Method of moments in the problem of wave front reconstruction from images

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New moment relationships between the function of wave front aberrations of an optical system and numerical characteristics (moments) of intensity distribution in the image are obtained. These are used as a basis for the method of wave front reconstruction from images of a source in several parallel planes.

Introduction

Among the methods of wave front (WF) reconstruction in an optical system (OS), the method based on analysis of images of a point source in several planes, parallel to the focal plane, is the simplest-to-realize and, at the same time, most poorly studied.

If WF distortions are presented in the form of the expansion into a series over some system of basis functions with unknown coefficients, then the problem of WF reconstruction reduces to the estimation of these coefficients from images or image functionals.

Let $G = A(\xi,\eta)\exp[k\Phi(\xi,\eta)]$ be the OS pupil function, describing the amplitude $A(\xi, \eta)$ and phase $\Phi(\xi, \eta)$ distortions of the wave field at any point (ξ, η) of the area of the exit pupil Ω; $k = 2\pi/\lambda$ is the wave number. Denote the intensity distribution in the image plane $z = \text{const}$ ($z = 0$ corresponds to the focal plane) as $I(x, y, z)$.

There are several approaches to WF reconstruction from images:

1. To find the wave function G from the known distributions $I(x, y, z_s)$, $s = \overline{1,S}$, in a limited area.
2. To find the function of aberrations Φ from the known distributions $I(x, y, z_s)$, $s = \overline{1,S}$, in a limited area and the amplitude A.
3. If the function of aberrations is specified by the partial sum of the series

$$\Phi = \sum c_k \varphi_k(\xi,\eta);$$

to estimate the coefficients $\tilde{c}_k$ (modes) in the form of image functionals $\tilde{c}_k(I)$ at the known or unknown amplitude.

The most flexible method for the solution of problems 1 and 2 is the method, reducing their solution to the geometrical problem of finding the common point of the given sets. Problem 1 can be solved in the geometrical treatment by the numerical Fienup algorithm[1] or the algorithm of increased dimensionality,[2] problem 2 can be solved by the Gerchberg–Saxton algorithm[3] and the algorithm of increased dimensionality. The solution of problem 3 can be retrieved from the solution of problems 1 and 2, or it can be solved independently by the iteration method.[4–6] Problem 3, however, allows the solution to be sought in the explicit form. One of such approaches is based on the calculation or measurement of image functionals of the form

$$\int\!\!\int_{-\infty}^{+\infty} \frac{\mathrm{d}^k I(0)}{\mathrm{d}z^k} x^p y^q \mathrm{d}x\mathrm{d}y \,. \qquad (1)$$

This method, called the moment method,[7,8] employs the linear relationships, connecting gradΦ with the values (1).

In this study, new relationships have been obtained. They have more general form than Eq. (1) and relate the function of WF aberrations to the numerical characteristics (moments) of the intensity distribution in the image. A mathematical apparatus based on the functions of a complex variable is proposed. For this reason, it has become simpler to find the explicit dependence of the estimates $\tilde{c}_k$ on the image functionals.

1. Generalized moment relationship

Let a remote point source be on the optical axis of an optical system and R be the radius of the ideal wave front (Gaussian sphere). The transverse aberrations $\bar{r}$ of the rays in the focal plane are related to the wave aberrations through the equality[9]:

$$\bar{r} = R\,\mathrm{grad}\Phi\,.$$

The point $\bar{r} = (x,y)$ in the focal plane conforms to a set of points in the pupil

$$\Omega(\bar{r}) = \{(\xi,\eta) \in \Omega: \; R\,\mathrm{grad}\Phi(\xi,\eta) = \bar{r}\}.$$

The set $\Omega(\bar{r})$ determines the elements of the distorted wave front, whose radiant energy is transported by the rays, passing through the point $\bar{r}$.

