OPTIMIZATION AND EFFICIENCY OF THE RADIATION CONTROL BY ADAPTIVE OPTICAL SYSTEMS WITH FLEXIBLE MIRRORS

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The performance of an adaptive optical system (AOS) of radiation focusing, which consists of a wavefront analyzer in the form of an array of the Hartmann sensors and a corrector which is a controllable flexible mirror. Under assumption of a complete statistical description of the light field being received, an optimal algorithm for control by the AOS intended for compensation for the distortions caused by the turbulent atmosphere is synthesized. The dependence of the efficiency of proposed algorithm on the main parameters of the AOS and on the environment is examined.

In a number of practically important areas, such as laser communication, laser detection and ranging, etc., there arises a problem of minimizing the losses of optical radiation when it propagates through the turbulent atmosphere.[1] This problem is usually solved by measuring the phase distortions of the wavefront (WF) caused by the random refraction due to the nonuniformities of the atmospheric refractive index and by their compensations with the help of the controllable elements of the adaptive optical systems (AOS's). Recently a large number of investigations has been devoted to the problems of compensation for the distortions in the light wave by the AOS's.[2-9]

From the viewpoint of minimizing the compensation errors and constructing the optimal algorithms for processing of the radiation being received, it is very important to use an *a priori* information about the statistical properties of the optical signal (in particular, about the spatial correlation of the phases of the wavefront on the aperture of the AOS) and about the instrumental noise . Meanwhile, in most of the above–indicated researches the compensation algorithms are constructed without taking into account the *a priori* statistical information. At the same time, methods of the theory of statistical decisions are adequate for the description of this problem of observations of the random fields against the background of noise radiation from the external sources.[10]

The operation efficiency of the AOS depends on the accuracy of compensation for the phase distortions of the wavefront. In its turn, the compensation accuracy is determined mainly by three factors: the accuracy of the measuring device which estimates the phases of the wavefront in the vicinity of the sensor, the errors in estimating the phase of the wavefront from the set of these measurements, and, finally, the errors in the reconstruction of the phase of the wavefront with the help of the controllable optical elements.

1. MEASUREMENTS OF THE PHASE OF THE WAVEFRONT

Measuring devices based on the Hartmann sensor are most widespread due to the simplicity of their design.[9] An elementary sensor usually consists of a very simple optical system, which forms an image, a photodetector, which records the spatial intensity distribution over this image, and a processing device for determining the coordinates of the centroid of the diffraction spot of the image being formed. As is well known,[11] under the ideal conditions such a processing procedure permits one to estimate the average (over the sensor aperture) tilt of the wavefront of radiation incident on the aperture. However, actually the intensity distribution, in addition to the desired laser signal, involves the background radiation, which deteriorates the image contrast and thereby results in the errors in the measurement of the wavefront tilt of the desired signal. For this reason, the normalized readings $\hat{\gamma}_i = [\hat{\gamma}_{x_i}, \hat{\gamma}_{y_i}]\, kR\, \frac{1+Q}{Q}$, which are recorded by the array of n Hartmann sensors, can be written in the form

$$\hat{\gamma}_i = \gamma_i + \xi_i,\ i = 1, \ldots, n\ , \qquad (1)$$

where $\hat{\gamma}_{x_i}$ and $\hat{\gamma}_{y_i}$ are the angular coordinates of the centroid of the formed image along $0X$ and $0Y$ axes of a chosen Cartesian coordinate system in the photodetection plane, k is the wave number, R is the radius of measuring aperture of the sensor, Q is the signal–to–noise ratio in the sensor, $\gamma_i = \frac{1}{\pi R}\int_{\Omega_i} \Omega_r \varphi(\mathbf{r}) d^2 r$ is the true average tilt of the wavefront (normalized by the diffraction size of the image kR) in the vicinity of the ith sensor, Ω_i is its input aperture, and ξ_i is the noise component of the normalized error of the measuring device. We may assume that the mean value of the error of the noise component is equal to zero, while the correlation matrix is diagonal, i.e., $\overline{\xi_i^T \xi_j} = \sigma_\xi^2 \delta_{ij}$, where the horizontal bar atop indicates an averaging over an ensemble of the background noise realizations, the superscript T stands for the matrix transposition, and σ_ξ^2 is the variance of the noise component of the error. When the measurement errors are limited by the radiation of the background[12] we have

$$\sigma_\xi^2 \approx \frac{8}{\pi Q m_t \sqrt{m_r}} \left(1 + \frac{\pi}{4} \frac{\sqrt{m_r}}{Q}\right), \qquad (2)$$

where m_t and m_r are the numbers of temporal and spatial noise modes recorded by the sensor.

