RADIUS OF CURVATURE OF A SOUND BEAM IN AN INHOMOGENEOUS MOVING MEDIUM

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An exact formula for the radius of curvature of a sound beam derived for the case of a moving inhomogeneous medium is presented in this paper. The case of plane wave was investigated in a nonorthogonal curvilinear coordinate system referred to a reference beam.

One of the well-known ways of describing the beam path is based on using the radius of the beam curvature $\rho(s)$ as a function of the path length s (see Refs. 1, 2, and 3). Formula for the radius of curvature for inhomogeneous stationary medium is well-known.[4] An approximate formula[2] for $\rho(s)$ derived for moving inhomogeneous medium is valid for small wind velocities. In this paper an exact formula for the radius of curvature of a sound beam is derived for the case of inhomogeneous moving medium using technique presented in Refs. 5 and 6.

The study was carried out for the case of two-dimensional space in a nonorthogonal curvilinear coordinate system in the vicinity of a reference beam. Implementation of the nonorthogonal coordinate system is physically justtified because when sound propagating in a moving medium the phase front is nonorthogonal to the beam direction (phase and group sound velocities are different).[7] This fact makes the mathematical representation of the initial equations more complicated, but the final forms of expressions become more simple and vivid that can be considered as justification of this approach.

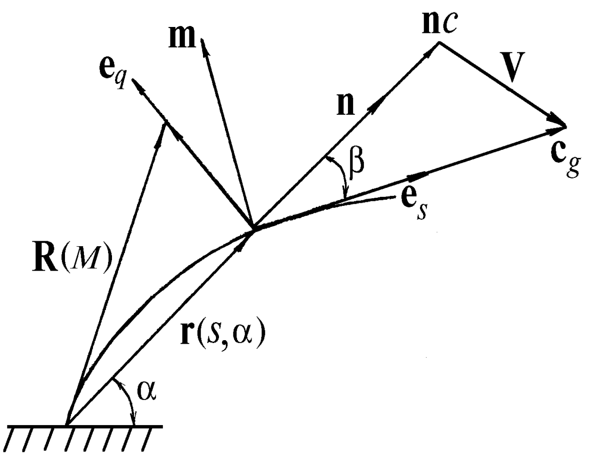

FIG. 1.

Let us describe the coordinate system referred to a reference beam $\mathbf{r}(s, \alpha)$ (see Fig. 1). The coordinate s varies along the unit vector $\mathbf{e}_s(s)$ which points the direction of the group velocity of sound in a reference beam; the coordinate q varies along the tangent line to the phase front at the point of its intersection with the reference beam (along the unit vector $\mathbf{e}_q(s)$); α is the angle of the beam outlet. Datum of the coordinate system $(\mathbf{e}_s(s); \mathbf{e}_q(s))$ is nonorthogonal. Radius-vector of an arbitrary point M in this coordinate system has the following form:

$$\mathbf{R}(M) = \mathbf{r}(s) + \mathbf{e}_q(s)\, q,$$

where s is the coordinate of the point M on the reference beam; $\mathbf{r}(s)$ is the radius-vector of the coordinate s; q is the lateral coordinate of the point M.

To determine the metric tensor q_{ij} for this coordinate system, we calculate the scalar product[1]

$$\left(\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s}, \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s}\right) = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} + \frac{\mathrm{d}q}{\mathrm{d}s}\,\mathbf{e}_q + q\,\frac{\mathrm{d}\mathbf{e}_q}{\mathrm{d}s}\right)^2. \tag{1}$$

Let us consider the term $\mathrm{d}\mathbf{e}_q/\mathrm{d}s$ in Eq. (1). From the geometry in Fig. 1 it follows that

$$\mathbf{e}_q(s) = \mathbf{m}(s)\cos\beta(s) - \mathbf{e}_s(s)\sin\beta(s), \tag{2}$$

where $\mathbf{m}(s)$ is the unit vector orthogonal to $\mathbf{e}_s(s)$; $\beta(s)$ is the angle between the directions of phase and group sound velocities (between $\mathbf{n}$ and $\mathbf{e}_s$). Since the Frenet formulas[3] are correct for the vectors $\mathbf{m}(s)$ and $\mathbf{e}_s(s)$

