METHOD OF TOMOGRAPHIC SOUNDING IN LIDAR STUDIES OF THE ATMOSPHERE

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Airborne lidar systems can be made more effective by processing the measurements using the methods of reconstructive tomography. A tomographic sounding scheme and an algorithm for interpreting lidar signals which implement the principle of computational tomography are described. The stabi1ity of the algorithm with respect to errors in the starting data is discussed. The results of the solution of model problems are presented.

A wide range of problems associated with the study of the spatial distribution of the aerosol component in the earth's atmosphere, and dynamic processes giving rise to transport of aerosol pollutants can be successfully solved using airborne lidars,[1–3] which enables real-time relocation of the lidars within the region of space under study. Space-based lidar systems that will enable global monitoring of atmospheric aerosol are under development.[4] The possibility of changing the position of the lidar relative to the volume of space under study makes it possible to consider the problem of determining the spatial structure of the optical characteristics from lidar observations from the viewpoint of reconstructive tomography.[5–7] This approach to the solution of the problem with sensing from aircraft was suggested in Ref. 6, where the term "tomographic lidar" was first introduced. The starting information in the tomographic study of the structure of objects consists, as is well known, of a collection of projections obtained for different angles of observation. Each projection consists of a family of integrals of the characteristics sought which are taken along lines of sight passing through the medium under study.

In contradistinction to conventional tomography the motion of the lidar relative to the medium under study makes it possible to obtain a sequence of projections determined by linear integrals of the spatial distribution of the attenuation coefficient of the medium on the interval from the radiation source to a point in the scattering volume and corresponding to different depths of the sounding beam in the volume of the medium. The specific feature of a lidar as a tomograph is that the recorded signals depend on the local values of the backscattering coefficient.

The similarity of the structure measurements in the case of laser sounding and in transmission tomography suggests that data obtained by scanning the lidar along different directions be employed to reconstruct the spatial distribution of the optical characteristics of the medium. In this paper one possible experimental implementation of this idea with airborne laser sounding is examined and an algorithm for interpreting the corresponding measurements is described.

Let the lidar be located on a platform moving in a straight line at some altitude H above the earth's surface, and let the sounding be performed in the direction of the lower hemisphere. The object of study are spatially inhomogeneous fields of the optical characteristics: the attenuation coefficient $\alpha(\vec{r})$ and the backscattering coefficient $\beta(\vec{r})$, where $\vec{r}$ is the radius vector of the point (x, z) in a rectangular coordinate system, fixed on the vertical scanning plane passing through the direction of motion of the platform; x and z are horizontal and vertical coordinates, respectively.

We affix to the lidar, which moves along the X-axis, a polar coordinate system characterized by the radius vector $\rho = (\rho\sin\varphi, \rho\cos\varphi)$ along the direction of sounding and the polar angle φ measured from the direction toward the nadir. If the lidar is located at the point $\vec{r}^* = (x^*, z^*)$, then the signal from the scattering volume located at the point $r = r^* + \rho$ is given in the single-scattering approximation by the following expression:

$$S(\vec{r}^*, \vec{\rho}, \varphi) = \beta(\vec{r}^*+\vec{\rho})\exp\left\{-2\int_0^{\rho}\alpha(\vec{r}^*+\vec{\rho}')d\vec{\rho}'\right\}, \qquad (1)$$

where $S(r^*, \rho, \varphi) = P(r^*, \rho, \varphi)^2\rho^2/(P_0A)$; P_0 and $P(r^*, \rho, \varphi)$ are the power of the transmitted signal and the power of the received signal, respectively; A is the instrumental constant; and, $\rho' = (\rho'\sin\varphi, \rho'\cos\varphi)$.

In the approach studied here the method of tomography consists of simultaneously sensing the medium from different directions at different polar angles φ and then solving a system of equations of the form (1) which arises in the process. In the simplest case it is sufficient to have lidar signals at two polar angles φ_1 and φ_2 in order to reconstruct the spatial distribution of characteristics of $\alpha(x, z)$ and $\beta(x, z)$. In this case there will be two equations for the two unknowns α and β at each point in space (x, z). We shall examine an algorithm for processing lidar signals in this case.