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The energy $I(x, y)\mathrm{d}x\mathrm{d}y$, transported by the rays from the set $\Omega(\bar{r})$, passes through the element $\mathrm{d}x\mathrm{d}y$, including the point $\bar{r}$. In geometrical optics, we have the energy equality

$$I(x,y)\mathrm{d}x\mathrm{d}y = \sum_{(\xi,\eta)\in\Omega(\bar{r})} A^2(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta.$$

Let $\varphi(\bar{r})$ be a function of (x, y). This function may be complex-valued. The energy element $I(x, y)\mathrm{d}x\mathrm{d}y$ has a moment

$$\varphi(x,y)I(x,y)\mathrm{d}x\mathrm{d}y = \varphi(R\mathrm{grad}\Phi) \sum_{(\xi,\eta)\in\Omega(\bar{r})} A^2(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta =$$

$$= \sum_{(\xi,\eta)\in\Omega(\bar{r})} \varphi[R\mathrm{grad}\Phi(\xi,\eta)]\, A^2(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta.$$

The total moment of all energy elements is

$$\int\limits_{-\infty}^{+\infty}\!\!\int \varphi(x,y)\, I(x,y)\mathrm{d}x\mathrm{d}y =$$

$$= \iint\limits_{\Omega} \varphi[R\mathrm{grad}\Phi(\xi,\eta)]A^2(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta. \qquad (2)$$

It is just this expression we call the generalized moment relationship. For the power-law function $\varphi = x^p y^q$, equality (2) was proved within the frames of the wave theory of light in Refs. 7 and 8 for a sufficiently smooth pupil function and in Ref. 8 using geometrical optics without the requirement of the smooth amplitude A.

Moment relationship in the complex form

The points of the pupil plane (ξ, η) and the focal plane (x, y) determine the complex variables $\zeta = (\xi + i\eta)/a = \rho\exp(i\theta)$ and $w = (a/\lambda R)(x + iy)$, where a is the characteristic size of the exit pupil. The function of aberrations can be represented in the form

$$\Phi = \lambda\mathrm{Re}\Psi\left(\zeta,\bar{\zeta}\right), \qquad (3)$$

where the function Ψ has the derivative with respect to its arguments. Then

$$\partial\Phi/\partial\xi = (\lambda/a)\mathrm{Re}(\partial\Psi/\partial\zeta + \partial\Psi/\partial\bar{\zeta}),$$

$$\partial\Phi/\partial\eta = (\lambda/a)\mathrm{Re}\,i(\partial\Psi/\partial\zeta - \partial\Psi/\partial\bar{\zeta}) =$$

$$= (\lambda/a)\mathrm{Im}(-\partial\Psi/\partial\zeta + \partial\Psi/\partial\bar{\zeta}).$$

The vector $\mathrm{grad}\Phi$ specifies a complex-valued function

$$(\lambda R/a)[\overline{(\partial\Psi/\partial\zeta)} + \partial\Psi/\partial\bar{\zeta}] =$$

$$= R(\partial\Phi/\partial\xi + i\partial\Phi/\partial\eta) = x + iy$$

of the variable ζ. Hence, it follows that WF distortions, deflecting light rays, define the function of a complex variable

$$w = w(\zeta) = \overline{(\partial\Psi/\partial\zeta)} + \partial\Psi/\partial\bar{\zeta},$$

which maps the points of the exit pupil into the points of the focal plane. The moment relationship (2) in the complex form is as follows

$$\int\limits_{-\infty}^{+\infty}\!\!\int \varphi(w) I(x,y)\mathrm{d}x\mathrm{d}y =$$

$$= a^2 \iint\limits_{\Omega} \varphi[w(\zeta)]A^2(\rho,\theta)\rho\mathrm{d}\rho\mathrm{d}\theta. \qquad (4)$$

Representation of the primary aberrations in a complex form

The general WF tilt introduces the following component into the function Φ:

$$\Phi_1^1/\lambda = A\rho\cos(\theta) + B\rho\sin(\theta) = \mathrm{Re}C\zeta,$$

where $C = A - iB$, and defines the transformation

$$w_1^1 = \overline{(\partial\Psi_1^1/\partial\zeta)} = \bar{C}.$$

Astigmatism introduces the component

$$\Phi_2^2/\lambda = A\rho^2\cos(2\theta) + B\rho^2\sin(2\theta) = \mathrm{Re}C\zeta^2$$

