OPTIMAL MODE EXPANSION OF PHASE RECONSTRUCTED FROM MEASUREMENTS OF WAVE FRONT TILTS IN A TURBULENT ATMOSPHERE. PART I. ABERRATION REPRESENTATION IN THE KARHUNEN–LOEVE–OBUKHOV BASIS

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In this paper we solve the problem on constructing optimal basis for mode representation of random phase of an optical wave in turbulent atmosphere. We propose an algorithm for obtaining eigenfunctions and eigenvalues of the integral operator with a difference kernel for the case of circular aperture and invariance of the form of eigenfunctions relative to rotation of a coordinate system. We also describe a technique of optimization of the classical Zernike expansion by optimizing coefficients of this expansion based on additional information about spatial correlation of the phase fluctuations.

An integral relation between the phase distribution within the confines of the given aperture and phase partial derivatives was suggested by us in Ref. 1. The magnitudes of such derivatives (wave front tilts) are the output signals of the Hartman sensors or the shift interferometers. The analytical relationships[1] for calculation of the coefficients of phase expansion in terms of Zernike polynomials from the results of measurement of the tilts are quite sufficient in the case of classical aberrations[2] or when phase distortions can be represented adequately by a few lower–order modes of Zernike expansion.

The self–sufficiency of the Zernike representation is violated in the case of higher order turbulent aberrations or in the case of aberrations caused by the adaptive system itself. The developers of adaptive optics systems should take into account that the Zernike expansion is close to the optimal one when approximating the atmospheric phase aberrations of not higher than fifth order.[3] For more accurate representation of the turbulent aberrations it is necessary to use the expansion in terms of the Karhunen–Loeve–Obukhov (K–L–O) eigenfunctions, which is universal for random fields.[4,5] However, this basis is of limited usefulness since it is generally agreed that "these functions should be calculated numerically because they can not be expressed analytically" (Ref. 6, p. 289) and "these functions have a complex structure and their applicability in the correcting devices is troublesome" (Ref. 7, p. 49).

In this paper we derive the analytical representation of the K–L–O basis, show the relation between this expansion and Zernike one (this makes it possible to use the traditional correcting devices). We also represent the coefficients of expansion of the atmospheric phase aberrations in K–L–O basis through the wave front tilts.

Let us represent the phase distribution $S(\rho)$ over the receiving aperture as the expansion in terms of the complete orthonormal set of functions $\Psi_k(\rho)$

$$S(\rho) = \sum_{k=0}^{\infty} b_k \, \Psi_k(\rho). \quad (1)$$

The coefficients of the series b_k in the framework of the hypothesis of frozen turbulence[8] are random variables since the phase in a turbulent atmosphere is a random one. We define the orthonormal system of functions $\Psi_k(\rho)$ so that norm of the ensemble–averaged error of wave front approximation be minimal. It is known that the problem of such basis constructing can be solved if the conditions of K–L–O theorem are fulfilled.[4,5] In accordance with this theorem the minimum norm of the error $\langle\varepsilon^2\rangle$ of random function approximation is achieved with the basis of N eigenfunctions corresponding to N maximum eigenvalues of the integral operator with a phase correlation function $B_s(\rho, \rho')$ as a kernel. If the random phase is to be approximated within the confines of the aperture with the pupil function $W(\rho)$, the problem of derivation of such eigenfunctions is reduced to the solution to the integral equation

$$\int\int W(\rho')\, B_s(\rho, \rho')\, \Psi_k(\rho')\, d^2 \rho' = \Lambda_k \, \Psi_k(\rho), \quad (2)$$

where

$$B_s(\rho, \rho') = \langle [S(\rho) - \langle S(\rho)\rangle]\, [S(\rho') - \langle S(\rho')\rangle]\rangle,$$

and the angular brackets denote the averaging over ensemble. The mean–square error of the approximation of random phase $\langle\varepsilon^2\rangle$ is defined by the following relationship:

$$\langle\varepsilon^2\rangle = \left\{ \int\limits_{-\infty}^{\infty} W(\rho') \langle [S(\rho') - \langle S(\rho')\rangle - \right.$$

$$\left. - \sum_{k=0}^{N} b_k \, \Psi_k(\rho')]^2\rangle \, d^2 \rho' \right\}_{\min} = \sum_{k=N}^{\infty} \Lambda_k \, . \quad (3)$$

The coefficients of the K–L–O expansion are uncorrelated and the phase expansion itself is the most informative at a given number of terms of expansion series as compared with the expansions of $S(\rho)$ in terms of any other basis with the same quantity of terms.