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2. ALGORITHM FOR ESTIMATING THE PHASE OF THE WAVEFRONT

As was shown in Ref. 4, the problem of radiation focusing into the turbulent atmosphere (when we control the phase of the radiation wavefront for a Gaussian statistics of the phase fluctuations) the optimal energy criterion (the maximum radiation intensity delivered at the given point) and information criterion (the minimum variance of the error in estimating the phase distortions of the wavefront) lead to one and the same optimal algorithm. If we use the criterion of the minimum of estimating the rms error of the unknown parameter, the so–called separation principle[13] is valid, in accordance with which the problem of seach for the optimal control is performed in two stages. At the first stage the phase of the wavefront is estimated and at the second the optimal control is found using the phase of the wavefront, which has been estimated at the previous optimization stage.

In order to construct the algorithm of the estimating the phase of the wavefront, it is convenient to use the mathematical apparatus of the statistical decision theory, which enables one to find both an optimal (in the sense of minimum variance of the error in estimating the phase of the wavefront at each point of the aperture) estimate of the phase of the wavefront and its variance.

Let us assume that the random Gaussian field $\boldsymbol{\varphi}(\mathbf{r})$, $\mathbf{r} \in \Omega_a$ with zero mean and the assigned structure function $D_\varphi(\mathbf{r}_1, \mathbf{r}_2)$ is prescribed on the aperture of the AOS. Let us change over from the continuous Gaussian field $\boldsymbol{\varphi}(\mathbf{r})$ to the Gaussian $1\times N$ random vector $\boldsymbol{\varphi}^T = [\varphi(r_1) \dots \varphi(\mathbf{r}_N)]$. It is necessary to obtain the estimate $\hat{j}$ of the vector $\boldsymbol{\varphi}$ based on the observed $1\times 2n$ vector of the measurements $\boldsymbol{\zeta}^T = \left[\hat{\gamma}_{x_1}, \dots, \hat{\gamma}_{x_n}; \hat{\gamma}_{y_1}, \dots, \hat{\gamma}_{y_n}\right]$.

Under the assumption of a large number of the temporal m_t and spatial m_r modes of the background radiation, which is recorded by the detector, the random vector $\boldsymbol{\zeta}$ may be regarded as Gaussian. The vector $[\boldsymbol{\varphi}|\boldsymbol{\zeta}]$ which is composed from the vectors $\boldsymbol{\varphi}$ and $\boldsymbol{\zeta}$ will be then also Gaussian. Therefore, using the theorem of normal correlation,[13] we can write at once the relation for estimating the unobservable part of the Gaussian vector (the *a posteriori* mean value) $[\boldsymbol{\varphi}|\boldsymbol{\zeta}]$ and of the correlation matrix $B_{\varepsilon 1} = \overline{\langle \boldsymbol{\varepsilon}_1^{\mathrm{T}} \times \boldsymbol{\varepsilon}_1 \rangle}$ of the estimation error $\varepsilon_1 = \hat{\boldsymbol{\varphi}} - \boldsymbol{\varphi}$

$$\begin{cases} \hat{\boldsymbol{\varphi}} = \mathbf{B}_{\varphi\zeta}\, \mathbf{B}_{\mathrm{ff}}^{-1} \cdot \zeta \, , \\ \mathbf{B}_{\varepsilon 1} = \mathbf{B}_{\varphi\varphi} - \mathbf{B}_{\varphi\zeta} \mathbf{B}_{\varphi\zeta}^{-1} \cdot \mathbf{B}_{\varphi\zeta}^{\mathrm{T}} \, , \end{cases} \tag{2'}$$