and defines the transformation

$$w_2^2 = 2\overline{C\zeta}.$$

Coma introduces the component

$$\Phi_3^1/\lambda = \rho^2[A\rho\cos(\theta) + B\rho\sin(\theta)] = \mathrm{Re}C\zeta^2\bar{\zeta}$$

and defines the transformation

$$w_3^1 = 2\bar{C}\zeta\bar{\zeta} + C\zeta^2.$$

Defocusing and spherical aberration introduce the components

$$\Phi_2^0/\lambda = A(2\rho^2 - 1) = A(2\zeta\bar{\zeta} - 1),$$

$$\Phi_4^0/\lambda = A(6\rho^4 - 6\rho^2 + 1) = A[6(\zeta\bar{\zeta})^2 - 6\zeta\bar{\zeta} + 1]$$

and define the transformations

$$w_2^0 = 4A\zeta \ \text{ and } \ w_4^0 = 12A\zeta(2\zeta\bar{\zeta} - 1).$$

Complex representation of transformations, corresponding to Zernike polynomials

In the case of a circular pupil of the radius a, the function of aberrations is often expanded into a series over the Zernike polynomials. Every component of the expansion in the real form looks like[9]:

$$V_n^m(\rho,\theta) = R_n^m(\rho)[A_n^m\cos(m\theta) + B_n^m\sin(\theta)],$$

where

$$R_n^m(\rho) = \rho^m \sum_{s=0}^{(n-m)/2} D_{n,s}^m (\rho^2)^{\frac{n-m}{2}-s} = \rho^m r_n^m(\rho^2),$$

$n - m \geq 0$ is an even number and

$$D^m_{n,s} = (-1)^s (n-s)!\Big/\left[s!\left(\frac{n+m}{2}-s\right)!\left(\frac{n-m}{2}-s\right)!\right].$$

Assuming $C^m_n = A^m_n - iB^m_n$, we obtain the complex form of the terms of the expansion over the Zernike polynomials:

$$V^m_n = \mathrm{Re}\, C^m_n \zeta^m r^m_n(\zeta\bar{\zeta}),$$

defined by the functions

$$\Psi^m_n(\zeta,\bar{\zeta}) = \zeta^m r^m_n(\zeta\bar{\zeta}).$$

The terms of the series define the transformations $w^m_n(\zeta)$, whose form depends on m and n. At $n > m = 0$

$$w^0_n = 2A^0_n \zeta (r^0_n)',$$

where $(r^0_n)'$ is the derivative with respect to the variable $\zeta\bar{\zeta}$.

At $n = m > 0$

$$w^m_m = m\bar{C}^m_n \bar{\zeta}^{m-1}.$$

At $n > m > 0$

$$w^m_n = \bar{C}^m_n \bar{\zeta}^{m-1}\left[m r^m_n + \zeta\bar{\zeta}(r^m_n)'\right] + C^m_n \zeta^{m+1}(r^m_n)'.$$

The transformations w^m_n, corresponding to the primary aberrations, were obtained in Ref. 9. The transformations w^m_n, presented in a complex form are more obvious and general. The circles ρ = const are converted with the aid of the transformations into the curves, referred to as aberration ones. The transformation w^0_n gives the identical extension to the radius vectors of the points on the circle ρ = const. Therefore, the aberration curves are circles. The transformation w^m_m performs the mentioned extension, rotation by an angle $\arg C^m_m + (m-2)\theta$, and symmetry about the real axis.

The transformation $w^l_n,\ n>1$, reduces to the following: uniform extension, rotation by an angle $\arg C^l_n + \theta$, and shift by the value of the lth term in the equation for w^l_n. Therefore, the aberration curves of the transformation w^l_n are shifted circles.

The transformation $w^m_n,\ n>m>1$, can be written in the form

$$w^m_n = (1/\zeta^{m-l})\left\{\bar{C}^m_n(\zeta\bar{\zeta})^{m-l}[m r^m_n + \zeta\bar{\zeta}(r^m_n)'] + C^m_n \zeta^{2m}(r^m_n)'\right\}$$

and it is a composition of two transformations:

$$w^m_n = (1/\zeta^{m-l}) \otimes (w^m_n)_l,$$

where the transformation $(w^m_n)_l$ is analogous to the transformation w^l_n, in which the rotation is performed by an angle $\arg C^m_n + (2m-l)\theta$.