The integral equation (2) allows us to construct the eigenfunctions of expansion for the phase centralized relative to its mean values. In this case for circular aperture of radius R at

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$$W(\rho') = \begin{cases} 1, & |\rho| \le R, \\ 0, & |\rho| > R \end{cases},$$

taking into account the orthonormality of Ψ_k we obtain instead of Eq. (1)

$$S(\rho) = \frac{1}{\pi R^2} \int_{-\infty}^{\infty} W(\rho')\, S(\rho')\, d^2\rho' + \sum_{k=1}^{\infty} b_k\, \Psi_k(\rho). \quad (4)$$

The first term of the representation describes the phase averaged over the aperture similarly Zernike expansion. As a rule, this parameter for adaptive optics is not important, therefore the first term of expansion (4) is usually omitted.

Using the relation between correlation and structure functions we can rewrite the relationship (2) in the form

$$B_s(\rho, \rho') = \frac{1}{2} B_s(\rho, \rho) + \frac{1}{2} B_s(\rho', \rho') - \frac{1}{2} D_s(\rho, \rho'),$$

where

$$D_s(\rho, \rho') = \langle [[S(\rho) - \langle S(\rho)\rangle] - [S(\rho') - \langle S(\rho')\rangle]]^2 \rangle.$$

Based on the orthogonality of the set Ψ_k and setting that the phase variance varies slightly within the confines of the receiving aperture we can write instead of (2)

$$-\frac{1}{2} \int \int d^2\rho'\, W(\rho')\, D_s(\rho, \rho')\, \Psi_k(\rho') = \Lambda_k\, \Psi_k(\rho), \quad (5)$$

$$\int d\rho'\, W(\rho')\, \Psi_k(\rho') = 0, \ \text{if } k \ne 0.$$

In addition, we require that the sought eigenfunctions should be of invariant form under rotation of a coordinate axes. The fulfilment of this requirement implies that $\Psi_k(\rho)$ should be of the following form[2]:

$$\Psi_k(\rho) = K^{\,l}(\rho) \exp(i\, l\, \theta),\ \rho = \{\rho, \theta\}.$$

Let the random field of the phase fluctuations be locally homogeneous and isotropic one ($D_s(\rho, \rho') = D_s(|\rho - \rho'|)$). Then we can rewrite the relationship (5) in the form

$$K^{\,l}(\rho) \exp(i\, l\, \theta)\, \Lambda_k =$$

$$= -\frac{1}{2} \int_0^R \rho'\, d\rho' \int_0^{2\pi} d\theta'\, D_s(\rho, \rho', \theta - \theta')\, K^{\,l}(\rho') \exp(i\, l\, \theta'). \quad (6)$$

We introduce the notation

$$M_l(\rho, \rho') = -\frac{1}{2} \int_0^{2\pi} d\theta' \exp(i\, l\, \theta')\, D_s(\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos\theta'}) \quad (7)$$

and after simple rearrangements instead of (6) we obtain the homogeneous integral Fredholm equation of the second type[9]

$$\int_0^R M_l(\rho, \rho')\, K^{\,l}(\rho')\, \rho'\, d\rho' = \lambda_l\, K^{\,l}(\rho). \quad (8)$$

Since the eigenvalues of integral equation (8) coincide with the eigenvalues of integral equation (5) then in (8) we rename the $\Lambda\Lambda_k$ as λ_l. Obtained by such a way equation will be named as adaptive integral equation.

In the paper[3] published previously the authors derive an integral equation akin to Eq. (8) and present the final results of its solution using known to them standard numerical methods of calculation of the eigenfunctions and eigenvalues of corresponding integral operator. Further we will try to obtain the analytical results using the numerical methods only in the exceptional cases. Let us present the phase structure function $D_s(\rho)$ by such a way that the kernel of integral equation be degenerate. To make this we expand the $D_s(\rho)$ in terms of Bessel functions

$$D_s(\rho) = \sum_{p=0}^{\infty} a_p\, J_0\left(\mu_p \frac{\rho}{2R}\right), \quad (9)$$

where $J_0(x)$ is the Bessel function of the first type of zeroth order, $a_p = \dfrac{2}{R^2\, [J_0'(\mu_p)]^2} \displaystyle\int_0^R \rho\, D_s(\rho)\, J_0\left(\mu_p \frac{\rho}{2R}\right) d\rho$, and μ_p are the roots of equation $J_0(\mu) = 0$.

