# APPLICABILITY LIMITS OF THE METHOD OF PHYSICAL OPTICS IN THE PROBLEMS OF LIGHT SCATTERING BY LARGE CRYSTALS. II. SCATTERING BY A RECTANGULAR PLATE

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*By means of physical optics method the formulas are obtained for extinction and scattering cross sections and efficiency factors for the case of normal incidence of a plane wave onto the base of a $2a\times 2b$ rectangular plate. An estimation of the relative error is obtained in the form of inequality, the right–hand side of which is a linear combination of two integrals being the functions of two diffraction parameters $p = ka$ and $q = kb$. The error is shown to be less than 3% for the rectangular plate with the minimum size of the base ten times exceeding the wavelength.*

Light scattered field in the near zone for an atmospheric crystal with a given orientation is a number of beams of parallel rays reflected from it. The size of the cross sections of the beams many times exceeds the wavelengths of the visible and IR radiation. So the method of physical optics is more optimum in describing the scattered field for such beams in the far zone. Applicability limits of the method of physical optics for a model of a crystal in the form of a round plate were formulated in Ref. 1. Within the framework of the model, the cross sections of the beams at normal incidence of a plane wave onto the plate base have the form of a circle, i.e. they are characterized by one size. As to some shapes of real crystals, e.g. for hexagonal columns, the cross sections of the beams should be described by two dimensions taking into account that one of them can considerably exceed the other. In this paper we generalize the results obtained in Ref. 1 for the case when the cross sections of scattered beams have two dimensions.

Let $2a$ and $2b$ be the dimensions of a rectangular plate along the $Ox$ and $Oy$ axes, respectively, $d$ be its thickness. Let the plane wave $\mathbf{E}\exp(ikz)$ be incident normally onto the plate base. Assume that the imaginary part, $\kappa$, of the complex refractive index $\tilde{n} = n + i\kappa$ equals zero, i.e. $\tilde{n} = n$. In Ref. 1 it is shown that the latter assumption does not lead to any loss of generality. The amplitude of the electric component of the incident field $\mathbf{E}$ is assumed to have the form

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{x}'_0 E_1 + \mathbf{y}'_0 E_2 = \mathbf{x}_0 E_{p_1} + \mathbf{y}_0 E_{p_2}, \qquad (1)$$

where

$$E_{p_1} = E_1\cos\xi - E_2\sin\xi;\quad E_{p_2} = E_1\sin\xi + E_2\cos\xi .$$

In other words, the components $\mathbf{E}_1$ and $\mathbf{E}_2$ forming a wave with elliptical polarization in the general case make an angle $\xi$ with the $Ox$ and $Oy$ axes in the $Oxy$ coordinate plane.

In the given formulation of the problem, we obtain formulas for the efficiency factors of extinction ($Q_{\text{ext}}$) and scattering ($Q_{\text{sca}}$) using the method of physical optics. Then we establish the applicability limits of the method by comparing local ($Q_{\text{ext}}$) and integral ($Q_{\text{sca}}$) characteristics of the scattered field.

The electric component of the whole electromagnetic field is represented in the form

$$\mathbf{E}_t = \mathbf{E}\exp(i\kappa z) - \mathbf{A}\exp(i\kappa r)/i\kappa r . \qquad (2)$$

In the general case the amplitude $\mathbf{A}$ of the scattered field is a sum of two orthogonal components $\mathbf{A}_1$ and $\mathbf{A}_2$. In the framework of the method of physical optics they are defined by the following expressions:

$$\begin{aligned}\mathbf{A}_1 &= (\boldsymbol{\upsilon}_0\cos\varphi - \boldsymbol{\varphi}_0\sin\varphi)\, S(\upsilon,\varphi)\, E_{p_1};\\ \mathbf{A}_2 &= (\boldsymbol{\upsilon}_0\sin\varphi + \boldsymbol{\varphi}_0\cos\varphi)\, S(\upsilon,\varphi)\, E_{p_2}.\end{aligned} \qquad (3)$$