where the correlation matrices $\mathbf{B}_{\varphi\varphi}$, $\mathbf{B}_{\varphi\zeta}$, and $\mathbf{B}_{\varphi\zeta}^{-1}$ are defined in the following way: $\mathbf{B}_{\varphi\varphi} = \langle \boldsymbol{\varphi\varphi}^{\mathrm{T}} \rangle$, $\mathbf{B}_{\varphi\xi} = \langle \boldsymbol{\varphi\zeta}^{\mathrm{T}} \rangle$, and $\mathbf{B}_{\xi\xi}$ = , the angular brackets denote an averaging over the ensemble of realizations of the phases of the wavefront. Proceeding in relations (2′) to the limit as $N \to \infty$ we will obtain the algorithm of estimating the phase of the wavefront $\hat{\varphi}(\mathbf{r})$ at any point of the aperture $\mathbf{r} \in \Omega_a$ and the structure function $D_{\varepsilon 1}(\mathbf{r}_1, \mathbf{r}_2) = \overline{\langle [\varepsilon(\mathbf{r}_1) - (\mathbf{r}_2)]^2 \rangle}$ of the estimation error $\varepsilon_1(\mathbf{r}) = \hat{\varphi}(\mathbf{r}) - \varphi(\mathbf{r})$

$$\begin{cases} \hat{\varphi}(\mathbf{r}) = \mathbf{H}^{\mathrm{T}}(\mathbf{r})\, \mathbf{B}_{\xi\xi}^{-1} \zeta \, , \\ D_{\varepsilon 1}(\mathbf{r}_1, \mathbf{r}_2) = D_\varphi(\mathbf{r}_1, \mathbf{r}_2) - [\mathbf{H}(\mathbf{r}_1) - \mathbf{H}(\mathbf{r}_2)]^{\mathrm{T}} \times \\ \times\, \mathbf{B}_\xi^{-1} [\mathbf{H}(\mathbf{r}_1) - \mathbf{H}(\mathbf{r}_2)] \, , \end{cases} \tag{3}$$

where $\mathbf{H}(\mathbf{r}) = \langle \varphi(\mathbf{r})\zeta \rangle$ is the column $(2n\times 1)$ vector. Relations (3) are one of the main results of Ref. 4.

As follows from Eqs. (3), the optimal estimate of the phase of the wavefront $\boldsymbol{\varphi}(\mathbf{r})$ at the arbitrary aperture point $\mathbf{r} \in \Omega_a$ is a weighted sum of the measurements performed by all of the elementary sensors of the system. The weight in this sum is determined by both the correlations between the phase values at different points of the aperture and the measurement errors of sensors.

3. ALGORITHM FOR RECONSTRUCTION OF THE MEASURED PHASE OF THE WAVEFRONT

Let us assume that in the plane optically conjugate to the plane in which the function $\hat{\varphi}(\mathbf{r})$ has been found a controllable flexible mirror is located. Let us designate the total number of the actuators of the mirror by m and their spatial response functions by $f_l(\mathbf{r})$ $(l = 1, \dots, m)$. The problem of reconstruction of the phase of the wavefront now reduces to expanding the function $(\mathbf{r})$ most suitably in a system of the functions $f_l(\mathbf{r})$, i.e., to finding a certain function

$$\tilde{\boldsymbol{\varphi}}(r) = \sum_{l=1}^{m} a_l\, f_l(\mathbf{r}) = \mathbf{f}^{\mathrm{T}}(\mathbf{r})\mathbf{a} \, , \tag{4}$$

which would be as close as possible to the function $(\mathbf{r})$. It is clear that subtracting from $\hat{\varphi}(\mathbf{r})$ its approximation $(\mathbf{r})$ we will compensate for the phase distortions.

The degree of proximity of the functions $(\mathbf{r})$ and $(\mathbf{r})$ can be characterized in different ways, but the following measure is most natural:

$$\frac{1}{S_a} \int_{\Omega_a} \varepsilon_2^2(\mathbf{r})\, d^2 r \, , \tag{5}$$

where $\varepsilon_2(\mathbf{r}) = (\mathbf{r}) - (\mathbf{r})$ is the error of control and S_a is the area of the aperture Ω_a. Such a control (i.e., choice of the coefficients a_l) is regarded as optimal by minimizing residual error (5). In order to find the unknown coefficients a_l, we should substitute Eq. (4) into expression (5), take the derivatives of the derived relation with respect to a_l $(l = 1, \dots, m)$, and set them equal to zero. The coefficients which have been thus found minimize expression (5). In a matrix representation the solution of the system has the form