$$\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}s} = \frac{1}{\rho}\,\mathbf{e}_s(s); \qquad \frac{\mathrm{d}\mathbf{e}_s}{\mathrm{d}s} = -\frac{1}{\rho}\,\mathbf{m}(s), \tag{3}$$

then using Eqs. (2) and (3) we obtain an important formula

$$\frac{\mathrm{d}\mathbf{e}_q}{\mathrm{d}s} = -1/\rho^*\,\mathbf{n}(s), \tag{4}$$

where there occurs the following notation: $1/\rho^*(s) = 1/\rho(s) - \mathrm{d}\beta/\mathrm{d}s$; $\mathbf{n}(s)$ is the unit vector normal to the phase front. Taking into consideration Eq. (4) and equality $\mathrm{d}\mathbf{r}/\mathrm{d}s = \mathbf{e}_s$ in Eq. (1) we obtain

$$(\mathrm{d}\mathbf{R}, \mathrm{d}\mathbf{R}) = \left(1 + 2\,\frac{q}{\rho^*}\cos\beta + \frac{q^2}{\rho^{*2}}\right)(\mathrm{d}s)^2 - 2\sin\beta\,\mathrm{d}q\,\mathrm{d}s + (\mathrm{d}q)^2.$$

Hence

$$g_{ij} = \begin{pmatrix} 1 + 2\,q/\rho^*\cos\beta + q^2/\rho^{*2}; & -\sin\beta \\ -\sin\beta; & 1 \end{pmatrix}.$$

To determine the sound beam path in moving media, it is necessary to find the relation $1/\rho^*(s)$ with the parameters

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of medium $c(s, q)$ and $\mathbf{v}(s, q)$. After that, using expression (4), we can determine the beam path in the plane (s, q).

The expression for $1/\rho^*(s)$ is obtained when solving an eikonal equation under conditions of curvilinear nonorthogonal coordinates in a small vicinity of the reference beam. This equation can be written down in the following form:[8]

$$(\nabla \theta)^2 = (1 - \nabla \theta, \mathbf{v}(s, q))^2 / c^2 (s, q). \tag{5}$$

If the eikonal $\theta(s, q)$ and all terms involved in Eq. (5) are expanded in the Taylor series with respect to lateral coordinate q, and all terms with equal powers are grouped together, the sought expression will be found at the first power of q.

Let us try to solve the problem. The terms constituting Eq. (5) should be represented in detail. Vector of the eikonal gradient has the covariant components. Hence

$$(\nabla \theta)^2 = g^{11} (\partial\theta/\partial s)^2 + 2\, g^{12}\, \partial\theta/\partial s\, \partial\theta/\partial q + g^{22} (\partial\theta/\partial q)^2, \tag{6}$$

where g^{ij} is the matrix inverse to the matrix g_{ij}. Since the components of the vector $\mathbf{v}(s, q)$ are covariant, the scalar product of $\nabla\theta$ and $\mathbf{v}(s, q)$ can be presented in the form

$$(\nabla \theta, \mathbf{v}) = \partial\theta / \partial s\, v^s(s, q) + \partial\theta / \partial s\, v^q(s, q), \tag{7}$$

where $v^s(s, q)$ and $v^q(s, q)$ are the components of the vector $\mathbf{v}(s, q)$ in the curvilinear nonorthogonal coordinate system.

We shall find the solution of eikonal equation (5) in the following form:

$$\theta(s, q) = \theta_0(s) + \frac{1}{2}\, \theta_2(s)\, q^2 + \dots ,$$

where $\theta_0(s) = \theta(s, 0)$, $\theta_2(s) = (\partial^2\theta/\partial^2 q)_{q=0}$. Since the coordinate q varies along the axis tangent to the phase front at the point $(s, q = 0)$ then $(\partial\theta/\partial q)_{q=0} = 0$.

