VARIATIONAL SYNTHESIS OF SIGNALS IN THE PROBLEM OF ACTIVE IMAGE RECONSTRUCTION

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In this paper we derive expressions which show that, from the point of view of reaching minimum in the rms error of phase estimate, the spatial distribution of sounding signal in the target plane as a finite function different from zero only within the target contour is optimal. This situation allows further development of the technique of active image reconstruction and stimulates euristic approach to synthesis of reconstruction algorithms in contrast to regular one since it does not impose any restrictions on sounding signals.

In Refs. 1 and 2 one can find a description of the technique proposed for reconstruction of a target image under conditions of phase distortions of spatial spectrum of a signal. This technique uses orthogonal spatiotemporal distributions of a sounding pulse over the transmitting aperture. It is characteristic of this technique that no source of reference emission in the target plane is needed.

In this connection it is important to know whether there exists or not some optimal distribution of the sounding signal, independent of the plane in which we consider it, either in the image plane of the target or in the plane of the transmitting aperture, because they are uniquely related to each other via the integral expression for Fraunhofer (Fresnel) diffraction. It is also important to elucidate how the solution derived in Refs. 1 and 2 relates to the optimal solution, provided that such a solution is found.

Taking the Fraunhofer model of diffraction, for certain, below we shall seek such a spatial distribution of the sounding signal in the target plane, which bring the highest value to a chosen quality index. After the target is irradiated, the signal in its plane may be presented in the following factorized form:

$$E_c(\mathbf{r}) = E_s(\mathbf{r})\, E(\mathbf{r}), \tag{1}$$

where $\mathbf{r}$ is the spatial coordinate in the target plane, $E_s(\mathbf{r})$ is the sounding signal; and $E(\mathbf{r})$ is the target image to be reconstructed.

However, one has first to choose a criterion of optimum. Since we aim at compensating for the destructive effect of phase distortions of the spatial spectrum of a signal while reconstructing the image, it appears to be reasonable to choose the rms error ($\sigma^2_{\varphi_n}$) of the estimate of phase distortions of the spatial spectrum of a signal as a criterion of quality.

Thus, a <good> modulating function, in the above sense, would be the function, which would permit the estimation of phase perturbations in the propagation medium to the highest accuracy. Therefore one needs to consider the synthesis of algorithms for optimal estimation of phase perturbations, which most often are introduced by the propagation medium itself.

Consider the functional of probability density for a field observed against a δ – correlated additive noise[3]

$$F(\varepsilon(\rho, t) / A_0, \varphi) = K_\varphi \exp\Big(-\frac{1}{2N_0}\int\limits_0^T\int\limits_\Omega \mathrm{d}\rho\, \mathrm{d}t\, \varepsilon^2(\rho, t) - q_0 A_0^2 + \frac{A_0}{N_0}\,\mathrm{Re}\int\limits_\Omega \mathrm{d}\rho\, \varepsilon_0^*(\rho)\, \varepsilon(\rho)\, \mathrm{e}^{-j\varphi(\rho)}\Big), \tag{2}$$

where

$$q_0 = \frac{T}{4N_0}\int\limits_\Omega \mathrm{d}\rho\, |\varepsilon(\rho)|^2; \tag{3}$$

$$\varepsilon_0(\rho) = \int\limits_0^T \mathrm{d}t\, \varepsilon(\rho, t)\, \mathrm{e}^{j\omega_0 t}; \tag{4}$$

$$\varepsilon_c(\rho, t) = \mathrm{Re}\,(A_0\, \varepsilon(\rho) \exp(j(\varphi(\rho) - \omega_0))); \tag{5}$$

$$\varepsilon(\rho, t) = \varepsilon_c(\rho, t) + n(\rho, t); \tag{6}$$

ρ is the spatial coordinate in the aperture plane; T is the time interval of observation; Ω is the integration domain in the aperture plane; ω_0 is carrier frequency; A_0 is the amplitude of the detected signal, which, in the general case is indeterminate; and, N_0 is related to $n(\rho, t)$ via the following relationship:

$$<n(\rho_1, t_1)\, n(\rho_2, t_2)> = N_0\, \delta(\rho_1 - \rho_2)\, \delta(t_1 - t_2). \tag{7}$$