$$\mathbf{a} = \mathbf{F}^{-1} \frac{1}{S_a} \int_{\Omega_a} \hat{\varphi}(\mathbf{r}) f(\mathbf{r})\, d^2 r \, , \tag{6}$$

where $\mathbf{F} = \frac{1}{S_a} \int_{\Omega_a} \mathbf{f}(\mathbf{r})\, \mathbf{f}^{\mathrm{T}}(\mathbf{r})\, d^2 r$ is the matrix of integrals of the overlap of the response functions. From physical considerations it is clear that the system of actuators and response functions should be chosen so that the diagonal elements of the matrix F_{il} were maximal, while for $i \neq l$ the elements F_{il} had to rapidly decrease as $|i - l|$ increases. This means that the matrix **F** is nondegenerated and the inverse matrix $\mathbf{F}^{-1}$ exists.

With an account of Eqs. (3), (4), and (6), the optimal algorithm for the reconstruction of the phase of the wavefront can be written in the form

$$\tilde{\varphi}(\mathbf{r}) = \mathbf{f}^{\mathrm{T}}(\mathbf{r})\mathbf{F}^{-1}\mathrm{F}\mathbf{B}_{\xi\xi}^{-1}\zeta \ , \qquad (7)$$

where $\Phi = \frac{1}{S_{\mathrm{a}}}\int\limits_{\Omega_{\mathrm{a}}}\mathbf{f}(\mathbf{r})\mathbf{H}^{\mathrm{T}}(\mathbf{r})\, d^2r$ is the $(m\times 2n)$ matrix which characterizes the degree of adjustment of the shape of the response function and the function of correlation between the reading and the phase.

Formally, Eq. (7) is a solution of the posed problem. This formula will be used in order to analyze the efficiency of the compensation algorithm.

4. ANALYSIS OF THE EFFICIENCY OF THE COMPENSATION ALGORITHM

As a measure of the efficiency of compensation for the phase distortions with the help of algorithm (7) for reconstruction of the phase of the wavefront we will use the Strehl factor[6]

$$K = \left\langle \overline{\left| \frac{1}{S_{\mathrm{a}}}\int\limits_{\Omega_{\mathrm{a}}} \mathrm{e}^{i\varepsilon(\mathrm{r})}\, d^2r \right|^2} \right\rangle , \qquad (8)$$

where averaging must be carried out over the ensembles of all the random mechanisms that are responsible for the random nature of the compensation error $\varepsilon(\mathbf{r})=\varepsilon_1(\mathbf{r}) + \varepsilon_2(\mathbf{r})=\varphi(\mathbf{r})-\tilde{(\mathbf{r})}$, $\mathbf{r} \in \Omega_{\mathrm{a}}$. By virtue of the fact that estimate (7) depends linearly on the readings ζ while the latter are the Gaussian random quantities, it is clear that the error $\varepsilon(\mathbf{r})$ obeys the normal distribution. For this reason,

$$K = \frac{1}{S_{\mathrm{a}}^2}\int\limits_{\Omega_{\mathrm{a}}} d^2r_1 \int\limits_{\Omega_{\mathrm{a}}} d^2r_2 \exp(-1/2D_{\varepsilon}(\mathbf{r}_1, \mathrm{r}_2)) \ , \qquad (9)$$

where, in accordance with Eq. (7), the structure function of the compensation error $D_{\varepsilon}(\mathbf{r}_1, \mathrm{r}_2)$ is

$$D_{\varepsilon}(\mathbf{r}_1, \mathbf{r}_2) = \overline{<[\varepsilon(\mathbf{r}_1) - (\mathbf{r}_2)]^2>} = D_{\varphi}(\mathbf{r}_1, \mathbf{r}_2) -$$

$$- 2\,[\mathbf{f}(\mathbf{r}_1) - \mathbf{f}(\mathbf{r}_2)]^{\mathrm{T}}\, \mathbf{F}^{-1}\mathbf{F}\, \mathbf{B}_{\xi\xi}^{-1}\, [\mathbf{H}(\mathbf{r}_1) - \mathbf{H}(\mathbf{r}_2)] \ +$$

$$+ [\mathbf{f}(\mathbf{r}_1) - \mathbf{f}(\mathbf{r}_2)]^{\mathrm{T}}\mathbf{F}^{-1}\mathbf{\Phi}\, \mathbf{B}^{-1}\Phi^{\mathrm{T}}\mathbf{F}^{-1}\, [\mathbf{f}(\mathbf{r}_1) - \mathbf{f}(\mathbf{r}_2)] \ . \qquad (10)$$