Determination of the Zernike modes from the intensity distribution in the image space

The moment relationship (4) is written for the focal plane. If a measurement is carried out in a non-focal plane $z \neq 0$, then at small z this measurement corresponds to the intensity in the focal plane if the pupil function is modified by the phase factor[9]:

$$\exp[-ik(a/2R)^2 z(2\rho^2 - 1)].$$

This change reduces to the change of the Zernike defocusing coefficient A^0_2, which should be replaced by

$$A^0_2(z) = A^0_2 - (a/2R)^2 z/\lambda = A^0_2 - (1/8\pi)\bar{z},$$

$$\bar{z} = k(a/R)^2 z.$$

Having the intensity $I(x, y, z)$, we have to find the Zernike modes C^m_n. One of the ways to solve this problem consists in the following. For different φ, we have to compose the system of equations from the relationships (4) with the known left-hand side for the determination of the modes. The right-hand side of Eq. (4) includes the amplitude A, which should be measured or determined from the intensity $I(x, y, z)$ by using the corresponding moment relationships.[8]

In the ideal case, to solve the problem on determination of the modes, we have to select such functions φ^m_n, at which the left-hand sides of the relationships (4) immediately give the estimates of the mode $\tilde{C}^m_n$. No solution of this form is known yet. However, for the case that the amplitude is independent of the angular coordinate $A = A(\rho)$, it will be shown that for any $m = m_0$ it is possible to construct such linear combinations of the moments (1), which depend only on the modes $C^{m_0}_n$. This means that the modes corresponding to different m can be determined separately.

Let the function of aberrations taking into account the controlled defocusing z be determined by the partial sum of the series

$$\Phi = \lambda \mathrm{Re}[\Psi(\zeta,\bar{\zeta}) - (1/8\pi)\bar{z}(2\zeta\bar{\zeta} - 1)],$$

where

$$\Psi(\zeta,\bar{\zeta}) = \sum_{m=0}^{M}\sum_{n=m}^{N(m)} C^m_n \Psi^m_n(\zeta,\bar{\zeta}).$$

The WF distortion corresponds to the transformation

$$w = w(\zeta,\bar{z}) = \sum_{m=0}^{M}\sum_{n=m}^{N(m)} w^m_n(\zeta) - (1/2\pi)\bar{z}\zeta =$$

$$= w(\zeta) - (1/2\pi)\bar{z}\zeta.$$

Take the function φ equal to $\varphi(w) = w^{k+l}\bar{w}^l,\ 0 \le k \le M$. It corresponds to the moment relationship (4) of the form

$$M_{k+2l}^{k}(\bar{z}) = \int\limits_{-\infty}^{+\infty}\int w^{k+l}\bar{w}^{l} I(x,y,\bar{z})\mathrm{d}x\mathrm{d}y =$$

$$= a^{2}\iint\limits_{\Omega} w^{k+l}(\zeta,\bar{z})\bar{w}^{l}(\zeta,\bar{z})A^{2}(\rho)\rho\mathrm{d}\rho\mathrm{d}\theta.$$

Since $w(\zeta,\bar{z})$ linearly depends on $\bar{z}$, the moment $M_{k+2l}^{k}(\bar{z})$ is a polynomial of degree $k + 2l$ in terms of $\bar{z}$.

With the aid of the Leibniz formula for the higher derivative of a product, we find that

$$[w^{k+l}(\zeta,\bar{z})\bar{w}^{l}(\zeta,\bar{z})]_{\bar{z}=0}^{(k+2l-1)} = \binom{l-1}{k+2l-1}(w^{k+l})^{(k+l)}(\bar{w}^{l})^{(l-1)} +$$

$$+\binom{l}{k+2l-1}(w^{k+l})^{(k+l-1)}(\bar{w}^{l})^{(l)} =$$

$$= \binom{l-1}{k+2l-1}(k+l)!\,l!\left(-\frac{1}{2\pi}\right)^{k+2l-1}\zeta^{k+l}\bar{\zeta}^{l-1}\bar{w}(\zeta) +$$

$$+\binom{l}{k+2l-1}(k+l)!\,l!\left(-\frac{1}{2\pi}\right)^{k+2l-1}\zeta^{k+l-1}\bar{\zeta}^{l}w(\zeta) =$$

$$= (k+2l-1)!(-1/2\pi)^{k+2l-1}\times$$

$$\times[l\zeta^{k+l}\bar{\zeta}^{l-1}\bar{w}(\zeta) + (k+l)\zeta^{k+l-1}\bar{\zeta}^{l}w(\zeta)].$$

