \le t \le \pi)$. It can be shown that

$$\int_0^\pi \exp[-(\alpha - \beta\cos t)^2]\,dt = \exp[-\alpha^2 - (\beta^2/2)] \times$$
$$\times \int_0^\pi \exp[(\beta^2/2)\cos 2t]\cosh(2\alpha\beta\sin t)\,dt. \quad (24)$$

The second integral can be calculated using the well-known equations[3]:

$$\cosh(z_1 \sin t) = I_0(z_1) + 2\sum_{n=1}^{\infty}(-1)^n I_{2n}(z_1)\cos 2nt,\ z_1 = 2\alpha\beta,$$

$$\int_0^\pi \exp(z_2\cos 2t)\cos 2nt\,dt = \pi I_n(z_2),\ z_2 = \beta^2/2, \quad (25)$$

in which $I_r(z)$ is the r-order Bessel function of an imaginary argument. Thus, taking the integral (23) with respect to t, we finally have

$$W_2(u_0, v_0) = \frac{|v_0|}{4(\pi m_{22}^3 \rho)^{1/2}} \times$$

$$\times \exp\left[-\frac{u_0^2 + (1+\rho)\, v_0^2}{4\rho m_{22}}\right] F(u_0, v_0), \qquad (26)$$

where

$$F(u_0, v_0) = I_0(z_1)\, I_0(z_2) +$$

$$+ 2 \sum_{n=1}^{\infty} (-1)^n I_{2n}(z_1)\, I_n(z_2);$$

$$z_1 = (3\varepsilon_2/4\rho m_{22})\, u_0\, v_0,\;\; z_2 = [(1-\rho)/4\rho m_{22}]\, v_0^2. \qquad (27)$$

In the isotropic case $\rho = F = 1$, therefore the distribution (26) coincides with that obtained earlier in Ref. 4. Apparently, using $W_2(u_0, v_0)$, we can find any moments of the invariants u_0, v_0, and $K_0 = u_0^2 - v_0^2$. The values of some of the moments are given in Eq. (7). At the same time, after changing variables, from Eq. (26) we can easily obtain other distributions, such as $W_2(s_0, K_0)$ or $W_2(k_{10}, k_{20})$, where $s_0 = k_{10}/k_{20}$ is curvature anisotropy, k_{10} and k_{20} are the principal curvature values at horizontal surface points.

3. Measurement of invariants

Since invariants of spectral moments depend only on external parameters, their measurement is a convenient method for studying the wave variability under the effect of wind and currents, as a result of interaction of surface and deep waves, in the presence of pollutant and surfactant films, and so on. Of particular interest in such studies are remote optical methods, which can be used for solution of a wide range of scientific and applied problems.[6–10]

It should be noted that measurements of higher invariants give the integral information about rather small variations of the spectrum in the high-frequency (capillary) region, where direct measurements are usually very difficult and unreliable. The results presented in Ref. 1 and in this paper allow us to give the exhaustive theoretical justification to already existing and new remote methods. In particular, as shown in Eq. (7), the invariants P_0, Q_0, S appearing in Refs. 9 and 10 can be rather easily expressed in terms of the invariants m_{22} and δ.

Assuming that the wavy surface is Gaussian, consider a simple optical method for simultaneous measurement of the second- and fourth-order invariants. This method is based on counting the number of mirror (reflection) surface points that are characterized by a certain slope angle θ in the preset direction φ. In practice, this method can be realized as counting the number of optical signals from reflection points when scanning the surface by a thin (1–5 mm in diameter) continuous-wave laser beam. Note that in lidar systems, which unite the source and the receiver of radiation in a single device, signals backscattered from the surface are received, and the angle of laser beam deflection from the vertical (in the scanning direction φ) coincides with the slope angle of mirror points. Below we assume that the measurements are conducted with a two-channel lidar having two independent (vertical and slant) optoelectronic systems for simultaneous emission of laser radiation and reception of reflected signals. Such a lidar design is most convenient for realization of the method suggested here.