The vectors $\boldsymbol{\upsilon}_0$ and $\boldsymbol{\varphi}_0$ are unit vectors of the spherical coordinate system $(r, \upsilon, \varphi)$ in which the angle $\upsilon$ is counted from $Oz$ axis and the angle $\varphi$ – from $Ox$ axis in $Oxy$ plane. In other words, the unit vectors of the spherical and rectangular Cartesian coordinate system are connected by the relations

$$\begin{aligned}\mathbf{r}_0 &= (\mathbf{x}_0\cos\varphi + \mathbf{y}_0\sin\varphi)\sin\upsilon + \mathbf{z}_0\cos\upsilon,\\ \boldsymbol{\upsilon}_0 &= (\mathbf{x}_0\cos\varphi + \mathbf{y}_0\sin\varphi)\cos\upsilon - \mathbf{z}_0\sin\upsilon,\\ \boldsymbol{\varphi}_0 &= -\mathbf{x}_0\sin\varphi + \mathbf{y}_0\cos\varphi.\end{aligned} \qquad (4)$$

The angular function $S(\upsilon, \varphi)$ has the same structure as in the case of a round plate,[1] i.e.

$$S(\upsilon,\varphi) = \frac{\kappa^2}{2\pi}\left(\frac{1+\cos\upsilon}{2}F_1(\upsilon,\varphi)(1-T) + \frac{1+\cos(\pi-\upsilon)}{2}F_2(\upsilon,\varphi)\,R\right), \qquad (5)$$

but here there is also a dependence on the azimuth angle $\varphi$ because of a lack of symmetry axis in the geometry of the problem. Remind that the complex values $T$ and $R$ are defined as Fresnel coefficients of transmission and reflection for a plane wave normally incident onto a semitransparent layer of thickness $d$. The formulas for them are given, in particular, in Ref. 1. The functions $F_1(\upsilon, \varphi)$ and $F_2(\upsilon, \varphi)$ are scattering characteristics in the far zone for beams emerging from the plate along the forward and backward directions. They are defined as Fraunhofer integrals for phase functions over the upper and lower bases of the plate. The integration is conducted here analytically, what allows us to obtain the following expressions for the functions $F_1(\upsilon, \varphi)$ and $F_2(\upsilon, \varphi)$:

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$$F_1(\upsilon, \varphi) = 4\,a\,b\,\frac{\sin(p\sin\upsilon\cos\varphi)}{p\sin\upsilon\cos\varphi}\,\frac{\sin(\theta\sin\upsilon\sin\varphi)}{q\sin u\sin\varphi},$$

$$F_2(\upsilon, \varphi) = 4\,a\,b\,\frac{\sin(p\sin(\pi-\upsilon)\cos\varphi)}{p\sin(\pi-\upsilon)\cos\varphi}\,\frac{\sin(q\sin(\pi-\upsilon)\sin\varphi)}{q\sin(\pi-\upsilon)\sin\varphi},$$

where $p = \kappa a$, $q = \kappa b$ are the diffraction parameters; $\kappa$ is the wave number.

Analyzing the aforementioned expressions one easily reveals that $F_1(\upsilon, \varphi) = F_2(\upsilon, \varphi)$. Let us introduce a new designation for these functions: $F(\upsilon, \varphi)$. As a result, for the angular function $S(\upsilon, \varphi)$ we have

$$S(\upsilon, \varphi) = \frac{\kappa^2}{2\,\pi}F(\upsilon, \varphi)\left(\frac{1+\cos\upsilon}{2}(1-T) + \frac{1-\cos\upsilon}{2}R\right). \quad (6)$$

To obtain the extinction cross section $\sigma_{ext}$, we use the extinction formula[2] for polarized fields

$$\sigma_{ext} = \frac{4\,\pi}{\kappa^2}\,\frac{\mathrm{Re}(\mathbf{E}^*\cdot\mathbf{A}|_{\tau=0})}{|\mathbf{E}|^2}. \quad (7)$$