Substituting the expansion (9) in Eq. (7), integrating, and using the formula of summing the Bessel function[10]

$$J_0(\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos\theta'}\,) =$$

$$= J_0(\rho)\, J_0(\rho') + 2 \sum_{m=1}^{\infty} J_m(\rho)\, J_m(\rho') \cos m\theta,$$

we obtain the following expression for kernel $M_l(\rho, \rho')$:

$$M_l(\rho, \rho') = -\pi \sum_{p=0}^{\infty} a_p\, J_l\left(\mu_p \frac{\rho}{2R}\right) J_l\left(\mu_p \frac{\rho'}{2R}\right). \quad (10)$$

After substitution of series (10) in Eq. (8) the latter can be rewritten as

$$\sum_{p=0}^{\infty} d_p^l\, J_l\left(\mu_p \frac{\rho}{2R}\right) = K^{\,l}(\rho), \quad (11)$$

with unknown constants

$$d_p^l = \frac{-\pi\, a_p}{\lambda_l} \int_0^R J_l\left(\mu_p \frac{\rho'}{2R}\right) K^{\,l}(\rho')\, \rho'\, d\rho'.$$

Let us select the coefficients d_p^l so that the function $K^{\,l}(\rho)$ defined by the formula (11) be the solution of the integral equation (8). To do this we use the expansion (11) in the both parts of Eq. (8) and since the Bessel functions $J_l(\alpha x)$ and $J_l(\beta x)$ are linearly independent we equate the coefficients of the same Bessel functions in the both parts. As a result we obtain the infinite homogeneous system of the linear algebraic equations for unknown constants d_p^l

$$\lambda_l\, d_p^l = \sum_{p'=0}^{\infty} d_{p'}^{\,l}\, c_{pp'}^{\,l}, \quad (12)$$

where

$$c_{pp'}^{\,l} = -\pi\, a_p \int_0^R J_l\left(\mu_p \frac{\rho}{2R}\right) J_l\left(\mu_{p'} \frac{\rho}{2R}\right) \rho\, d\rho =$$

$$= \frac{-\pi\, a_p\, 4R^2}{(\mu_p)^2 - (\mu_{p'})^2} [\mu_{p'} J_{l-1}(\mu_{p'}/2)\, J_l(\mu_p/2) - \mu_p J_{l-1}(\mu_p/2) J_l(\mu_{p'}/2)]$$

or if $p = p'$ then

$$= \frac{R^2}{2}\left\{[J'_l(\mu_p/2)]^2 - \left(1 - \frac{l^2}{2\mu_p^2}\right)[J_l(\mu_p/2)]^2\right\},\ l > -1.$$

Being limited in the expansions (9)–(11) by the terms of Pth order we rewrite the relation (12) in the operator form

$$\mathrm{C}\,\mathbf{d} = \lambda_l\,\mathbf{d}, \tag{13}$$

where $\mathrm{C} = (c^{\,l}_{pp'})$ is the Gramm matrix and $\mathbf{d} = (d^l_1, d^l_2, ..., d^l_p)^{\mathrm{T}}$ is the vector–column. Determinant of the system (13) is represented in the form

$$D(\lambda_l) = \left\| \begin{matrix} c^l_{11} - \lambda_l & c^l_{12} & \dots & c^l_{1P} \\ c^l_{21} & c^l_{22} - \lambda_l & \dots & c^l_{2P} \\ \dots & \dots & \dots & \dots \\ c^l_{P1} & c^l_{P2} & \dots & c^l_{PP} - \lambda_l \end{matrix} \right\|$$

or $D(\lambda_l) = \det(\mathrm{C} - \lambda_l I)$, where I is the unit matrix. $D(\lambda_l)$ is the polynomial of P power with respect to variables λ_l. Let us find its roots $\lambda^{(1)}_l, \lambda^{(2)}_l, ..., \lambda^{(j)}_l, ..., \lambda^{(P)}_l$ diagonalizing the Gramm matrix and thereafter the components of eigenvectors $d^{\,lj}_p$ of the C matrix which are the coefficients of the expansion K–L–O in terms of Bessel functions

$$\Psi_k(\rho) = K^{\,l}_j(\rho)\exp(i\,l\,\theta) = \sum_{p=0}^{P} d^{\,lj}_p\, J_l(\mu_p\,\rho/2R)\exp(i\,l\,\theta). \tag{14}$$