As foolows from Ref. 3, further investigations into the algorithms for estimation of the parameters of targets located under conditions of phase distortions introduced by a medium mainly used two approximations of the function describing the distortions. The first class of approximating functions is as follows:

$$\varphi(\rho) = \sum_{l=1}^{L} \varphi_l\, \chi_l(\rho - \rho_l), \tag{8}$$

where φ_l may be arbitrary, and

$$\chi_l(\rho - \rho_l) = \begin{cases} 1, & \forall\ \rho \in \Delta_r, \\ 0, & \forall\ \rho \notin \Delta_r; \end{cases} \tag{9}$$

while the second class is described by the equality

$$\varphi(\rho) = \sum_{l=1}^{L} (\varphi_l - \kappa_l(\rho - \rho_l))\, \chi_l(\rho - \rho_l), \tag{10}$$

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where κ_l is an arbitrary vector; ρ_l is the radius-vector of the correlation cell Δ_l. The whole domain of definition of the function $\varphi(\rho)$ is divided into L subregions Δ_l.

Use of such approximations is aimed at elimination of an explicit dependence of the phase distortions (estimated in the course of implementation of the adaptive Bayes algorithms) on the integrand parameter ρ. Thus the estimation of a continuous distribution $\varphi(\rho)$ is reduced to estimation of a finite-dimensional vector $\{\varphi_l\}$, $l = 1,\dots, L$, according to the expression:

$$\nabla_\varphi \ln F(\varepsilon(\rho, t) / A_0, \varphi) = 0, \tag{11}$$

which, in the case of approximation (8) is reduced to the form

$$\frac{\partial}{\partial\varphi_l} \mathrm{Re} \sum_{l=1}^{L} \exp(j\varphi_l) \int_{\Delta_l} d\rho\, \varepsilon_0^*(\rho)\, \varepsilon(\rho) = 0, \tag{12}$$

which yields the following estimation algorithm:

$$\hat{\varphi}_l = -\arg \int_{\Delta_l} d\rho\, \varepsilon_0^*(\rho)\, \varepsilon(\rho) \pm 2\pi n, \tag{13}$$

$n = 0; \pm 1; \pm 2; \dots;\ l = 1, \dots, L.$

Meanwhile, from the standpoint of synthesizing adaptive algorithms, the class of functions used to approximate phase distortions is indecisive since variational calculus enables one to find an extremum of the functional (2) by directly varying function to be estimated. In this case the following expression corresponds to Eq. (11):

$$\frac{\delta}{\delta\varphi(\rho)} \ln F(\varepsilon(\rho', t) / A_0, \varphi(\rho')) = 0. \tag{14}$$

Following the rules of variational calculus (see, for example, Ref. 4), we obtain from Eqs. (14) and (2):

$$\frac{\delta}{\delta\varphi(\rho)} \ln F(\varepsilon(\rho', t) / A_0, \varphi(\rho')) =$$

$$= \frac{A_0}{N_0} \mathrm{Re} \int_\Omega d\rho' \frac{\delta}{\delta\varphi(\rho)} \varepsilon_0^*(\rho')\, \varepsilon(\rho')\, e^{j\varphi(\rho')} =$$

$$= \frac{A_0}{N_0} \mathrm{Re} \int_\Omega d\rho'\, \varepsilon_0^*(\rho')\, \varepsilon(\rho')\, \delta(\rho - \rho')\, e^{j\varphi(\rho')} =$$

$$= \frac{A_0}{N_0} \mathrm{Re}\, j\, \varepsilon_0^*(\rho)\, \varepsilon(\rho)\, e^{j\varphi(\rho)} = 0. \tag{15}$$

It follows from Eq. (15) that

$$\hat{\varphi}(\rho) = \arg \varepsilon_0^*(\rho)\, \varepsilon(\rho) \pm 2\pi n, \tag{16}$$