When $\upsilon = 0$, it is easy to obtain from Eq. (4) that

$$\boldsymbol{\upsilon}_0\cos\varphi - \boldsymbol{\varphi}_0\sin\varphi = \mathbf{x}_0,$$
$$\boldsymbol{\upsilon}_0\sin\varphi + \boldsymbol{\varphi}_0\cos\varphi = \mathbf{y}_0.$$

As a result, the amplitude $\mathbf{A}$ of the scattered field is transformed to the form

$$\mathbf{A}\big|_{\upsilon=0} = \mathbf{A}_1\big|_{\upsilon=0} + \mathbf{A}_2\big|_{\upsilon=0} = (E_{p_1}\mathbf{x}_0 + E_{p_2}\mathbf{y}_0)\,S(0, \varphi) =$$
$$= \mathbf{E}\,S(0, \varphi) = \mathbf{E}\,\frac{2}{\pi}\,p\,q\,(1-T). \quad (8)$$

Using Eq. (8) and taking into account that

$$|\mathbf{E}|^2 = \mathbf{E}^*\cdot\mathbf{E} = |E_{p_1}|^2 + |E_{p_2}|^2 = |E_1|^2 + |E_2|^2, \quad (9)$$

we obtain the following expression for the extinction cross section

$$\sigma_{ext} = \frac{8\,p\,q}{\kappa^2}(1-\mathrm{Re}(T)). \quad (10)$$

In the given formulation of the problem the square of the geometrical shadow equals $4ab$. As a result, the formula for the extinction efficiency factor is

$$Q_{ext} = \sigma_{ext}/4ab = 2(1-\mathrm{Re}(T)). \quad (11)$$

Thus, the extinction efficiency factors for the rectangular and round plates are defined by one and the same relation (11).

To find the scattering cross section $\sigma_{sca}$, we use the following formula[2]:

$$\sigma_{sca} = \int_0^{2\pi}\int_0^{\pi}\frac{|\mathbf{A}|^2}{\kappa^2\,|\mathbf{E}|^2}\sin\upsilon\,d\upsilon\,d\varphi. \quad (12)$$

In the given formulation of the problem, the formula (12) permits a simplification. Taking into account that

$$|\mathbf{A}|^2 = \mathbf{A}^*\cdot\mathbf{A} = (|E_{p_1}|^2 + |E_{p_2}|^2)\,|S(\upsilon, \varphi)|^2 =$$
$$= (|E_1|^2 + |E_2|^2)|S(\upsilon, \varphi)|^2,$$

we transform Eq. (12) to

$$\sigma_{sca} = \int_0^{2\pi}\int_0^{\pi}\frac{|S(u, \varphi)|^2}{\kappa^2}\sin\upsilon\,d\upsilon\,d\varphi. \quad (13)$$

Let us take into consideration a new function

$$f(\upsilon, \varphi) = \frac{\sin X}{X}\,\frac{\sin Y}{Y},$$

where $X = p\sin\upsilon\cos\varphi$, $Y = q\sin\upsilon\sin\varphi$. Taking into account that $F(\upsilon, \varphi) = 4\,abf(\upsilon, \varphi)$ we define the angular function $S(\upsilon, \varphi)$ and the square of its module $|S(\upsilon, \varphi)|^2$ by the following expressions:

$$S(\upsilon, \varphi) = \frac{2\,p\,q}{\pi}f(\upsilon, \varphi)\left(\frac{1+\cos\upsilon}{2}(1-T) + \frac{1-\cos\upsilon}{2}R\right),$$

$$|S(\upsilon, \varphi)|^2 = \frac{4\,p^2\,q^2}{\pi^2}f^2(\upsilon, \varphi)\left[\left(\frac{1+\cos\upsilon}{2}\right)^2|1-T|^2 + \right.$$
$$+\left(\frac{1-\cos\upsilon}{2}\right)^2|R|^2 +$$
$$\left. + 2\,\mathrm{Re}[(1-T)R^*]\left(\frac{1+\cos\upsilon}{2}\right)\left(\frac{1-\cos\upsilon}{2}\right)\right]. \quad (14)$$