At the same time, we expand all terms of eikonal equation (5) in a power series with respect to q

$$\partial\theta/\partial s = \theta_0'(s) + \frac{1}{2}\, \theta_2'(s)\, q^2 + \dots , \tag{8}$$

where the prime denotes the derivative with respect to ds: $\theta_0' = \mathrm{d}\theta_0/\mathrm{d}s$ and so on.

$$\partial\theta/\partial q = \theta_2 (s) + \frac{1}{2}\, \theta_3 (s)\, q^2 + \dots , \tag{9}$$

Taking into account Eqs. (8) and (9) and expanding all components of the tensor in power series with respect to q

$$g^{ij} = g^{-1}_{ij} = \frac{1}{g} \begin{pmatrix} 1; & \sin\beta \\ \sin\beta; & 1 + 2\, q/\rho^* \cos\beta + q^2/\rho^{*2} \end{pmatrix},$$

where $g = \det g_{ij} = (\cos\beta + q/\rho^*)^2$, we obtain

$$(\nabla \theta)^2 = \frac{1}{\cos^2\beta} \left\{\theta_0'^2 - 2\, \frac{1}{\rho_c}\, \theta_0'^2\, q + (\theta_0'\, \theta_2' + 3\, \frac{1}{\rho_c^2}\, \theta_0'^2\,)q^2\right\} +$$

$$+ 2\, \frac{\tan\beta}{\cos\beta} \left\{\theta_0'\, \theta_2\, q + (\frac{1}{2}\, \theta_0'\, \theta_3 - 2\, \frac{1}{\rho_c}\, \theta_0'\, \theta_2) q^2\right\} + \frac{1}{\cos^2\beta}\, \theta_2^2\, q^2. \tag{10}$$

Here three terms of Eq. (6) are successively expanded in the series correct to the second power of q, $\rho_c = \rho^* \cos\beta$.

The components of vector of wind velocity $v^s(s, q)$ and $v^q(s, q)$ and sound velocity are to be also expanded in the Taylor series

$$v^s(s, q) = v_0^s(s) + v_1^s(s)\, q + \frac{1}{2}\, v_2^s(s)\, q^2 + \dots ; \tag{11}$$

$$v^q(s, q) = v_0^q(s) + v_1^q(s)\, q + \frac{1}{2}\, v_2^q(s)\, q^2 + \dots ; \tag{12}$$

$$c(s, q) = c_0(s) + c_1(s)\, q + \frac{1}{2}\, c_2(s)\, q^2 + \dots , \tag{13}$$

where the following notations are introduced:

$$v_0^s(s) = v^s(s, 0); \quad v_1^s(s) = (\partial v^s/\partial q)_{q=0}; \quad v_2^s(s) = (\partial^2 v^s/\partial q^2)_{q=0},$$

(in the same way we use the designations for the component $v^q(s, q)$),

$$c_0(s) = c(s, 0); \quad c_1(s) = (\partial c/\partial q)_{q=0}; \quad c_2(s) = (\partial^2 c/\partial q^2)_{q=0}.$$

Using Eqs. (8), (9), (11), and (12), write expression (7) as the series

$$(\nabla \theta, \mathbf{v}) = \theta_0'\, v_0^s + (\theta_0'\, v_1^s + \theta_2\, v_0^q)\, q +$$

$$+ \left[\frac{1}{2}\, (\theta_0'\, v_2^s + \theta_2'\, v_0^s) + (\theta_2\, v_1^q + \frac{1}{2}\, \theta_3\, v_0^q)\right] q^2. \tag{14}$$

From Eq. (13) we obtain

$$\frac{1}{c^2(s, q)} = \frac{1}{c_0^2(s)} \left(1 - 2\, \frac{c_1(s)}{c_0(s)}\, q + 3\, \frac{c_1^2(s)}{c_0^2(s)}\, q^2 - \frac{c_2(s)}{c_0(s)}\, q^2\right). \tag{15}$$

For brevity, let us introduce the following auxiliary notations:

$$A = \theta_0'\, v_0^s; \quad B = (\theta_0'\, v_1^s + \theta_2\, v_0^q);$$

$$D = \frac{1}{2}\, (\theta_0'\, v_2^s + \theta_2'\, v_0^s) + (\theta_2\, v_1^q + \frac{1}{2}\, \theta_3\, v_0^q). \tag{16}$$