$n = 0; \pm 1; \pm 2; \dots;\ l = 1, \dots, L.$

By comparing Eqs. (13) and (16) one can see that expression (16) is a limit of the expression (13) under condition that Δ_l vanishes. Really, by multiplying both sides of Eq. (12) by the factor $1/S_l$, where $S_l = \int_{\Delta_l} d\rho$ is the area of a Δ_l subregion, we have

$$\frac{\partial}{\partial\varphi_l} \mathrm{Re} \sum_{l=1}^{L} \frac{1}{S_l} \exp(j\varphi_l) \int_{\Delta_l} d\rho\, \varepsilon_0^*(\rho)\, \varepsilon(\rho) = 0. \tag{17}$$

Differentiating it and making use of the mean value theorem,[5] we have

$$\mathrm{Re} \frac{1}{S_l} j\, \varepsilon_0^*(\rho_l')\, \varepsilon(\rho_l')\, e^{j\varphi(\rho_0')} \int_{\Delta_l} d\rho = 0, \tag{18}$$

where $\rho_l' \in \Delta_l$. Then, in the limiting case of vanishing Δ_l we have

$$\lim_{S_l \to 0} \mathrm{Re} \frac{1}{S_l} j\, \varepsilon_0^*(\rho_l')\, \varepsilon(\rho_l')\, e^{j\varphi(\rho_0')} \int_{\Delta_l} d\rho =$$

$$= \frac{A_0}{N_0} \mathrm{Re}\, j\, \varepsilon_0^*(\rho)\, \varepsilon(\rho)\, e^{j\varphi(\rho)} = 0, \tag{19}$$

which exactly coincides with Eq. (15), since $\rho_l' \to \rho$ at $\Delta_l \to 0$.

Thus the approximations (8) and (10) are not principle limitations on the synthesis of optimal processing schemes, representing only the way following which the phase distortions are written in a discrete form. However, no discrete form of this function is urgently needed when performing thus synthisized algorithms on an analog device.

The meaning of the optimal scheme for estimating phase distortions directly follows from Eq. (16), and it is in calculation of the phase difference between the received field and the field from a known source in the absence of phase distortions.

Let us now find such a distribution $E_s(\mathbf{r})$, which would make the estimation algorithm (16) most accurate.

We define the limiting accuracy of estimating the phase distortions using the functional analog of the Fischer matrix, that is, a limiting transition to continuum

$$-\left\langle \frac{\delta^2 \ln F(\varepsilon(\rho, t) / A_0, \varphi)}{\delta\varphi(\rho')\, \delta\varphi(\rho'')} \right\rangle. \tag{20}$$

By differentiating Eq. (2) we obtain

$$\frac{\delta^2 \ln F(\varepsilon(\rho, t) / A_0, \varphi)}{\delta\varphi(\rho')\, \delta\varphi(\rho'')} =$$

$$= \frac{A_0}{N_0} \mathrm{Re}(j)^2\, \varepsilon_0^*(\rho')\, \delta(\rho' - \rho'')\, \varepsilon(\rho')\, e^{j\varphi(\rho')}. \tag{21}$$

Let now integrate Eq. (21) over the aperture. The assumption on spatial ergodicity of the random field $\varphi(\rho)$ makes this procedure statistically equivalent to averaging over ensemble at a high angular resolution, i. e, to the case when the aperture is much larger than the interval of phase fluctuations. The limiting accuracy then takes the form

$$\sigma_\varphi^{-2} = \frac{A_0}{N_0} \mathrm{Re} \int_\Omega d\rho\, \varepsilon_0^*(\rho)\, \varepsilon(\rho)\, e^{j\varphi(\rho)}. \tag{22}$$

Taking into account the mathematical model assumed

$$\varepsilon_c(\rho) = e^{j\varphi(\rho)} \int_{\Omega_0} d\mathbf{r}\, E_s(\mathbf{r})\, E(\mathbf{r})\, e^{j(k/R)\mathbf{r}\rho}, \tag{23}$$

which, in fact, is the Frauhnhofer approximation written assuming the presence of a thin phase screen, we obtain from Eq. (22)

$$\sigma_\varphi^{-2} = \frac{A_0}{N_0} \mathrm{Re} \int_\Omega d\rho\, \varepsilon_0^*(\rho)\, e^{j\varphi(\rho)} \int_{\Omega_0} d\mathbf{r}\, E_s(\mathbf{r})\, E(\mathbf{r})\, e^{j(k/R)\mathbf{r}\rho}. \tag{24}$$

Here k is the wave number, and R is the distance between the aperture plane and the target plane.