For definiteness assume that the sounding is performed at the nadir ($\varphi_1 = 0$) and at an angle $\varphi_2 = \varphi$ to the nadir with a pulse repetition frequency ν. Then discrete readings can be determined along the X and Z axes (Fig. 1): $x_j = j\Delta x$, $j = 0.1, \ldots, n$; $z_i = i\Delta z$; $i = 0.1, \ldots, M$, where $\Delta x = V/\nu$, $\Delta z = \Delta x \operatorname{ctg}\varphi$ (V is the velocity of the lidar). For these readings a finite-difference analog of the system of equations (1) can be constructed in the form of a system of grid equations[8]:

$$S_{ij} = \beta_{ij} T_{i-1,j} \exp[-(\alpha_{i-1,j}+\alpha_{ij})\Delta z],$$

$$S^{\varphi}_{ij} = \beta_{ij} T^{\varphi}_{i-1,j-1} \exp[-(\alpha_{i-1,j-1}+\alpha_{ij})\Delta z/\cos\varphi], \tag{2}$$

$$T_{ij} = \exp\left[-\frac{\Delta z}{2}\sum_{k=0}^{i}\omega_k\alpha_{kj}\right], \quad T_{0j} = 1;$$

$$T^{\varphi}_{ij} = \exp\left[-(\Delta z/2\cos\varphi)\sum_{k=0}^{i}\omega_k\alpha_{k,j-i+k}\right], \quad T^{\varphi}_{0j} = 1,$$

$$i = 1, 2, \ldots, M; \; j = i, i+1, \ldots, n; \; M \le n;$$

$$\omega_0 = \omega_i = 1, \; \omega_k = 2 \; (k \ne 0; \; k \ne i).$$

Here the following notation has been adopted: characteristics with a superscript φ are obtained by replacing Eq. (1) at $\varphi_2 = \varphi$ by its discrete analog; variables with the pair of subscripts (i, j) refer to the point (x_j, z_i). The system of grid equations (2) is defined for $j \ge i$ and $2N$ unknowns, where the number of nodes $N = M\,n - M(M-1)/2$.

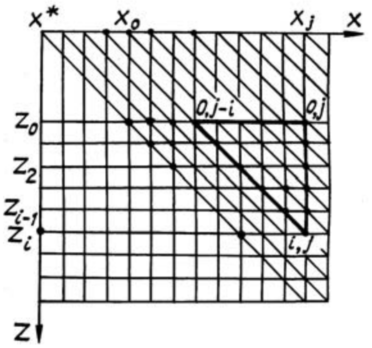


FIG. 1. A discrete scheme for two-beam tomographic sounding of the atmosphere.

The general approach to the solution of the system (2) is as follows. A solution is obtained sequentially for each layer z_i, $i = 1, 2, \ldots$ starting with $i = 1$. This requires a priori knowledge of the boundary values α_{0j}. If such data are not available, the problem can be solved by setting $\alpha_{0j} = \alpha_{1j}$. The solution at the i-th layer will be determined in terms of the solution at the preceding layers. Then at the i-th layer for a given value of j the system (2) will be defined for a pair of unknowns α_{ij} and β_{ij} and reduces to a system of two linear algebraic equations of the form

$$\begin{aligned} &\ln\beta - \alpha\Delta z = b_1; \\ &\ln\beta - \alpha\Delta z/\cos\varphi = b_2. \end{aligned} \tag{3}$$

In Eq. (3) the following notation is adopted:

$$\beta = \beta_{ij}; \quad \alpha = \alpha_{ij}; \quad b_1 = \ln G_{ij}; \quad b_2 = \ln G^{\varphi}_{ij};$$

$$G_{ij} = S_{ij}\exp[\alpha_{i-1,j}\Delta z]/T_{i-1,j}; \tag{4}$$

$$G^{\varphi}_{ij} = S^{\varphi}_{ij}\exp[\alpha_{i-1,j-1}\Delta z/\cos\varphi]T^{\varphi}_{i-1,j-1}.$$

The repeated solution of a system of equations of the form (3) is the main feature of the processing of lidar signals in the problem under study. We shall discuss briefly the special features of the inversion of the system (3). As the step Δz is made to approach zero the system (3) becomes degenerate, and its determinant $D = \Delta z\,(\cos\varphi - 1)/\cos\varphi$ approaches zero. However this is not reflected in the reconstruction of the quantity In β from the system (3):

$$\ln\beta = (b_2\cos\varphi - b_1)/(\cos\varphi - 1), \tag{5}$$

which does not depend on Δz and remains stable. On the other hand, the calculation of the attenuation coefficient a from the formula

$$\alpha = (b_2 - b_1)/D \tag{6}$$

will be unstable, since in the limit $\Delta z \to 0$, the expression (6) is an indefinite form of the type 0/0. This becomes clearer when the errors in the solution of the system (3) are analyzed. Let the exact right side of (3) (b_1^0, b_2^0) correspond to the exact solution $(\ln\beta_0, \alpha_0)$. It can be shown that the error in the reconstruction of $\ln\beta$ when the right side of Eq. (3) b_1^δ, b_2^δ contains an error will be characterized by the quantity

$$\delta_\beta = (\delta_1^2 + \delta_2^2\cos^2\varphi)^{1/2}/(1-\cos\varphi), \tag{7}$$

where

$$\delta_i^2 = \overline{(b_i^\delta - b_i^0)^2}, \quad \delta_\beta^2 = \overline{(\ln\beta_\delta - \ln\beta_0)^2}.$$