The derived equations (9) and (10) permit us, generally speaking, to calculate the Strehl factor k when the shape of the response function $\mathbf{f}(\mathbf{r})$ and the shape of the structure function of the phases of the wavefront of the initial signal $D_{\varphi}(\mathbf{r}_1, \mathbf{r}_2)$ are arbitrary. In this case, however, the main difficulty is in calculating the elements of the matrices $\mathbf{H}(\mathbf{r})$, $\mathbf{B}_{\xi\xi}$, and $\mathbf{\Phi}$. Such calculations are given in Appendix for a locally isotropic field of the phase distortions which are described by the structure function of the form $D_{\varphi}(\rho) = 2(\rho/\rho_c)^{5/3}$, where ρ_c is the coherence radius of the wave being received and by the response functions, which have an identical Gaussian shape and differ only by the point of exerting of the deforming force $\mathbf{r}_l$: $f_l(\mathbf{r}) = \exp(-\pi/S_T|\mathbf{r} - \mathbf{r}_l|^2)$, where S_T is the parameter which describes the spatial scale of the response function.

The resulting Strehl factors, which have been calculated using the formulas from Appendix, are given in Tables I–IV. In so doing, it was assumed that the center points of the apertures of the Hartmann sensors and the clamping points of the actuators were located equidistantly over a square aperture and their number was, generally speaking, different.

Table I illustrates the Strehl factor as a function of the number of the control channels m for the following fixed parameters of the problem: the number of the sensors of the measuring device $n = 9$, the signal–to–noise ratio in a sensor $Q = 10$, the number of background temporal modes $m_t = 10^3$ and of background spatial modes $m_r = 10^2$ (which are recorded by a sensor), and the number of coherence spots in the initial field over the AOS aperture $N_n = S_{\mathrm{a}}/\pi\rho_k^2 = 3$.

TABLE I.

m	4	9	16	∞	Note
K	0.359	0.505	0.588	0.935	$K_{\mathrm{n}} = 0.250$

The column of the table which corresponds to $m \to \infty$, describes the situation in which the reconstruction of the phases of the wavefront based on their estimated realizations (with errors) is performed absolutely correctly with the help of algorithm (2), and K_{n} denotes the Strehl factor of a nonadaptive system. The data of Table I indicate that algorithm (7) of reconstruction of the phase of the wavefront is efficient enough. In addition, the deterioration of the quality of compensation for the phase distortions of the wavefront associated with the flexible mirror control algorithm are much more significant than that at the stage of estimation of the phase of the wavefront. As the results of calculations show, such a situation lasts as far as the relation $N_{n/n}$ 1 is valid.

Which is more important for algorithm (2) either to measure the local characteristics, i.e., to choose small–size sensor apertures and separate them ($h = S/(S_a/n) < 1$, where S is the area of the sensor aperture and S/n is the area of the responsibility zone of an individual sensor) or to increase the signal–to–noise ratio at the expense of increasing the sensor aperture, is solved in accordance with Table II in favor of choosing a dense packing ($h = 1$) of the sensor subapertures.

TABLE II.

m	1	3	10	Note
0.3	0.536	0.351	0.198	$Q = 0.1$
	0.944	0.857	0.683	$Q = 10$
1	0.879	0.821	0.706	$Q = 0.1$
	0.973	0.935	0.833	$Q = 10$

When the number of the control channels m is fixed it is also important to provide the required width of the response function of the corrector in order to minimize the compensation error. The data of Table III indicate that, when varying the width of the response function (with the parameter $h_{\mathrm{T}} = S_{\mathrm{T}}/\left(\frac{S_{\mathrm{a}}}{n}\right)$ which has the sense of the degree of filling of the responsibility zone of the actuator by the response function), the optimum lies near $h_T = 1$. The physical interpretation is more or less obvious: it is necessary to arrange the actuator in such a way that any large gaps between the individual peaks of the response functions be absent (this is obvious to be bad), but their strong overlap is also unadmissible, since it does not enable processing of the small scales of the phase fluctuations.