Index j in representation of $K^{\,l}_j(\rho)$ in contrast to $K^{\,l}(\rho)$ from Eqs. (5) to (11) is introduced to denote the splitting of radial mode $K^{\,l}(\rho)$ over index j.

To put into correspondence the functions $\Psi_k(\rho)$, where $k = 1,2, ..., K$ with the obtained functions $K^{\,l}_j(\rho)\exp(i\,l\,\theta)$ we arrange eigenvalues $\lambda^{(j)}_l$ corresponding to them in order of decreasing of their magnitudes $\Lambda_1 > \Lambda_2 > \Lambda_3 > ... > \Lambda_K$ basing on condition $\Lambda_1 = \max \lambda^{(j)}_l$. In line with the derived order we sort the eigenfunctions $K^{\,l}_j(\rho)\exp(i\,l\,\theta)$. The obtained sequence of functions $\Psi_k(\rho)$, $k = 1, 2, ..., K$ is the sequence of K–L–O modes.

Before the numerical realization of this approach we obtain the formulas which relate the mode representation of wave front aberrations on the basis of Zernike polynomials and in terms of K–L–O eigenfunctions.

In existing adaptive systems[6,7] the devices correcting the turbulent distortions of optical wave operate in regimes of mode or zonal step compensation. The Zernike basis is used, as a rule, in the case of mode compensation and therewith final–control apparatus correct tilts, defocusing of wave front, and higher–order aberrations. Let us set the task of optimizing the corrector which operates on Zernike basis through the K–L–O functions.

Turning back to expressions for functions $\Psi_n(\rho)$ and Zernike polynomials $Z_n(\rho)$

$$\Psi_n(\rho) = K^{\,l}_j(\rho)\exp(i\,l\,\theta),\quad Z_n(\rho) = R^l_j(\rho)\exp(i\,l\,\theta), \tag{15}$$

we see that their azimuth parts coincide. It remains to find the relationship between radial parts. To do this, we rewrite (8) in the following form:

$$\int_0^R M_l(\rho, \rho')\,K^{\,l}_j(\rho')\,\rho'\,d\rho' = \lambda^{(j)}_l\,K^{\,l}_j(\rho). \tag{16}$$

We will seek a solution in the form of expansion

$$K^{\,l}_j(\rho) = \sum_{n=1}^{\infty} w^{\,l}_{jn}\,R^l_n(\rho). \tag{17}$$

Let us multiply Eq. (16) by $R^l_n(\rho)\rho$ and integrate the result within the confines of aperture of radius R. Using the orthonormality of functions of $R^l_n(\rho)$

$$\int_0^R R^l_n(\rho'/R)\,R^l_k(\rho'/R)\,\rho'\,d\rho' = \frac{R^2}{2(n+1)}\,\delta_{kn},$$

where $\delta_{k\,n}$ is the Kroneker–symbol, the known relationship[2]

$$\int_0^R R^l_n(\rho'/R)\,J_l\left(\gamma\frac{\rho'}{R}\right)\rho'\,d\rho' = (-1)^{(n-l)/2}\,R^2\,\frac{J_{n+1}(\gamma)}{\gamma},$$

as well as the representation of the kernel (10) we obtain for the vector–line $\mathbf{w}^{\,l}_j\,(w^{\,l}_{j1}, w^{\,l}_{j2}, ..., w^{\,l}_{jN})$ at finite number of terms in series (17) the following system of equations

$$\sum_{n=1}^{N} w^{\,l}_{jn}\left[\beta^{\,l}_{kn} - \lambda^{(j)}_l\,\frac{\delta_{kn}}{2(n+1)}\right] = 0,\ k = 1, 2, ..., K,\ K = N, \tag{18}$$

where the matrix $\beta^{\,l}$ has the following elements:

$$\beta^{\,l}_{kn} = -4\pi\sum_{p=0}^{P} a_p\,(-1)^{(k+n-2\,l)/2}\,\frac{J_{k+1}(\mu_p)\,J_{n+1}(\mu_p)}{\mu_p^2}.$$