Analogously for α_δ we have

$$\delta_\alpha = \cos\varphi(\delta_1^2 + \delta_2^2)^{1/2}/[\Delta z(1-\cos\varphi)]. \tag{8}$$

It is obvious from Eqs. (7) and (8) that as $\Delta z \to 0$ the error δ_α increases without limit, while the quantity $\delta\beta$ is always bounded as long as $\varphi \neq 0$. Thus the problem under study is stable only with respect to the backscattering coefficient $\beta(x, z)$. The stable reconstruction of the spatial distribution of the attenuation coefficient is achieved by using regularization methods.[9] This is most simply achieved by restricting the value of the norm of $\alpha(x, z)$ separately for each i-th layer, defining the stabilizing functional in the form

$$\|\alpha\|^2 = \sum_{j=i}^{n} \alpha_{ij}^2 / n', \quad n' = n-i+1.$$

The application of Tikhonov's method of regularization ultimately gives for each i-th layer $n' = n - i + 1$ independent pairs of equations. Each pair of equations is the regularized analog of the system (3) with a common regularization parameter γ for all equations. For uncorrelated errors δ_1 and δ_2 the estimate of the optimal value of the regularization parameter has the form

$$\gamma^* = \varepsilon^2/\|\alpha_0\|^2, \tag{9}$$

where ε is the relative error in $G_{ij}(G_{ij}^\varphi)$ (for simplicity assumed to be constant for all j). It is obvious from the formula (9) that to determine the optimal value of γ^* with a fixed error ε in the initial data it is sufficient to evaluate *a priori* the average value of the square of the attenuation coefficient a over the i-th layer. The relative mean-square error in the reconstruction of α in the i-th layer with the optimal value of the regularization parameter γ is given by the expression

$$\varepsilon_\alpha^2 = \dot{\varepsilon}^2/[\varepsilon^2 + D^2\|\alpha_0\|^2/2]. \tag{10}$$

It follows from Eq. (10) that as $\varepsilon \to 0$ the error in the solution also approaches zero. In addition, the error in the solution remains bounded as $D \to 0$. The spatial distribution of the attenuation coefficient α is reconstructed more accurately as the density of the medium increases and as the discretization step Δz and the viewing angle φ increase. Setting an admissible level of discretization of the solution ε_α, the admissible discretization step for media with different optical density can be determined from formula

$$\Delta z = \frac{\varepsilon}{\varepsilon_\alpha \|\alpha_0\|} \sqrt{2(1-\varepsilon_\alpha^2)} \left| \frac{\cos\varphi}{1-\cos\varphi} \right|. \tag{11}$$

For example, it follows from Eq. (11) that, for $\varepsilon = 10\%$ and $\varphi = \pi/3$ the attenuation coefficient α can be reconstructed with an error of 20–30% and a spatial resolution of the order of 1 km with a visibility range exceeding 10 km. The admissible discretization step Δz is proportional to the visibility range. The general relationship between the inversion error and the spatial resolution follows from formulas (10) and (11): a gain in the accuracy in the reconstruction of α will be accompanied by a gain in the resolution and *vice versa*.

Substituting Δz from Eq 11) into the expression for δ_a without regularization (8) gives

$$\delta_\alpha = \varepsilon_\alpha \|\alpha\| / \sqrt{1-\varepsilon_\alpha^2}. \tag{12}$$

The relation (12) gives a relation between the error obtained in the regularized solution and the error obtained without regularization with the same discretization step Δz. It is obvious from Eq. (12) that these errors differ by the factor $\sqrt{1-\varepsilon_\alpha^2}$, which for small values of ε_α is close to unity. The discretization steps Δz for regularized and unregularized problems are in the same ratio. Therefore the solution obtained without regularization will be close to the regularized solution, if the step Δz is chosen based on the error in the initial data. Based on the remarks regarding the choice of Δz the algorithm for inverting the system (2) can be written in the f orm

$$\alpha_{ij} = \frac{1}{\Delta z} \ln\left[\frac{G_{ij}}{G_{ij}^\varphi}\right] \frac{\cos\varphi}{1-\cos\varphi}, \quad \beta_{ij} = G_{ij}\left[\frac{G_{ij}}{G_{ij}^\varphi}\right]^{\frac{\cos\varphi}{1-\cos\varphi}}, \tag{13}$$

where G_{ij} and G_{ij}^φ are defined in Eqs. (4).

In the algorithm (13) the discretization step along X-axis is related with the discretization step along the Z axis: $\Delta x = \Delta z \cdot \tan\varphi$. The limit on the spatial resolution established for the z coordinate is transferred also to the x coordinate.