Therefore

$$[M_{k+2l}^{k}(0)]^{(k+2l-1)} = (k+2l-1)!(-1/2\pi)^{k+2l-1}a^{2}\times$$

$$\times\int\limits_{0}^{2\pi}\int\limits_{0}^{1}[l\zeta^{k+l}\bar{\zeta}^{l-1}\bar{w}(\zeta) + (k+l)\zeta^{k+l-1}\bar{\zeta}^{l}w(\zeta)]A^{2}(\rho)\rho\mathrm{d}\rho\mathrm{d}\theta. \quad (5)$$

Consider the integral over θ in the equality (5):

$$\int\limits_{0}^{2\pi}[\ldots]\mathrm{d}\theta =$$

$$= \sum_{m=0}^{M}\sum_{n=m}^{N(m)}\int\limits_{0}^{2\pi}[l\zeta^{k+l}\bar{\zeta}^{l-1}\bar{w}_{n}^{m}(\zeta) + (k+l)\zeta^{k+l-1}\bar{\zeta}^{l}w_{n}^{m}(\zeta)]\mathrm{d}\theta =$$

$$= \sum_{m=0}^{M}\sum_{n=m}^{N(m)}[l\int\limits_{0}^{2\pi}\zeta^{k+l}\bar{\zeta}^{l-1}\Big\{C_{n}^{m}[\zeta^{m-1}mr_{n}^{m} + \zeta^{m}\bar{\zeta}(r_{n}^{m})'] +$$

$$+\bar{C}_{n}^{m}\bar{\zeta}^{m+1}(r_{n}^{m})'\Big\}\mathrm{d}\theta +$$

$$+(k+l)\int\limits_{0}^{2\pi}\zeta^{k+l-1}\bar{\zeta}^{l}\Big\{\bar{C}_{n}^{m}[\bar{\zeta}^{m-1}mr_{n}^{m} + \bar{\zeta}^{m}\zeta(r_{n}^{m})'] +$$

$$+C_{n}^{m}\zeta^{m+1}(r_{n}^{m})'\Big\}\mathrm{d}\theta] =$$

$$= \sum_{m=0}^{M}\sum_{n=m}^{N(m)}[l\int\limits_{0}^{2\pi}\Big\{C_{n}^{m}[\zeta^{m+k+l-1}\bar{\zeta}^{l-1}mr_{n}^{m} + \zeta^{m+k+l}\bar{\zeta}^{l}(r_{n}^{m})'] +$$

$$+\bar{C}_{n}^{m}\zeta^{k+l}\bar{\zeta}^{m+l}(r_{n}^{m})'\Big\} +$$

$$+(k+l)\int\limits_{0}^{2\pi}\Big\{\bar{C}_{n}^{m}[\zeta^{k+l-1}\bar{\zeta}^{m+l-1}mr_{n}^{m} + \zeta^{k+l}\bar{\zeta}^{m+l}(r_{n}^{m})'] +$$

$$+C_{n}^{m}\zeta^{m+k+1}\bar{\zeta}^{l}(r_{n}^{m})'\Big\}\mathrm{d}\theta].$$

A nonzero contribution comes from the integrals, which depend only on the product $\zeta\bar{\zeta}$.