As was shown in Ref. 2, the mean number $\eta(\varphi, \theta)$ of reflection points per unit length of the surface in the direction φ is determined by the equation

$$\eta(\varphi, \theta) = N(\varphi) \exp\,[-\tan^2\theta/2m_2(\varphi)];$$

$$N(\varphi) = \frac{1}{\pi}\,[m_4(\varphi)/m_2(\varphi)]^{1/2}, \qquad (28)$$

where $N(\varphi) \equiv \eta(\varphi, 0)$ is the mean number of horizontal points (wave peaks and dips) per unit length. Substituting, according to Eq. (14), direction moments m_2 and m_4 by their representations in terms of invariants, we obtain

$$N(\varphi) = Q\left[\frac{1+\varepsilon_2\;\cos 2\varphi}{1+\varepsilon_1\;\cos 2\varphi}\right]^{1/2},\;\; Q = \frac{1}{\pi}\,(p_4/p_2)^{1/2}, \qquad (29)$$

where the angle $\varphi = 0$ corresponds to the principal wave direction. Denote then

$$\alpha_m = \ln(N_m/\eta_m),\;\; N_m \equiv N(\varphi_m),\;\; \eta_m \equiv \eta(\varphi_m, \theta),$$

$$\varphi_m = \varphi_1 + 45°\,(m-1),\;\; 1 \le m \le 3, \qquad (30)$$

where m is the number of measurement (scanning). Consider the situation that the direction of waves is well-known (for example, at studies in a watercourse or a tray). Assuming in this case that $\varphi_1 = 0$, we have

$$p_2 = \tan^2\theta/\alpha_2 = (\alpha_1 + \alpha_3)\tan^2\theta/2\alpha_1\alpha_3,$$

$$p_4 = \pi^2 Q^2 p_2,$$

$$Q^2 = N_2^2 = (N_1^2\,\alpha_3 + N_3^2\,\alpha_1)/(\alpha_1 + \alpha_3),$$

$$\varepsilon_1 = (\alpha_3 - \alpha_1)/(\alpha_1 + \alpha_3),$$

$$\varepsilon_2 = (N_1^2\,\alpha_3 - N_3^2\,\alpha_1)/(N_1^2\,\alpha_3 + N_3^2\,\alpha_1). \qquad (31)$$

Consequently, here it is sufficient to carry out the surface scanning in the directions $\varphi_m = 0$, 45, 90° with respect to the principal wave direction.

However, in field measurements, the principal direction is usually unknown. Keeping this fact in mind, for an arbitrary angle φ_1 we have

$$p_2 = (\alpha_1 + \alpha_3)\tan^2\theta/2\alpha_1\alpha_3,\quad p_4 = \pi^2 Q^2 p_2,$$

$$Q^2 = (N_1^2\,\alpha_3 + N_3^2\,\alpha_1)/(\alpha_1 + \alpha_3),$$

$$\varepsilon_1 = A/\alpha_2\,(\alpha_1 + \alpha_3),\quad \varepsilon_2 = B/\alpha_2\,(N_1^2\,\alpha_3 + N_3^2\,\alpha_1),$$

$$\cos 2\varphi_1 = \alpha_2(\alpha_3 - \alpha_1)/A = \alpha_2(N_1^2\alpha_3 - N_3^2\alpha_1)/B, \quad (32)$$

where the coefficients A and B are determined by the equations

$$A^2 = 4\alpha_1^2\alpha_3^2 + 2\alpha_2^2(\alpha_1^2 + \alpha_3^2) - 4\alpha_1\alpha_2\alpha_3(\alpha_1 + \alpha_3),$$

$$B^2 = 4\alpha_1^2\alpha_3^2 N_2^4 + 2\alpha_2^2(\alpha_1^2 N_3^4 + \alpha_3^2 N_1^4) - - 4\alpha_1\alpha_2\alpha_3 N_2^2(\alpha_1 N_3^2 + \alpha_3 N_1^2). \quad (33)$$

Thus, in the general case, all invariants of the second and fourth orders along with the principal wave direction can be determined by scanning the surface in three directions with the interval of 45°.