TABLE III.

h_T	0.3	0.6	1	2	3	Note
K	0.242	0.344	0.505	0.425	0.324	$Nn = 3$ $n = m = 9$ $Q = 10$

It follows from Table II that in order to reach the Strehl factor K at least one sensor for one coherence spot is required (i.e., $N_n/h \le 1$). In addition, the signal–to–noise ratio on the sensor aperture Q must be of the order of 10.

The given formulas make it possible to take into account quite a large number of factors, which affect the AOS efficiency, both at the stage of estimating the phase of the wavefront (e.g., the parameters of the turbulent atmosphere are taken into account by means of N_n, the signal–to–noise ratio – by Q, the number of sensors – by n, and the packing density of the latter – by h) and at the stage of control with the flexible mirror (the number of the actuators is taken into account by m and their separation – by h_T).

APPENDIX

For the isotropic (locally isotropic) field $\varphi(\mathbf{r})$, it is convenient to calculate the matrix elements $\mathbf{H}(\mathbf{r})$, $\mathbf{B}_{\xi\xi}$, and $\mathbf{\Phi}$ using the spatial spectrum of phase fluctuations of the light wave $G_\varphi(\kappa)$, which describes the correlation properties of the wave. In this case, the cross–correlation functions of the phase of the wavefront at the point $\mathbf{r}$ on the receiving aperture and the readings of the wavefront tilt $H_i(\mathbf{r})$ of the arbitrary ith sensor as well as the correlation matrix elements of the measurement errors $\mathbf{B}_{\xi\xi}$, including both the dynamic and the background components, assume the forms

$$H_i(\mathbf{r}) = \frac{2\pi R}{|\mathbf{r} - \mathbf{r}_i|}\begin{bmatrix} x - x_i \\ y - y_i \end{bmatrix} \times$$

$$\times \int_0^\infty G_\varphi(k) G_0(k) J_1(k|\mathbf{r} - \mathbf{r}_1|) k^2 dk , \quad (11)$$

and

$$\mathbf{B}_{\xi\xi} = \mathbf{R}^2 \begin{bmatrix} \partial^2/\partial x_i \partial x_j & \partial^2/\partial x_i \partial y_i \\ \partial^2/\partial y_i \partial x_j & \partial^2/\partial y_i \partial y_j \end{bmatrix} \times$$

$$\times \int_0^\infty G_\varphi(k)\, G_0^2(k)\, J_0(k|\mathbf{r}_i - \mathbf{r}_j|) k dk +$$

$$+ \frac{1}{2}\sigma_\xi^2 \, E , \; i = 1, \ldots, n ; \; j = 1, \ldots, n . \quad (12)$$

The following notations are used in Eqs. (11) and (12): $\mathbf{r}_i = (x_i, y_i)$ is the radial distance of the aperture center of the ith sensor, $G_0(\kappa) = \dfrac{2J_1(kR)}{kR}$ is the filtering function of the receiving aperture of the sensor with radius R, $J_0(\ldots)$ and $J_1(\ldots)$ are the zeroth and first order Bessel functions, and $\mathbf{E}$ is the $(2n\times 2n)$ unit matrix.

When calculating the matrix elements Φ we will integrate between the infinite limits thereby neglecting the edge effects (which is justified for $m > 1$). In this case,

$$\Phi_{il} = \frac{2\pi R}{|\mathbf{r}_i - \mathbf{r}_l|}\begin{bmatrix} x_i - x_l \\ y_i - y_l \end{bmatrix} \times$$

$$\times \frac{1}{S_a}\int_0^\infty G_\varphi(k)\, G_0(k)\, G_f(k)\, J_1(k|\mathbf{r}_1 - \mathbf{r}_l|) k^2 dk , \quad (13)$$

where $G_f(\kappa) = \int_{-\infty}^{\infty} f(r) \exp(i\kappa \mathbf{r} d^2 r)$ is the spatial spectrum of the response function.