At $k = 0$

$$\int\limits_{0}^{2\pi}[\ldots]\mathrm{d}\theta = 8\pi\sum_{n=0}^{N(0)} lA_{n}^{0}(\zeta\bar{\zeta})^{l}(r_{n}^{0})'.$$

At $k \neq 0$

$$\int\limits_{0}^{2\pi}[\ldots]\mathrm{d}\theta = 2\pi\sum_{n=0}^{N(k)} C_{n}^{k}\Big\{l(\zeta\bar{\zeta})^{k+l}(r_{n}^{k})' +$$

$$+(k+l)[k(\zeta\bar{\zeta})^{k+l-1}r_{n}^{k} + (\zeta\bar{\zeta})^{k+l}(r_{n}^{k})']\Big\} =$$

$$= 2\pi\sum_{n=0}^{N(k)} C_{n}^{k}[(k+2l)(\zeta\bar{\zeta})^{k+l}(r_{n}^{k})' + (k+l)k(\zeta\bar{\zeta})^{k+l-1}r_{n}^{k}].$$

Taking into account the integrals over θ, we obtain the following form of the derivatives at $k = 0$:

$$[M_{2l}^{0}(0)]^{2l-1} =$$

$$= -\frac{(2l-1)!}{(-2\pi)^{2l-2}}4a^{2}l\sum_{n=0}^{N(0)}A_{n}^{0}\int\limits_{0}^{1}\rho^{2l}(r_{n}^{0})'A^{2}(\rho)\rho\mathrm{d}\rho, \quad (6)$$

at $k \neq 0$

$$[M_{k+2l}^{k}(0)]^{k+2l-1} = -\frac{(k+2l-1)!}{(-2\pi)^{k+2l-2}}a^{2}\times$$

$$\times\sum_{n=0}^{N(k)}C_{n}^{k}\int\limits_{0}^{l}\Big[(k+2l)\rho^{2}(r_{n}^{0})' + k(k+l)r_{n}^{k}\Big]\rho^{2(k+l-1)}A^{2}(\rho)\rho\mathrm{d}\rho. \quad (7)$$

The relationships (6) and (7), written at different l, serve for separate determination of the coefficients C_{n}^{k} at a given k. The relationship (6) results from Eq. (7) after multiplication by two.

Determination of the primary aberrations from the moment relationships

Let the wave front be determined only by the primary aberrations

$$\Phi = \lambda\,\mathrm{Re}\Big\{A_{2}^{0}(2\zeta\bar{\zeta}-1) + A_{4}^{0}[6(\zeta\bar{\zeta})^{2} - 6\zeta\bar{\zeta} + 1] +$$

$$+C_{1}^{1}\zeta + C_{3}^{1}(3\zeta\bar{\zeta}-2)\zeta + C_{2}^{2}\zeta^{2}\Big\}.$$

Then at $k = 0$ and $l = 1, 2$ we have a system for the determination of A_2^0 and A_4^0:

$$[M_2^0(0)]' =$$

$$= -8a^2\left[A_2^0\int_0^1\rho^2A^2(\rho)\,\rho d\rho + 3A_4^0\int_0^1\rho^2(2\rho^2-1)\,A^2(\rho)\,\rho d\rho\right],$$

$$[M_4^0(0)]''' = -\frac{3!}{(2\pi)^2}16a^2\times$$

$$\times\left[A_2^0\int_0^1\rho^4A^2(\rho)\,\rho d\rho + 3A_4^0\int_0^1\rho^4(2\rho^2-1)\,A^2(\rho)\,\rho d\rho\right].$$

At $k = 1$ and $l = 1, 2$ we have a system for the determination of $\bar{C}_1^1$ and $\bar{C}_3^1$:

$$M_1^1(0) = 2\pi a^2\left[\bar{C}_1^1\int_0^1A^2(\rho)\rho d\rho + \bar{C}_3^1\int_0^1(6\rho^2-2)A^2(\rho)\rho d\rho\right],$$

$$[M_3^1(0)]'' =$$

$$= a^2/\pi\left[2\bar{C}_1^1\int_0^1\rho^2A^2(\rho)\rho d\rho + \bar{C}_3^1\int_0^1\rho^2(15\rho^2-4)A^2(\rho)\rho d\rho\right].$$

At $k = 2$ and $l = 0$ we have equations for the determination of C_2^2:

$$[M_2^2(0)]' = -4a^2C_2^2\int_0^1\rho^2A^2(\rho)\,\rho d\rho.$$

2. Application of the complex representation of WF to the problem on the reconstruction of modes from the data acquired with a Hartmann sensor

In the Hartmann sensor, the WF local tilts are measured at a discrete set of points of the exit pupil ω. The problem is to estimate the function of aberration from its gradient at a discrete set ω. There are several methods to solve this problem. One is to represent the function Φ as a section of a series over some system of basis functions with unknown coefficients, which are found by the least-squares method. Selecting the system of basis functions essentially effects the calculation of the serial coefficients. In this section, we propose such a series expansion of the function Φ, which, in our opinion, significantly simplifies the calculation of the sought coefficients.