The formula (13) with regard to Eq. (14) is transformed to

$$\sigma_{sca} = \frac{4\,p^2\,q^2}{\pi^2\,\kappa^2}\left(|1-T|^2\int_0^{2\pi}\int_0^{\pi}f^2(\upsilon, \varphi)\left(\frac{1+\cos u}{2}\right)^2\right.$$
$$\sin\upsilon\,d\upsilon\,d\varphi +$$
$$+ |R|^2\int_0^{2\pi}\int_0^{\pi}f^2(\upsilon, \varphi)\left(\frac{1-\cos u}{2}\right)^2\sin\upsilon\,d\upsilon\,d\varphi +$$
$$\left. + 2\,\mathrm{Re}[(1-T)R^*]\int_0^{2\pi}\int_0^{\pi}f^2(\upsilon, \varphi)\frac{1+\cos\upsilon}{2}\,\frac{1-\cos\upsilon}{2}\sin\upsilon\,d\upsilon\,d\varphi\right). \quad (15)$$

By changing the variable $\upsilon$ in one of the integrals it is easy to show that

$$\int_0^{2\pi}\int_0^{\pi}f^2(\upsilon, \varphi)\left(\frac{1+\cos\upsilon}{2}\right)^2\sin\upsilon\,d\upsilon\,d\varphi =$$
$$= \int_0^{2\pi}\int_0^{\pi}f^2(\upsilon, \varphi)\left(\frac{1-\cos\upsilon}{2}\right)^2\sin\upsilon\,d\upsilon\,d\varphi. \quad (16)$$

Proceeding from scattering cross section to the scattering efficiency factor and taking into account Eq. (16) we finally obtain

$$Q_{sca} = \frac{\sigma_{sca}}{4\,a\,b} = (|1-T|^2 + |R|^2)\,A(p, q) +$$

$$+ 2\,\mathrm{Re}[(1 - T)\,R^*]\,B(p,\,q)\,, \qquad (17)$$

where

$$A(p,\,q) = \frac{p\,q}{\pi^2}\int\limits_0^{2\pi}\int\limits_0^{\pi} f^2(\upsilon,\,\varphi)\left(\frac{1+\cos\upsilon}{2}\right)^2 \sin\upsilon\,d\upsilon\,d\varphi\,,$$

$$B(p,\,q) = \frac{p\,q}{\pi^2}\frac{1}{4}\int\limits_0^{2\pi}\int\limits_0^{\pi} f^2(\upsilon,\,\varphi)\,\sin^3\upsilon\,d\upsilon\,d\varphi\,.$$

The formula (17) for the scattering efficiency factor has the same structure as in the case of a round plate[1] and differs from it only by the form of the functions $A$ and $B$ which depend on the two diffraction parameters $p$ and $q$ in the given case. Remind that the functions $A$ and $B$ bear the information about the degree of scattered intensity localization near the directions of refracted beams emergence and reach limiting values 1 and 0, respectively,[1] under unlimited growth of the cross–sectional size of the plate. Following general representations for the functions $A$ and $B$, it is natural to assume that

$$\lim_{\substack{p \to \infty \\ q \to \infty}} A\,(p,\,q) = 1,\ \lim_{\substack{p \to \infty \\ q \to \infty}} B\,(p,\,q) = 0. \qquad (18)$$

The following identity is established in Ref. 1 for a nonabsorbing plate ($\kappa = 0$)

$$2(1 - \mathrm{Re}\,(T)) = |1 - T|^2 + |R|^2. \qquad (19)$$

The relations (18) and (19) allow us to write the well–known equality

$$Q_{\mathrm{ext}} = Q_{\mathrm{sca}}\,, \qquad (20)$$

connecting the local and integral values of the angular function $S(\upsilon,\,\varphi)$ of a scattered field for a nonabsorbing particle.