The exponential spectral density $G_\varphi(\kappa) = \dfrac{5}{3}\dfrac{\Gamma(11/6)}{\Gamma(1/6)}\dfrac{2^{2/3}}{\pi\rho_c^{5/3}}\kappa^{-11/3}$ corresponds to the structure function of the phases the wavefront of the form $D_\varphi(\rho) = 2(\rho/\rho_c)^{5/3}$, which is employed here. Although here we do not succeed in finding the exact values of the integrals, which enter in Eqs. (11), (12), and (13), we can, however, derive the related approximate formulas. Let us give the detailed calculations of the integral of Eq. (11) as an example

$$I(|\mathbf{r} - \mathbf{r}_i|, R) = \int_0^\infty k^{-8/3} J_1(k|\mathbf{r} - \mathbf{r}_i|)\, J_1(kR)\, dk . \quad (14)$$

This integral is a symmetric function about the parameters $|\mathbf{r} - \mathbf{r}_i|$ and R. For this reason, when approximately calculating it, it is convenient to consider two regions: $|\mathbf{r} - \mathbf{r}_i| \le R$ (i.e., $\mathbf{r} - \mathbf{r}_i \in \Omega_i$) and $|\mathbf{r} - \mathbf{r}_i| > R$ ($\mathbf{r} - \mathbf{r}_i \notin \Omega_i$). If $|\mathbf{r} - \mathbf{r}_i|/R \le 1$, then it is important to take the tails of the function $J_1(\kappa R)$ into account while to expand the function $\mathbf{J}_1(\kappa|\mathbf{r} - \mathbf{r}_i|)$ in a series, and vice versa, if $|\mathbf{r} - \mathbf{r}_i|/R > 1$, then $J(\kappa|\mathbf{r} - \mathbf{r}_i|))$ should be preserved while the second multiplier should be expanded in a series. Taking into account the first three terms of the series we will obtain

$$H_i(\mathbf{r}) \approx \frac{5}{3}\left(\frac{N_p h}{n}\right)^{5/6}\begin{bmatrix} \dfrac{x - x_i}{R} \\ \dfrac{y - y_i}{R} \end{bmatrix} \times$$

$$\times \begin{cases} 1 - \dfrac{5}{72}\left(\dfrac{|\mathbf{r}-\mathbf{r}_i|}{R}\right)^2 - \dfrac{35}{15552}\left(\dfrac{|\mathbf{r}-\mathbf{r}_i|}{R}\right)^4 ; \\ \text{for } \left(\dfrac{|\mathbf{r}-\mathbf{r}_i|}{R}\right) \le 1 \\ \left(\dfrac{R}{|\mathbf{r}-\mathbf{r}_i|}\right)^{1/3}\left[1 - \dfrac{5}{72}\left(\dfrac{R}{|\mathbf{r}-\mathbf{r}_i|}\right)^2 - \right. \\ \left. - \dfrac{35}{1555}\left(\dfrac{R}{|\mathbf{r}-\mathbf{r}_i|}\right)^4\right]; \text{ for } \left(\dfrac{|\mathbf{r}-\mathbf{r}_i|}{R}\right) > 1, \end{cases} \quad (15)$$

where $N_n = S_a/\pi\rho$ is the number of coherence spots on the aperture of the AOS and $h = S/(S_a/n)$ is the ratio of the area of the sensor aperture $S = \pi R^2$ to the area of responsibility zone S_a/n of an individual sensor.

It is obvious that the largest error in calculating integral (14) is obtained when $|\mathbf{r} - \mathbf{r}_i|/R = 1$. However, in this case the integral can be calculated exactly

$$I(R, R) = \frac{30}{33}\frac{2^{-8/3}}{R^{1/3}}\frac{\Gamma(1/6)\,\Gamma(5/3)}{\Gamma^3(11/6)}. \qquad (16)$$

Comparing the latter relation with the approximate one we obtain that the error of the approximate relation does not exceed 0.2%.