In operator form the expression (18) can be rewritten as

$$(\beta^{\,l} - \lambda^{(j)}_l\,I)\,\mathbf{w}^{\,l}_j = 0. \tag{19}$$

The solution to the system (19) allows us to find the unknown coefficients of expansion of radial part of K-L-O functions (17) using the radial part of Zernike polynomials. Hence, we have

$$\Psi_k(\rho) \cong \exp(i\,l\,\theta)\sum_{n=1}^{N} w^{\,l}_{jn}\,R^l_n(\rho). \tag{20}$$

Taking into account that the set of indexes l, j corresponding to the eigenvalues $\lambda^{(j)}_l$ arranged in descending order is in agreement with the values of $k = 1, 2, ..., K$ we can rewrite the expression (20) as

$$\Psi_k(\rho) = \exp(i\,l\,\theta)\sum_{n=1}^{N} w^{\,k}_n\,R^l_n(\rho), \tag{21}$$

or in the generalized form

$$\Psi_k(\rho) = \sum_{n=1}^{N} v_n^k Z_n(\rho) . \quad (22)$$

The use of the formulas (4) and (20)–(22) allows us to construct the phase expansion in terms of the Zernike polynomials with the coefficients optimized in line with the information on randomly inhomogeneous atmosphere. Such an information is derived from the known correlation function of phase fluctuations $B_s(\rho, \rho')$.

Let now the coefficients of the expansion of random phase in terms of Zernike polynomials be known

$$S(\rho) = a_0 + \sum_{k=1}^{K} a_k Z_k(\rho). \quad (23)$$

Let us obtain the optimal phase expansion, that is, the representation of the phase in K–L–O basis

$$S(\rho) = b_0 + \sum_{m=1}^{M} b_m \Psi_m(\rho) ,$$

where $a_0 = b_0$ is the phase averaged over aperture. Considering the Eq. (22) and the orthonormality of Zernike polynomials we derive the coefficients b_m of the given expansion

$$b_m = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^{R} S(\rho, \theta) \Psi_m(\rho, \theta) \rho \, d\rho \, d\theta =$$

$$= \frac{1}{\pi R^2} \sum_{k=1}^{K} a_k \int_0^{2\pi} \int_0^{R} Z_k(\rho, \theta) \Psi_m(\rho, \theta) \rho \, d\rho \, d\theta =$$

$$= \frac{1}{\pi R^2} \sum_{k=1}^{K} a_k \sum_{n=1}^{N} v_n^m \int_0^{2\pi} \int_0^{R} Z_k(\rho, \theta) Z_m(\rho, \theta) \rho \, d\rho \, d\theta =$$

$$= \sum_{k=1}^{K} a_k \sum_{n=1}^{N} v_n^m \delta_{nk} = \sum_{n=1}^{N} v_n^m a_n , \text{ if } N \le K . \quad (24)$$

Rewriting the (24) in the matrix form we have

$$\mathbf{b} = V \mathbf{a}, \text{ if } M = N, \quad (25)$$

where $\mathbf{b} = (b_1, b_2, ..., b_N)$ are the coefficients of phase expansion in K–L–O basis and $\mathbf{a} = (a_1, a_2, ..., a_N)^{\mathrm{T}}$ are the coefficients of phase expansion in terms of Zernike polynomials; $V = (v_n^m)$ is the transformation matrix from the Zernike basis to the K–L–O basis, T is the sign of transposition. The transformation matrix $V\{v_n^m\}$ allows us to turn from Zernike expansion, the coefficients of which are determined by "classical" methods to expansion in terms of Zernike polynomials, the coefficients of which are optimized in a sense of minimization of the mean value of norm (3). The finding of matrix V enables us to calculate the coefficients of this optimized expansion directly from the measurements of tilts of wave front with Hartman sensors or shift interferometers based on the analytically obtained relation between them and the coefficients of phase expansion into a classical Zernike series.[1]

Thus, this paper is devoted mainly to description of a theoretical approach to the problem of obtaining an optimal mode expansion of phase. The second part of this paper contains the algorithms of numerical simulation of the approach described and the results of numerical experiments.

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