As follows from Eq. (24), the limiting accuracy is linear with respect to the variable function $E_s(\mathbf{r})$. Hence the question on its best form may only be considered for the case when there are some nonlinear, for example, energy limitations,

$$\int_{\Omega_0} d\mathbf{r}\, E_s(\mathbf{r})\, E(\mathbf{r})\, E_s^*(\mathbf{r})\, E^*(\mathbf{r}) = E_0\,, \quad (25)$$

where E_0 is the energy in the target plane. Then, following the Lagrange factor technique[5], we may construct an auxiliary functional

$$\Phi(E_s(\mathbf{r})) = \frac{A_0}{N_0} \mathrm{Re} \int_{\Omega} d\rho\, \varepsilon_0^*(\rho)\, e^{j\varphi(\rho)} \int_{\Omega_0} d\,\mathbf{r}\, E_s(\mathbf{r})\, E(\mathbf{r})\, e^{j\,(k/R)\mathbf{r}\,\rho} +$$

$$+ \lambda\Big(\int_{\Omega_0} d\mathbf{r}\, |E_s(\mathbf{r})\, E(\mathbf{r})|^2 - E_0\Big), \quad (26)$$

where λ is the Lagrange factor. By differentiating Eq. (26) with respect to $|E_s(\mathbf{r})|$ and arg $E_s(\mathbf{r})$, we obtain:

$$\frac{\delta\, \Phi(E_s(\mathbf{r}))}{\delta\, |E_s(\mathbf{r})|} =$$

$$= \frac{A_0}{N_0} \mathrm{Re} \int_{\Omega} d\rho\, \varepsilon_0^*(\rho)\, e^{j\varphi(\rho)}\, E(\mathbf{r})\, e^{j \arg E_s(\mathbf{r})}\, e^{j(k/R)\mathbf{r}\,\rho} +$$

$$+ 2\lambda\, |E(\mathbf{r})|^2\, |E_s(\mathbf{r})| = 0, \quad (27)$$

$$\frac{\delta\, \Phi(E_s(\mathbf{r}))}{\delta \arg E_s(\mathbf{r})} = \frac{A_0}{N_0} \mathrm{Re}\, j \int_{\Omega} d\rho\, \varepsilon_0^*(\rho)\, e^{j\varphi(\rho)}\, e^{j(k/R)\mathbf{r}\,\rho}\, E(\mathbf{r})\, E_s(\mathbf{r}) =$$

$$= \frac{A_0}{N_0} \mathrm{Re}\, e^{j((\pi/2)-2\pi n)} E(\mathbf{r})\, E_s(\mathbf{r}) \int_{\Omega} d\rho\, \varepsilon_0^*(\rho)\, e^{j\varphi(\rho)} e^{j(k/R)\mathbf{r}\,\rho} = 0. \quad (28)$$

Solving Eqs. (27) and (28) with respect to $|E_s(\mathbf{r})|$ and arg $E_s(\mathbf{r})$, we have

$$\arg E_{s\,\mathrm{opt}}(\mathbf{r}) = -\arg E(\mathbf{r}) - \arg \int_{\Omega} d\rho\, \varepsilon_0^*(\rho)\, e^{j\varphi(\rho)} e^{j(k/R)\mathbf{r}\,\rho} + 2\pi n, \quad (29)$$

$$|E_{s\,\mathrm{opt}}(\mathbf{r})| = -\frac{A_0}{N_0} \frac{1}{2\lambda} \frac{1}{|E(\mathbf{r})|} \left| \int_{\Omega} d\mathbf{r}\, \varepsilon_0^*(\mathbf{r})\, e^{j\varphi(\rho)}\, e^{j(k/R)\mathbf{r}\,\rho} \right|, \quad (30)$$