The obvious advantage of the proposed method is the simplicity of the recording system, which, operating in the key mode, only counts light signals without their analog processing. To obtain the high spatial-angular resolution needed to separate signals from neighboring reflection points, the receiver's filed of view $\Delta\theta$ should be taken rather small based on the estimate $\Delta\theta \le d/H$, where d is the spatial resolution ($d \sim 1$–5 cm), and H is the lidar height above the surface. Besides, the lidar receiver should provide for reliable recording of optical signals with the intensity varying by four to five orders of magnitude. Note also that the rate of signal accumulation is proportional to the scanning rate (lidar speed), and in the slant channel of the lidar it also significantly depends on θ.

Conclusion

In this paper, it has been shown that at the Gaussian statistics of the surface the higher than second spectral moments and the higher than second derivatives of the correlation function are degenerate, only three of them are linearly independent, and only two invariants U_{2n} and $V_{2n}^{(1)}$ are nonzero.

As a consequence, the spectrum of the Gaussian surface has the second-order rotation axis and is characterized by symmetry properties of the moments given by Eqs. (12)–(14). All these peculiarities should apparently be taken into account in theoretical models or empirical approximations of the established (developed) spectrum of sea roughness.

Some limiting values of the angular structure (anisotropy coefficients) of the spectrum corresponding to its decomposition into some simple (one-dimensional) systems have been found. A joint statistical distribution of the mean and differential curvatures at horizontal surface points has been obtained.

Within the framework of the Gaussian wave model, a simple optical method that allows simultaneous measurement of the second- and fourth-order moments has been suggested. It has been shown that for measurements of the invariants it is sufficient to scan the surface in three directions with the interval of 45°.

References

1. A.N. Dubovik, Atmos. Oceanic Opt. **15**, No. 8, 645–650 (2002).
2. M.S. Longuet-Higgins, Philos. Trans. Roy. Soc. A **249**, 321–387, (1957).
3. G.A. Korn and T.M. Korn, *Mathematical Handbook for Scientists and Engineers* (McGraw-Hill, New York, 1961).
4. A.N. Dubovik, Izv. Ros. Akad. Nauk, Ser. Fiz. Atmos. Okeana **32**, No. 4, 523–527 (1996).
5. L.N. Sretenskii, *Theory of Wave Motion of Fluid* (Nauka, Moscow, 1977), 342 pp.
6. J. Wu, J. Opt. Soc. Am. **61**, No. 7, 852–858 (1971).
7. J.L. Bufton, F.E. Hoge, and R.N. Swift, Appl. Opt. **22**, No. 17, 2603–2618 (1983).
8. F.V. Bunkin, K.I. Volyak, and A.I. Malyarevskii, in: *Remote Sensing of Ocean* (GPI AN Proc.. Vol. 1) (Nauka, Moscow, 1986), pp. 3–22.
9. A.N. Dubovik, Izv. Ros. Akad. Nauk, Ser. Fiz. Atmos. Okeana **33**, No. 1, 137–144 (1997).
10. A.N. Dubovik, Izv. Ros. Akad. Nauk, Ser. Fiz. Atmos. Okeana **33**, No. 3, 389–393 (1997).
11. V.I. Ivanov and A.N. Lazarchik, Izv. Ros. Akad. Nauk, Ser. Fiz. Atmos. Okeana **28**, No. 3, 319–324 (1992).
12. R.G. Gardashov and T.G. Gardashova, Izv. Ros. Akad. Nauk, Ser. Fiz. Atmos. Okeana **37**, No. 1, 99–103 (2001).