Let the function Φ be specified by Eq. (3), in which the complex function Ψ is represented by the partial sum of the series of the following form:

$$\Psi = \sum_{m=0}^{M}C^m\left(\zeta\bar{\zeta}\right)\zeta^m. \qquad (8)$$

To be determined are the real C^0 and complex C^m coefficients, depending on the argument $\zeta\bar{\zeta} = \rho^2$. The representation of Φ by the equalities (3) and (8) includes the expansion over the Zernike polynomials. The Hartmann data define the values of the transformation $w(\zeta)$ on ω. Taking into account Eq. (8),

$$w(\zeta) = 2\zeta(C^0)' + \sum_{m=1}^{M}\bar{\zeta}^{m-1}[m\bar{C}^m + \zeta\bar{\zeta}(\bar{C}^m)'] + \zeta^{m+1}(C^m)'.$$

It is assumed that from the values of $w(\zeta)$ on ω it is possible to estimate $w(\zeta)$ at the points $\zeta_k = \rho\exp(i2\pi k/N)$, $0 \le k \le N-1$, where the natural number $N = 2M + 1$ is determined by the sampling theorem. Then at the point ζ_k

$$(\bar{\zeta}w)_k = \bar{\zeta}_k w(\zeta_k) = 2(C^0)'\rho^2 +$$
$$+\sum_{m=1}^{M}\exp(-i2\pi km/N)a_m(\rho) +$$
$$+\exp[-i2\pi k(N-m)/N]\,a_{N-m}(\rho),$$

where we used the designation

$$a_m(\rho) = \rho^m[m\bar{C}^m + \rho^2(\bar{C}^m)'],\quad a_{N-m}(\rho) = \rho^{m+2}(C^m)'.$$

Assume that $a_0 = 2(C^0)'\rho^2$, then

$$(\zeta w)_k = a_0(\rho) + \sum_{m=1}^{M}\exp(-i2\pi km/N)\,a_m(\rho) +$$
$$+\sum_{m=1}^{M}\exp(-i2\pi k(N-m)/N)\,a_{N-m}(\rho) =$$
$$= \sum_{m=0}^{N-1}\exp(-i2\pi km/N)\,a_m(\rho),\quad k = \overline{0, N-1}.$$

It follows from this equality that the values of $(\bar{\zeta}w)_k$ are the coordinates of the vector of the discrete Fourier transform of the vector $(a_0(\rho), ..., a_{N-1}(\rho))$, and therefore

$$a_m(\rho) = (1/N)\sum_{k=0}^{N-1}\exp(i2\pi km/N)(\bar{\zeta}w)_k,\quad m = \overline{0, N-1}.$$

Thus, from the values of $w(\zeta_k)$ it is possible to find the values of $a_m(\rho)$ and from them to obtain

$$2\rho^2(C^0)' = a_0 \qquad (9)$$

and

$$m\rho^mC^m(\rho^2) = \bar{a}_m(\rho) - a_{N-m}(\rho),\quad m = \overline{1, M}. \qquad (10)$$

The equalities (10) directly determine the values of $\rho^mC^m(\rho^2)$, entering into the expansions (3) and (8).

The equality (9) determines only the derivative of the coefficient $C^0(\rho^2)$, which should be used to estimate the coefficient itself, characterizing the rotation-symmetric component of the wave front.

Conclusions

The method of derivation of the moment relationships (2) and their applications form the theoretical basis for the development and justification of algorithms for reconstruction of WF modes from images of a source. The expansions (3) and (8) are rational in the sense that the Hartmann data and the coefficients of the expansion are related by a simple dependence, *viz.*, by discrete Fourier transform. In this case, the integration is needed only to find the coefficient, corresponding to the zero frequency.

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