The integrals in Eq. (13) are calculated in the same way

$$\Phi_{il} = \frac{5}{3}\left(\frac{N_{\mathrm{p}}}{m}\,h_{\mathrm{T}}\right)^{5/6}\cdot\sqrt{\frac{h\cdot h_{\mathrm{T}}}{n\cdot m}}\begin{bmatrix}\dfrac{x_i - x_l}{R_{\mathrm{T}}}\\ \dfrac{y_i - y_l}{R_{\mathrm{T}}}\end{bmatrix}\times$$

$$\begin{cases} {}_1F_1\!\left(1/6, 2; -\dfrac{h}{h_{\mathrm{T}}}\dfrac{m}{n}\right) - \dfrac{\mathrm{B}}{12}\,{}_1F_1\!\left(7/6, 2; -\dfrac{h}{h_{\mathrm{T}}}\dfrac{m}{n}\right) + \\ + \dfrac{7}{432}\mathrm{B}^2\,{}_1F_1\!\left(13/6, 2; -\dfrac{h}{h_{\mathrm{T}}}\dfrac{m}{n}\right)\ \text{for}\ \dfrac{|\mathrm{r}_i - \mathrm{r}_l|}{R} \le 1; \\ {}_1F_1(1/6, 2; -\mathrm{B}) - \dfrac{1}{12}\dfrac{h}{h_{\mathrm{T}}}\dfrac{m}{n}\,{}_1F_1(7/6, 2; -\mathrm{B}) + \\ + \dfrac{7}{432}\left(\dfrac{h}{h_{\mathrm{T}}}\dfrac{m}{n}\right)^2 {}_1F_1(13/6, 2; -\mathrm{B})\ \text{for}\ \dfrac{|\mathrm{r}_i - \mathrm{r}_l|}{R} > 1, \end{cases} \qquad (17)$$

where $\mathrm{B} = \dfrac{\pi|\mathbf{r}_i - \mathbf{r}_l|}{S_{\mathrm{T}}}$, $h_{\mathrm{T}} = S_{\mathrm{T}}/(S_{\mathrm{a}}/n)$ is the coefficient of filling of the responsibility zone of the actuator by the response function, and ${}_1F_1(\alpha, \beta; x)$ is the degenerated hypergeometric function. The largest error of calculating integral (14) occurs when $(|\mathbf{r}_i - \mathbf{r}_l| = R$ as $h_{\mathrm{T}}\to 0$. But an exact value of the integral in this situation has already been calculated and is given by formula (16), and since in this case $\Phi_{il} = H_i(\mathbf{r}_i)$, the error of approximate relation (17) also does not exceed 0.2%.

We succeed in calculating the integral in Eq. (13) with the use of the quite exact approximation for the squared filtering function of the sensor aperture $G_0^2(\kappa) \simeq \exp\left(-\dfrac{\mathrm{k}^2\mathrm{R}^2}{4}\right)$ (see Ref. 11). In so doing, instead of Eq. (12) we obtain

$$B_{\zeta\zeta} = \frac{5}{3}\,\Gamma(11/6)\left(\frac{N_{\mathrm{p}}}{n}\,h\right)^{5/6}\begin{bmatrix}P(x_i, x_j) & s(x_i, y_j)\\ s(y_i, x_j) & P(y_i, y_j)\end{bmatrix} + \frac{1}{2}\,\sigma_\zeta^2 E, \qquad (18)$$

where

$$P(x_i, x_j) = {}_1F_1\!\left(1/6,\ 2;\ -\frac{|\mathbf{r}_i - \mathbf{r}_j|^2}{R^2}\right) - \frac{1}{6}\left(\frac{x_i - x_j}{R}\right)^2 {}_1F_1\!\left(7/6,\ 3;\ -\frac{|\mathbf{r}_i - \mathbf{r}_j|^2}{R^2}\right);$$

$$s(x_i, y_j) = s(y_i, x_j) = -\frac{1}{6}\frac{(x_i - x_j)\,(y_i - y_j)}{R^2}\times {}_1F_1\!\left(7/6,\ 3;\ -\frac{|\mathbf{r}_i - \mathbf{r}_j|^2}{R^2}\right).$$

It follows from Eq. (18) that the diagonal elements of the matrix of the measurement errors are equal to $\dfrac{5}{3}\,\Gamma(11/16)\left(\dfrac{N_n h}{n}\right)^{5/6} + \dfrac{\sigma_\zeta^2}{2}$. At the same time, it is easy to find an exact expression for the diagonal elements which is equal to $\dfrac{50}{33}\dfrac{\Gamma(5/3)}{\Gamma^2(11/6)}\left(\dfrac{N_{\mathrm{n}} h}{n}\right)^{5/6} + \dfrac{\sigma_\zeta^2}{2}$. Thus, the error of the approximate relation for the elements of the correlation matrix (13) does not exceed 1.5% (at least, in calculating the variances).

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