The functions $A$ and $B$ differ, although slightly, from 1 and 0 for plates with large cross–sectional size. So in the general case the relation (20) is fulfilled approximately. It is easy to prove the following inequality for arbitrary diffraction parameters $p$ and $q$ (by analogy with Ref. 1):

$$(A - B)\,Q_{\mathrm{ext}} \le Q_{\mathrm{sca}} \le (A + B)\,Q_{\mathrm{ext}}\,. \qquad (21)$$

Let us take into consideration the parameter

$$\Delta = (Q_{\mathrm{ext}} - Q_{\mathrm{sca}})\,/\,Q_{\mathrm{ext}}\,, \qquad (22)$$

which will be taken as the relative error of the method of physical optics within the framework of which the relation for the angular function $S(\upsilon,\,\varphi)$ is obtained. Uniting (22) and the left–hand side of the inequality (21), we obtain the following inequality for the parameter $\Delta$:

$$\Delta(p,\,q) \le 1 - A(p,\,q) + B(p,\,q), \qquad (23)$$

which coincides in its form with the analogous inequality obtained in Ref. 1 for a round plate. The integral expressions for the functions $A$ and $B$ permit simplifications. As a result of simple transformations, one can considerably restrict the domain of integration and write the final formulas for these functions in the form

$$A(p,\,q) = \frac{2\,p\,q}{\pi^2}\int\limits_0^{\pi/2}\int\limits_0^{\pi/2} f^2(\upsilon,\,\varphi)(1 + \cos^2\upsilon)\,\sin\upsilon\,d\upsilon\,d\varphi\,,$$

$$B(p,\,q) = \frac{2\,p\,q}{\pi^2}\int\limits_0^{\pi/2}\int\limits_0^{\pi/2} f^2(\upsilon,\,\varphi)\sin^3\upsilon\,d\upsilon\,d\varphi\,. \qquad (24)$$

Thus, the formula (23) in which the values of the integrals $A$ and $B$ depend on the two diffraction parameters defines applicability limits of the method of physical optics for the problems when the cross sections of the beams should be given by two linear dimensions.

The values of integrals $A$ and $B$ for plates with square cross sections are depicted in Figs. 1*a* and 2*a*. The presented dependences of $A$ and $B$ on the diffraction parameter $p$ qualitatively follow the analogous dependences obtained in Ref. 1 for a round plate. So approximately the same values as in Ref. 1 are obtained here for the relative error of the method of physical optics. In particular, the value of $\Delta$ does not exceed 0.1 (10%) for diffraction parameter $p > 10$; $\Delta < 0.05$ (5%) for $p > 20$; $\Delta < 0.02$ (2%) for $p > 50$; $\Delta < 0.01$ (1%) for $p > 100$.