Taking into consideration Eqs. (14), (15), and (16), we obtain the expressions for the terms of eikonal equation (5)

$$2\, \frac{(\nabla \theta, \mathbf{v})}{c^2(s, q)} = \frac{2}{c_0^2(s)} \left\{A - \left(2\, A\, \frac{c_1(s)}{c_0(s)} - B\right) q + \right.$$

$$\left. + \left(3\, A\, \frac{c_1^2(s)}{c_0^2(s)} - A\, \frac{c_2(s)}{c_0(s)} - 2B\, \frac{c_1(s)}{c_0(s)} + D\right) q^2\right\}, \tag{17}$$

$$\frac{(\nabla \theta, \mathbf{v})^2}{c^2(s, q)} = \frac{1}{c_0^2(s)} \left\{A^2 + 2\left(A\, B - A^2\, \frac{c_1(s)}{c_0(s)}\right) q + \right.$$

$$\left. + \left(3\, A^2\, \frac{c_1^2(s)}{c_0^2(s)} - A^2\, \frac{c_2(s)}{c_0(s)} - 4\, AB\, \frac{c_1(s)}{c_0(s)} + 2\, AD + B^2\right) q^2\right\}. \tag{18}$$

Now group together all components of Eq. (5) for equal powers of q. For zero power of q we obtain the equation

$$\frac{1}{\cos^2\beta}\, \theta_0'^2 + 2\, \frac{1}{c_0^2(s)}\, \theta_0'\, v_0^s - \frac{1}{c_0^2(s)}\, \theta_0'^2\, v_0^{s2} = \frac{1}{c_0^2(s)}\, .$$

Solving the quadratic equation with respect to θ_0' we obtain

$$\theta_0' = \frac{\cos\beta\,(s)}{c_0(s) + v_0^s(s)\cos\beta\,(s)}\,. \qquad (19)$$

It is necessary to take in mind that the group velocity of sound of the reference beam can be determined from the relation[7]

$$\mathbf{c}_g(s,\,0) = \mathbf{e}_s\, c_g(s,\,0) = c(s,\,0)\,\mathbf{n}(s) + \mathbf{v}(s,\,0).$$

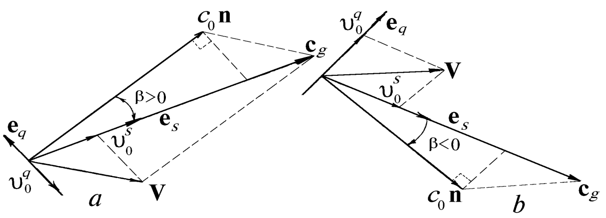

FIG. 2.

Then from the geometry of Fig. 2 one can see that the expression in the denominator of the right–hand part of Eq. (19) is the modulus of group velocity of sound c_g(s). Equation (19) can be presented in the following form:

$$\theta_0 = \int\limits_0^s \frac{\mathrm{d}s}{c_g(s)}\,. \qquad (20)$$

Equation (20) makes clear the physical meaning of the first term in expansion of the eikonal function in the Taylor series. The value θ_0 is equal to the time of sound propagation along the reference beam from the point (0, 0) to the point (s, 0).

Write down the equation for the first power of q

$$\frac{1}{\cos^2\beta\,\rho_c}\,\theta_0'^2 - \frac{\tan\beta}{\cos\beta}\,\theta_0'\,\theta_2 - \frac{1}{c_0^2}\left[(\theta_0'\,v_1^s + \theta_2\,v_0^q) - 2\,\theta_0'\,v_0^s\,\frac{c_1}{c_0}\right] +$$

$$+ \frac{1}{2\,c_0^2}\left\{\theta_0'\,v_0^s\left[(\theta_0'\,v_1^s + \theta_2\,v_0^q) - 2\,\theta_0'\,v_0^s\,\frac{c_1}{c_0}\right] +\right.$$

$$\left. + \,\theta_0'\,v_0^s\,(\theta_0'\,v_1^s + \theta_2\,v_0^q)\right\} = \frac{1}{c_0^2}\,\frac{c_1}{c_0}\,. \qquad (21)$$

Take into account that $\beta \in [-\pi,\,\pi]$ as well as $\beta > 0$ in counting anticlockwise from $\mathbf{c}_s$ but in the opposite case $\beta < 0$. Consideration must be also taken of the fact that $v_0^q/c_0 = -\tan\beta$ for an ascending beam, while $v_0^q/c_0 = \tan\beta$ for descending one (see Figs. 2a and b, respectively). In this case, the sum of terms of Eq. (21) containing θ_2 is equal to zero. Remaining expression (21) can be solved with respect to $1/\rho_c = 1/(\rho^*\cos\beta)$, using the equality $\theta_0' = \cos\beta\,/(c_0 + v_0^s\cos\beta)$. As a result, we obtain the following expression for the radius of sound beam curvature in an inhomogeneous moving medium

$$\frac{1}{\rho^*(s)} = \frac{\partial c/\partial q + \partial v^s/\partial q\,\cos\beta\,(s)}{c(s,\,0)}\,\cos\beta\,(s), \qquad (22)$$

where $1/\rho^*(s) = 1/\rho(s) - \mathrm{d}\beta/\mathrm{d}s$. All derivatives are taken on the reference beam at the point (s, 0).