$n = 0;\ \pm 1;\ \pm 2;\ \dots;\ \ l = 1, \dots, L.$

By substituting Eq. (30) into limitation (25) we have

$$\frac{A_0^2}{N_0^2} \frac{1}{4\lambda^2} \int_{\Omega_0} d\mathbf{r} \left| \int_{\Omega} d\rho\, \varepsilon_0^*(\rho)\, e^{j\varphi(\rho)}\, e^{j(k/R)\mathbf{r}\,\rho} \right|^2 = E_0\,, \quad (31)$$

from which it follows that

$$\lambda_{1,2} = \pm \frac{A_0}{2N_0} \sqrt{\frac{1}{E_0} - \int_{\Omega_0} d\,\mathbf{r} \left| \int_{\Omega} d\rho\, \varepsilon_0^*(\rho)\, e^{j\varphi(\rho)}\, e^{j(k/R)\mathbf{r}\,\rho} \right|^2} \quad .(32)$$

Note from a comparison of Eqs. (30) and (32), that only the solution corresponding to $\lambda < 0$ is valid because of the physical condition that $|E_s(\mathbf{r})| > 0$. Then the final expression is

$$|E_{s\,opt}(\mathbf{r})| = \frac{\sqrt{E_0}}{|E(\mathbf{r})|} \sqrt{\frac{\left| \int_{\Omega} d\rho\, \varepsilon_0^*(\rho)\, e^{j\varphi(\rho)}\, e^{j(k/R)\mathbf{r}\,\rho} \right|^2}{\int_{\Omega_0} d\mathbf{r} \left| \int_{\Omega} d\rho\, \varepsilon_0^*(\rho)\, e^{j\varphi(\rho)}\, e^{j(k/R)\mathbf{r}\,\rho} \right|^2}}. \quad (33)$$

Since the obtained functional is quadratic, one can show that thus derived solution is just the solution we have initially sought.

Let now the integration be carried out over infinite planes, and $\varepsilon_0(\rho)$ be the ideal estimate of the signal spectrum, multiplied by phase distortions introduced by the medium. It is obvious that in this case we have from Eq. (33)

$$|E_{s\,\mathrm{opt}}(\mathbf{r})| = \mathrm{const}, \quad (34)$$

since

$$\lim_{\substack{\Omega_0\to\infty \\ \Omega\to\infty \\ N_0\to\infty}} \int_{\Omega_0} d\rho \left| \int_{\Omega} d\rho\, \varepsilon_0^*(\rho)\, e^{j\varphi(\rho)}\, e^{j(k/R)\mathbf{r}\,\rho} \right|^2 \sim E_0\,, \quad (35)$$

$$\lim_{\substack{\Omega_0\to\infty \\ \Omega\to\infty \\ N_0\to\infty}} \int_{\Omega} d\rho\, \varepsilon_0^*(\rho)\, e^{j\varphi(\rho)}\, e^{j(k/R)\mathbf{r}\,\rho} = E(\mathbf{r}). \quad (36)$$

Thus it is shown that the view of a sounding signal in the target plane does not play a decisive role in the problem of attainment of the highest accuracy of phase distortions estimation, since no *a priori* assumptions on the target image E(r) are needed, and it is only desirable, naturally, the signal to be concentrated in the area occupied by the target itself. As a consequence, we may state that the technique of estimating (compensating for) the phase distortions introduced by a turbulent medium proposed in Refs. 1 and 2 provides maximum accuracy of such estimates and can be successfully used in problems on reconstructing images distorted by a turbulent medium.

However, since the algorithm of active reconstruction appears to be not optimal because the sounding signal is scattered in the regions outside the target, it is worthwhile to consider the <energy crisis> noted in this paper and the ways to overcome it.

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