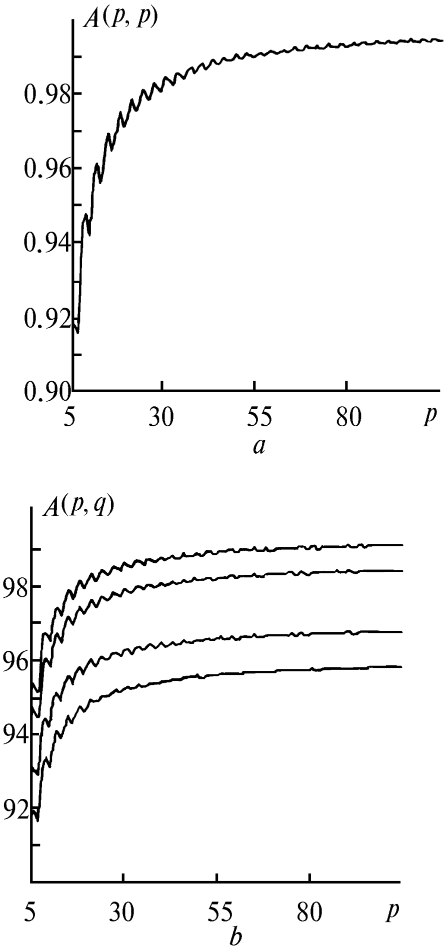


*FIG. 1. Dependences of the values of integrals A on the diffraction parameter p for a square (a) and rectangular (b) plates. The diffraction parameter q for A(p, q) takes the values 5, 10, 20, and 40 (curves with larger values of A correspond to larger q).*

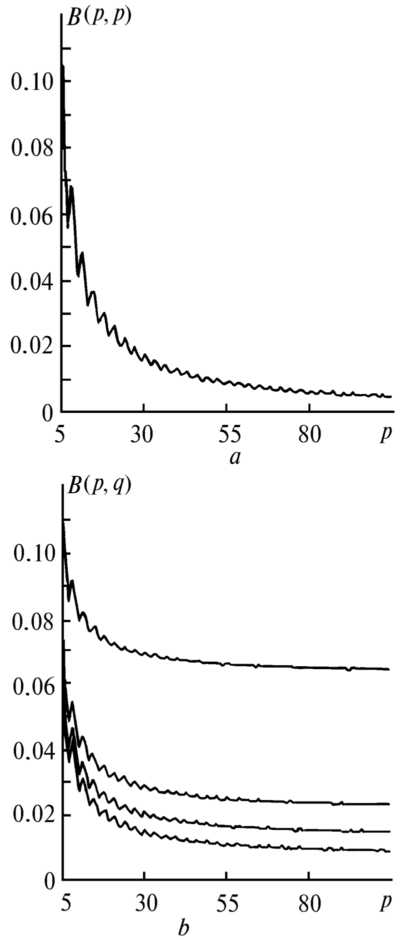


*FIG. 2. Dependences of the values of integrals B on the diffraction parameter p for a square (a) and a rectangular (b) plates. The diffraction parameter q for B(p, q) takes the values 5, 10, 20, and 40 (curves with smaller values of B correspond to larger q).*

The values of integrals $A$ and $B$ are depicted in Figs. 1$b$ and 2$b$ as a function of diffraction parameters $p$ and $q$ for rectangular plates. The curves have the same shape as in the case of square plates, but they are limited from above or below by different horizontal asymptotes whose values depend on the parameter $q$. In other words, by increasing only one of the diffraction parameters, one can always decrease the relative error of the method but only to a certain limit. The limiting value of the parameter $\Delta$ for $q$ = const is found by the following formula

$$\Delta \le 1 - A(\infty,\, q) + B(\infty,\, q)\ ,$$

where $A(\infty, q)$ and $B(\infty, q)$ are asymptotic values for the curves $A = A(p, q)$ and $B = B(p, q)$ including the curves depicted in Figs. 1$b$ and 2$b$.

By analyzing the dependences represented in Figs. 1 and 2 one can draw a conclusion that the maximum increase of values of functions $A$ and $B$ are within the variability of the argument $p$ from 5 to 30. Analogous conclusion can be drawn for the argument $q$ because the functions $A$ and $B$ are symmetric with respect to their arguments. As a result, the relative error of the method at $p$ = 30 slightly decreases with the increase of $p$. On the other hand, this error is about 3% for a square plate with $p$ = 30. Moreover, it becomes less for a rectangular plate whose minimum diffraction parameter is 30 ($2b \approx 9.6\lambda$, where $\lambda$ is the wavelength).

Thus, the error of the method of physical optics does not exceed 3% for rectangular plates whose minimum size of the cross section ten times exceeds the wavelength.

## REFERENCES

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