For $\mathbf{v}(s,\,q) = 0$ Eq. (22) coincides with the corresponding expression for the radius of the beam curvature for a stationary medium.[1,2,4] In this case the vectors $\mathbf{e}_s$ and $\mathbf{e}_q$ are perpendicular, $\beta(s) = 0$, and $1/\rho^*(s) = 1/\rho(s)$. Note that the effect of motion of medium on the curvature radius can be expressed through $\partial v^s/\partial q\cos\beta(s)$ which is the projection of the derivative of the velocity component $v^s(s,\,q)$ onto the normal to the phase front at the point (s, $q = 0$). The effect of vector $\mathbf{v}(s,\,0)$ is expressed by the value of $\cos\beta(s)$. The radius of beam curvature does not depend on the derivatives of component of the wind velocity along the tangent to the phase front $v^q(s,\,q)$. Using Eqs. (22) and (4) and parameters of medium $c(s,\,q)$ and $\mathbf{v}(s,\,q)$ as the coordinate functions it becomes possible to construct the beam paths.

To construct the beams in an inhomogeneous moving medium, the integral beam equation (9) is generally used. Compared to this method, the construction of beams based on the expression for the radius of beam curvature has the following advantages:

1. Parameters c and $\mathbf{v}$ can be the functions of two variables (or even three ones in generalizing results of this paper for the case of three-dimensional space). The integral form of the beam equation takes place only for a stratified medium.

2. The point of the beam bend (in using this method) is not a singular point of the beam path. Construction of the beam in its vicinity is carried out in the same way as in any point of the beam. Construction of the beam from the integral equation in the vicinity of the bend point is associated with the well-known difficulties.

Long-time calculations are the disadvantage of this method.

Above-mentioned well-known expression for the radius of the sound beam curvature in an inhomogeneous moving medium was obtained within the framework of linear aproximation with respect to $\mathbf{v}/c$ (Ref. 2). This approximation results in the errors in constructing the beam path, and expression for the curvature radius is rather awkward.

Remark. The expression for the curvature radius was obtained in constructing the Gaussian beam in an inhomogeneous moving medium. To construct this beam, it is necessary to solve the equation for the second power of q for θ_2. For θ_2, it must be the Riccati equation as in the case of stationary medium.[1,5,6] To find the amplitude of the Gaussian beam, it is necessary to study the transfer equation in the vicinity of reference beam, for an inhomogeneous moving medium[8] as well.

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REFERENCES

1. V.M. Babich and M.M. Popov, Akust. Zh. **27**, No. 6, 828–835 (1981).
2. R.F. Myulali, *Radio–Wave Ray Approximation and Propagation,* (Nauka, Moscow, 1971), 311 pp.
3. S.P. Novikov and A.T. Fomenko, *Differential Geometry and Topology* (Nauka, Moscow, 1987), 432 pp.

4. V.M. Babich and V.S. Buldyrev, *Asymptotic Formulas in Problems of the Short Waves Diffraction* (Nauka, Moscow, 1972), 453 pp.
5. V.M. Babich, V.S. Buldyrev, and I.A. Molotkov, *Spatiotemporal Ray Method: Linear and Nonlinear Waves* (Leningrad State University Publishing House, Leningrad, 1985), 272 pp.
6. V.M. Babich and M.M. Popov, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **32**, No. 12, 1447–1466 (1989).
7. D.I. Blokhintsev, *Acoustics of Inhomogeneous Moving Medium* (Nauka, Moscow, 1981), 206 pp.
8. N.S. Grigor'eva, *Asymptotic Methods Use in Problems on Sound Propagation in Inhomogeneous Moving Medium* (Leningrad State University Publishing House, Leningrad, 1991), 240 pp.
9. V.E. Ostashev, *Sound Propagation in Moving Media* (Nauka, Moscow, 1992), 208 pp.