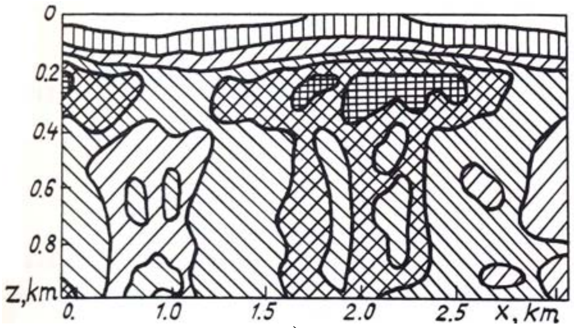
A simple modification of the algorithm (13) makes it possible to eliminate the limits on the spatial resolution along the X axis. If the step Δx is reduced by a factor of m, $\Delta x' = \Delta x/m$, then for the new readings $x_j = j\,\Delta x'$, $J = 1, 2, \ldots$, in the algorithm (13) the quantities G_{ij}^φ are given by the formula

$$G_{ij}^\varphi = S_{ij}^\varphi \exp\{\alpha_{i,j-m} \Delta z/\cos\varphi\} T_{i-1,j-m}^\varphi, \tag{14}$$

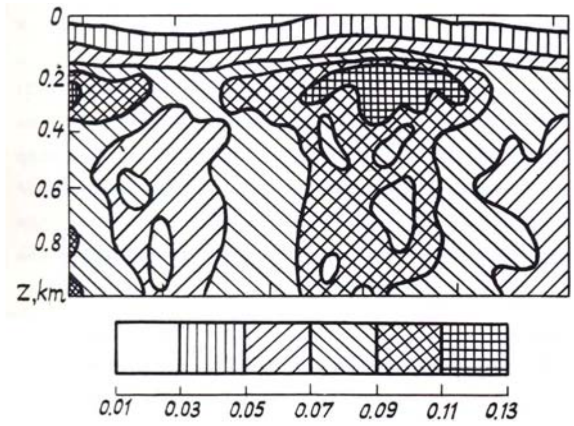
where $J = im$, $im = 1, \ldots, n$.

Within the layer $z_i \le z \le z_{i+1}$ the backscattering coefficient $\beta(x, z)$ can be determined by introducing a correction for transmission through the top boundary of the layer to the level z by interpolating $\alpha(x, z)$ at each point on the boundaries z_i and z_{i+1}.

As an example of the processing of lidar signals Fig. 2 shows the results of the numerical modeling of the problem of tomographic sounding of a weakly turbid atmosphere. Figure 2a shows the initial spatial distribution of the backscattering coefficient $\beta(x, z)$. The average value of the attenuation coefficient α was equal to about 0.1 km^{-1}. The result of reconstruction of the field $\beta(x, z)$ with an error of $\varepsilon = 10\%$ in the experimental data is presented in Fig. 2b for the following conditions of simulation of the experiment: the angle $\varphi = 45°$, $\Delta x' = 0.1$ km, and $m = 10$. Figure 3 shows the error, averaged over the layer, in constructing $\beta(x, z)$ from the penetration depth in the scattering medium with 5% and 10% errors in the signals.



a)



b)

FIG. 2. The reconstruction of the two-dimensional spatial distribution of the backscattering coefficient $\beta(x, z)$ in a numerical experiment: a) model distribution $\beta_0(x, z)$; b) result of the solution of the inverse problem with a 10%-error in the lidar signals.

We shall point out some characteristic features of the solution obtained by means of lidar tomographic sounding. First, to construct the solution it is not necessary to use any *a priori* assumptions regarding the characteristics sought, as is usually done in the solution of the lidar equation. Second, since the signals S_{ij} and S_{ij}^{φ}. appear in the form of a ratio in the formulas (13) determining the solution for the attenuation coefficient a_{ij} it is obvious that their values can be given to within a constant factor, i.e., in reconstructing the attenuation coefficient field $\alpha(x, z)$ the lidar needs not to be calibrated.

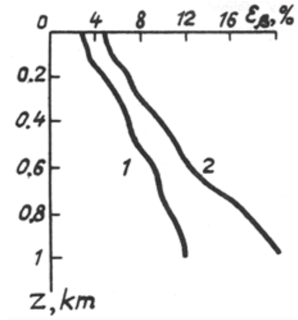


FIG. 3. The error, averaged over the layer, in the reconstruction of $\beta(x, z)$ versus the penetration depth in the scattering medium: $\varepsilon = 5\%$ (1) and $\varepsilon = 10\%$ (2).

It may be expected that the method studied here will have an advantage over the traditional methods for processing signals for a range of problems associated with sounding of optically dense media when there is a high degree of uncertainty in the lidar ratio. The proposed method can be recommended for airborne lidar studies of extended aerosol nonuniformities, such as, for example, smoke plumes from forest fires or of industrial origin, volcanic emissions, etc. The method can be easily extended to the case of multiangle and three-dimensional tomography.

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